



Asymptotic mean value Laplacian in metric measure spaces

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ABSTRACT

We use the mean value property in an asymptotic way to provide a notion of a pointwise Laplacian, called AMV Laplacian, that we study in several contexts including the Heisenberg group and weighted Lebesgue measures. We focus especially on a class of metric measure spaces including intersecting submanifolds of \mathbb{R}^n , a context in which our notion brings new insights; the Kirchhoff law appears as a special case. In the general case, we also prove a maximum and comparison principle, as well as a Green-type identity for a related operator.

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1. Introduction and definitions

Harmonic functions defined on Euclidean domains are well-known to have the mean value property: for any domain $\Omega \subset \mathbb{R}^n$, a function $u \in C^2(\Omega)$ such that $\Delta u = 0$ (where $\Delta = \sum_{k=1}^n \partial_{kk}$) satisfies

$$u(x) = \int_{B_r(x)} u \tag{1.1}$$

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for any solid ball $B_r(x) \subseteq \Omega$ centered at x with radius $r > 0$. We call this the *strong mean value property*. Here, $\int_{B_r(x)} u$ is defined as the average integral $\mathcal{L}^n(B_r(x))^{-1} \int_{B_r(x)} u \, d\mathcal{L}^n$, where \mathcal{L}^n denotes the Lebesgue measure. This celebrated result is credited to Gauss [9].

The converse implication was first studied by Koebe [14]. He proved that if u is continuous in Ω and satisfies the mean value property on every sphere, then u is harmonic, i.e., $\Delta u = 0$. Moreover, Koebe's arguments show that the same statement is true if the mean value property is satisfied at every $x \in \Omega$ only for some radii $\{r_i(x)\}_i$ such that $\inf_i r_i(x) = 0$. Later on, Volterra [18] for regular domains Ω and then Kellogg [12] for general domains proved that (1.1) is enough for a *single* radius, i.e., a function $u \in C(\overline{\Omega})$ is harmonic if it satisfies

$$u(x) = \int_{B_{r(x)}(x)} u \quad \forall x \in \Omega,$$

where r is a positive function on Ω with $r(x) < d(x, \Omega^c)$. We call this the *weak mean value property*. To sum up, under suitable conditions, we have the following equivalences.

$$\begin{aligned} u \text{ is harmonic.} &\iff u \text{ has the strong mean value property.} \\ &\iff u \text{ has the weak mean value property.} \end{aligned}$$

For the interested reader, we refer to the exhaustive survey by Netuka and Veselý [17], and the work of Llorente [15].

Moving from Euclidean domains to a general metric measure space (X, d, μ) , the above picture led Gaczkowski and Górká [8] to study the properties of locally integrable functions $u : X \rightarrow \mathbb{R}$ such that

$$u(x) = \int_{B_r(x)} u \, d\mu$$

for any ball $B_r(x) \subseteq \Omega$ centered at x with radius $r > 0$. Such functions are called *strongly harmonic* by Adamowicz, Gaczkowski and Górká [1]. Also *weakly harmonic* functions are introduced, namely those locally integrable functions $u : X \rightarrow \mathbb{R}$ satisfying, for every x , the mean value property for a single radius, denoted $r(x)$:

$$u(x) = \int_{B_{r(x)}(x)} u \, d\mu.$$

It turns out that, in this context in which a pointwise definition of a Laplacian is delicate to set, strongly and weakly harmonic functions share some common properties with harmonic functions on Euclidean domains. For instance, the maximum principle and the Harnack inequality hold under rather general assumptions.

However, weak harmonicity does not imply strong harmonicity [1, Example 1]:

$$u \text{ is weakly harmonic.} \not\Rightarrow u \text{ is strongly harmonic.}$$

Moreover, as shown by Bose [5], there exist functions in \mathbb{R}^n satisfying a weighted Laplace equation without being strongly harmonic. In hindsight this might not be too surprising due to \mathbb{R}^n being homogeneous seen as a medium. For a general measure, local properties (think partial derivatives) might have little to do with macroscopic properties (think strong harmonicity). We will return to this issue in Section 2.

In this article, we propose another approach based on the asymptotic fulfillment of the mean value property. This idea has been in the air for some time — see for instance the works of Manfredi et al. [16,7] and of Burago et al. [6] — but perhaps has not been completely crystallized yet.

We want to point out that during the finalization of this article, it came to our attention via private communication with Adamowicz that he, Kijowski and Soultanis [2] independently have come up with the same definition. They have investigated several related problems, e.g. the Hölder regularity of continuous asymptotic mean value harmonic functions, and the dimension of the space of continuous asymptotic mean value harmonic functions with polynomial growth.

It should be noted that when we refer to a metric measure space (X, μ, d) , we always consider a strictly positive and locally finite metric measure space, i.e., $\mu(B_r(x)) < \infty$ for $r > 0$ small enough, and $\mu(\Omega) > 0$ for every non-empty open subset $\Omega \subset X$. To be able to discuss pointwise properties of L^1_{loc} functions, we use the convention of choosing the representative \tilde{u} of $u \in L^1_{loc}(X, \mu)$ defined at x as the limit of $\int_{B_r(x)} u(y) d\mu(y)$ as $r \rightarrow 0^+$, whenever the limit exists, which it does μ -a.e.

Definition 1.1 (AMV Laplacian). Let (X, d, μ) be a metric measure space and $u : X \rightarrow \overline{\mathbb{R}}$ be locally integrable. Then the *asymptotic mean value Laplacian* (AMV Laplacian for short) of u is defined as

$$\Delta_\mu^d u(x) := \lim_{r \rightarrow 0^+} \frac{1}{r^2} \int_{B_r(x)} u(y) - u(x) d\mu(y)$$

for any $x \in X$ for which the limit exists. We also define for μ -a.e. x

$$\Delta_{\mu,r}^d u(x) := \frac{1}{r^2} \int_{B_r(x)} u(y) - u(x) d\mu(y)$$

for any $r > 0$.

If u is defined only on a subset $\Omega \subset X$, then we set

$$\Delta_{\mu,r}^d u(x) := \frac{1}{r^2} \int_{B_r(x) \cap \Omega} u(y) - u(x) d\mu(y).$$

Note that this definition does not require Ω to be open, only that $\mu(B_r(x) \cap \Omega) > 0$ for all positive r .

Having a notion of pointwise Laplacian at our disposal, we can define harmonicity in the following manner.

Definition 1.2 (AMV harmonic function). Let (X, d, μ) be a metric measure space and $u \in L^1_{loc}(X, \mu)$. We say that u is *asymptotically mean value harmonic* (AMV harmonic for short) in $\Omega \subset X$ if $\Delta_\mu^d u(x) = 0$ for all $x \in \Omega$.

Remark 1.3. There are evident versions $\overline{\Delta}_\mu^d u(\cdot)$ and $\underline{\Delta}_\mu^d u(\cdot)$ in which the limit is replaced by \limsup and \liminf , for which one can define asymptotically mean value sub- and superharmonic functions, see Section 4.

AMV harmonic functions have some advantages over strongly harmonic ones. Indeed, the latter are trivially seen to be AMV harmonic. Moreover, AMV harmonicity comes along with a natural notion of Laplacian, which is absent in the context of strongly harmonic functions. Therefore, it is possible to consider the corresponding Poisson equation, $\Delta_\mu^d u(x) = f$, heat equation or Helmholtz equation, etc.

A natural question is the relation of the AMV Laplacian with other notions of Laplacians. When $u \in C^2(\mathbb{R}^n)$, one can easily show that a second order Taylor expansion gives

$$\Delta_{\mathcal{L}^n}^{d_e} u = \frac{1}{2(n+2)} \Delta u,$$

where d_e stands for the Euclidean distance. Using normal coordinates, this calculation can be adapted to the setting of a C^2 Riemannian manifold (M, g) in order to prove that for any $u \in C^2(M)$ and any interior point $x \in M$,

$$\Delta_{\text{vol}_g}^{d_g} u(x) = \frac{1}{2(n+2)} \Delta_g u(x),$$

where d_g , vol_g and Δ_g are the canonical Riemannian distance, the Riemannian volume measure and the Laplace–Beltrami operator of (M, g) respectively. In Section 2, we consider more examples: the Heisenberg group, and the Euclidean space equipped with a weighted Lebesgue measure, showing that in both cases the AMV Laplacian is comparable to the corresponding Laplace operator, namely the Kohn Laplacian and the weighted Laplacian respectively.

More interestingly, our definition permits to deduce some results on spaces which, to the best of our knowledge, has not been proposed yet. We focus especially on metric measure spaces for which the measure has a dimension that depends on the part of the space we are looking at. This is formalized through the notion of a stratified measure, see Section 3. Our results (Theorem 3.2 and Corollary 3.4) show that the AMV Laplacian at a point in the intersection of the supports of measures with different Ahlfors dimension only takes into account the *lowest* dimension. Here by Ahlfors dimension we mean a number $Q \geq 0$ such that the measure is Q -Ahlfors regular, see Section 3 for details. We apply these results in the context of submanifolds of \mathbb{R}^n intersecting each other, see Corollary 3.7. Note that our example in Subsection 2.3, namely the Euclidean space equipped with the Lebesgue measure plus a Dirac mass, can be seen as a particular case of these stratified spaces.

In Section 4 we define AMV sub- and superharmonic functions, and show that the maximum of upper semicontinuous AMV subharmonic functions is attained on the boundary (see Theorem 4.3 for precise assumptions). A symmetrical argument goes through for lower semicontinuous AMV superharmonic functions, and a comparison principle is obtained as a corollary.

Finally, we prove a Green-type identity for the operators $\Delta_{\mu,r}^d$ restricted to a suitable weighted L^2 space. This formula suggests to define the weak AMV Laplacian $\Delta_{\mu}^d u$ as the measure ν such that

$$\int_X \phi \, d\nu = \lim_{r \rightarrow 0^+} \int_X \phi \Delta_{\mu,r}^d u \, d\mu$$

holds for any $\phi \in C_c(X)$, see Definition 5.5. It is worth pointing out that with this definition, pointwise AMV harmonic functions might fail to be weakly AMV harmonic. An example can be found in the paragraph following Definition 5.5.

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2. Examples

In this section, we get some familiarity with the AMV Laplacian by looking at three different examples for which it is possible to do explicit computations.

2.1. Heisenberg group

Let \mathbb{H} be the Heisenberg group that we interpret here as \mathbb{R}^3 equipped with the following group law:

$$(x, y, t) \circ (x', y', t') := (x + x', y + y', t + t' + 2(yx' - y'x)) \quad \forall (x, y, t), (x', y', t') \in \mathbb{R}^3.$$

We equip \mathbb{H} with the classical vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = -4 \frac{\partial}{\partial t},$$

which provide a sub-Riemannian structure to \mathbb{H} . We denote by d_{CC} the associated intrinsic metric, called the Carnot–Carathéodory metric (see e.g. [4, 5.2]). We recall that $B_r(p) = p \circ \delta_r(B_1(o))$ for any $p \in \mathbb{H}$ and any $r > 0$, where δ_r is the dilation $(x, y, t) \mapsto (rx, ry, r^2t)$ and $o = (0, 0, 0)$ is the origin.

For any $p_o = (x_o, y_o, t_o) \in \mathbb{H}$, let L_{p_o} be the left translation $\mathbb{H} \ni p \mapsto p_o \circ p$. Then L_{p_o} is smooth and its Jacobian matrix in $q = (x, y, t)$ is

$$J_{p_o}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2y_o & 2x_o & 1 \end{pmatrix};$$

in particular $|\det(J_{p_o}(q))| = 1$.

In the next proposition, we show that for C^3 functions the AMV Laplacian on $(\mathbb{H}, d_{CC}, \mathcal{L}^3)$ coincides with the usual Kohn Laplacian $\Delta_{\mathbb{H}} = X^2 + Y^2$ up to a multiplicative constant. This result is already known from the work of Ferrari et al. [7], but we provide a slightly different proof for the reader’s convenience.

Proposition 2.1. *Let $u \in C^3(\mathbb{H})$. Then*

$$\Delta_{\mathcal{L}^3}^{d_{CC}} u = c \Delta_{\mathbb{H}} u,$$

where $c = \frac{1}{2} \int_{B_1(o)} x^2 d\mathcal{L}^3$.

Proof. Let $u \in C^3(\mathbb{H})$ and $p_o = (x_o, y_o, t_o) \in \mathbb{H}$. Note that in this proof, all the balls are taken with respect to the Carnot–Carathéodory metric. For any given $r > 0$, by the change of variable $p = L_{p_o}(q)$ we have

$$\begin{aligned} \Delta_{\mathcal{L}^3, r}^{d_{CC}} u(p_o) &= \frac{1}{r^2} \int_{B_r(p_o)} u(p) - u(p_o) dp = \frac{1}{r^2} \int_{B_r(o)} [u(L_{p_o}(q)) - u(p_o)] |\det(J_{p_o}(q))| dq \\ &= \frac{1}{r^2} \int_{B_r(o)} [v(q) - v(o)] dq, \end{aligned} \tag{2.1}$$

where we have set $v = u \circ L_{p_o}$. Let us write the Taylor expansion of v at o (see [4, p. 743]¹): for $q = (x, y, t) \in B_r(o)$,

$$\begin{aligned} v(q) &= v(o) + (Xv)(o)x + (Yv)(o)y + (Tv)(o)t \\ &\quad + \frac{1}{2}(X^2v)(o)x^2 + \frac{1}{2}(Y^2v)(o)y^2 + [2(Tv)(o) + (XYv)(o)]xy + O(r^3). \end{aligned}$$

¹ It is at this point we need $u \in C^3(\mathbb{H})$.

Since the vector fields X , Y and T commute with left translations,

$$\begin{aligned} v(q) &= v(o) + (Xu)(p_o)x + (Yu)(p_o)y + (Tu)(p_o)t \\ &\quad + \frac{1}{2}(X^2u)(p_o)x^2 + \frac{1}{2}(Y^2u)(p_o)y^2 + [2(Tu)(p_o) + (XYu)(p_o)]xy + O(r^3). \end{aligned}$$

Plugging this expression into (2.1) and using the fact that $\mathcal{L}^3(B_r(p_o)) = \mathcal{L}^3(B_r(o))$, we obtain

$$\begin{aligned} \Delta_{\mathcal{L}^3, r}^{\text{d}_{CC}} u(p_o) &= \frac{1}{r^2} \left((Xu)(p_o) \int_{B_r(o)} x \, dq + (Yu)(p_o) \int_{B_r(o)} y \, dq + (Tu)(p_o) \int_{B_r(o)} t \, dq \right. \\ &\quad + \frac{1}{2}(X^2u)(p_o) \int_{B_r(o)} x^2 \, dq + \frac{1}{2}(Y^2u)(p_o) \int_{B_r(o)} y^2 \, dq \\ &\quad \left. + [2(Tu)(p_o) + (XYu)(p_o)] \int_{B_r(o)} xy \, dq \right) + O(r). \end{aligned}$$

Now, it is known that a Carnot–Carathéodory ball centered at the origin of \mathbb{H} is symmetric around the t -axis and also symmetric with respect to the xy -plane, hence

$$\int_{B_r(o)} x \, dq = \int_{B_r(o)} y \, dq = \int_{B_r(o)} t \, dq = \int_{B_r(o)} xy \, dq = 0.$$

Moreover, a Carnot–Carathéodory ball centered at the origin is invariant under rotations around the z -axis, consequently

$$\int_{B_r(o)} x^2 \, dq = \int_{B_r(o)} y^2 \, dq.$$

Therefore we get

$$\begin{aligned} \Delta_{\mathcal{L}^3, r}^{\text{d}_{CC}} u(p_o) &= \frac{(X^2u)(p_o) + (Y^2u)(p_o)}{2r^2} \int_{B_r(o)} x^2 \, dq + O(r) \\ &= \Delta_{\mathbb{H}} u(p_o) \frac{1}{2r^2} \int_{B_r(o)} x^2 \, dq + O(r) \end{aligned}$$

Since $B_r(o) = \delta_r(B_1(o))$ and the Jacobian determinant of δ_r is constant equal to r^4 , the change of variable $q = \delta_r(q')$ provides $\mathcal{L}^3(B_r(o)) = r^4 \mathcal{L}^3(B_1(o))$ and $\int_{B_r(o)} x^2 \, dq = r^4 \int_{B_1(o)} (rx)^2 \, dq$. Therefore

$$\Delta_{\mathcal{L}^3, r}^{\text{d}_{CC}} u(p_o) = \frac{\Delta_{\mathbb{H}} u(p_o)}{2} \int_{B_1(o)} x^2 \, dq + O(r),$$

hence the result. \square

2.2. Weighted Lebesgue measures in \mathbb{R}^n

In an earlier work by A.K. Bose [5], the author considers weighted Lebesgue measures

$$\mu = w\mathcal{L}^n$$

on connected open sets Ω in \mathbb{R}^n , where $w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n)$ is nonnegative and such that $\mu(B) > 0$ for any ball $B \subset \Omega$. He shows that when $w \in C^1(\Omega)$, any function $u \in L^1_{\text{loc}}(\Omega, \mu)$ which satisfies the mean value property,

$$u(x) = \int_{B_r(x)} u \, d\mu \quad \forall B_r(x) \subset \Omega,$$

is a C^2 function and a solution to the partial differential equation

$$L_w u := w\Delta u + 2\nabla w \cdot \nabla u = 0 \tag{2.2}$$

in Ω , hence a harmonic function for the weighted Laplacian L_w (see [13] for the case of a Sobolev regular weight function). The converse is not true:

Example 2.2 ([5]). Let $w(x, y) = (x + y)^2$ and $u(x, y) = x^2 - 3xy + y^2$ in \mathbb{R}^2 . A direct computation shows that u satisfies (2.2), but

$$\int_{B_r(x,y)} u \, d\mu = u(x, y) + \frac{r^4}{6(r^2 + 2(x + y)^2)} \neq u(x, y)$$

for all $(x, y) \in \mathbb{R}^2$ and $r > 0$. However, with our notation, we have

$$\Delta_{\mu,r}^{de} u(x, y) = \frac{r^2}{6(r^2 + 2(x + y)^2)}$$

for any $(x, y) \in \mathbb{R}^2$ and $r > 0$. Letting r tend to 0 shows that u is AMV harmonic outside the diagonal $\{x = -y\}$, which coincides with $\{w = 0\}$. More generally, we have the following.

Proposition 2.3. Let $w \in C^1(\Omega)$ and $u \in C^2(\Omega)$. Then for any $x \in \Omega$ such that $w(x) \neq 0$,

$$\Delta_{\mu}^{de} u(x) = c_n \frac{L_w u(x)}{w(x)}, \tag{2.3}$$

where $c_n = 2^{-1}(n + 2)^{-1}$. Moreover, for any $x \in \Omega$ such that $w(x) = 0$, assume that $b^r(x) := \frac{1}{r^2} \int_{B_r(x)} (y - x)w(y) \, dy$ and $a_{ij}^r(x) := \frac{1}{2r^2} \int_{B_r(x)} (y - x)_i (y - x)_j w(y) \, dy$ converge to $b(x) \in \mathbb{R}^n$ and $a_{ij}(x) \in \mathbb{R}$ respectively when $r \rightarrow 0^+$. Then

$$\Delta_{\mu}^{de} u(x) = \sum_{i,j=1}^n a_{ij}(x) \partial_{ij} u(x) + b(x) \cdot \nabla u(x). \tag{2.4}$$

Proof. Let $x \in \Omega$ be such that $w(x) \neq 0$. By first and second order Taylor expansions of w and u respectively, we know that there exists a function $E : (0, +\infty) \rightarrow (0, +\infty)$ such that $E(r) \rightarrow 0$ when $r \rightarrow 0^+$ and for any $r > 0$ and $y \in B_r(x)$,

$$\begin{aligned}(u(y) - u(x))w(y) &= w(x)\nabla u(x) \cdot (y - x) + \frac{w(x)}{2}\nabla^2 u(x) \cdot (y - x, y - x) \\ &\quad + [\nabla u(x) \cdot (y - x)][\nabla w(x) \cdot (y - x)] + E(r)r^2.\end{aligned}$$

Note that for any $v \in \mathbb{R}^n$, the antisymmetry of $y \mapsto v \cdot (y - x)$ with respect to the hyperplane v^\perp implies that $\int_{B_r(x)} v \cdot (y - x) dx = 0$, and that $\int_{B_r(x)} (y - x)_i (y - x)_j dy = 0$ for $i \neq j$ and $\int_{B_r(x)} (y - x)_i^2 dy = 2c_n \mathcal{L}^n(B_r(x))r^2$ for any $1 \leq i, j \leq n$. From this, a direct computation shows that

$$\Delta_{\mu,r}^d u(x) = c_n \frac{\mathcal{L}^n(B_r(x))}{\mu(B_r(x))} (w(x)\Delta u(x) + 2\nabla u(x) \cdot \nabla w(x) + o(1)).$$

Since w is C^1 , $\mu(B_r(x))/\mathcal{L}^n(B_r(x)) \rightarrow w(x)$ as $r \rightarrow 0^+$, hence (2.3).

Now take $x \in \Omega$ such that $w(x) = 0$ and the required assumptions are satisfied. By a similar expansion as above, but with respect to u only gives that

$$\Delta_{\mu,r}^{d_e} u(x) = \nabla u(x) \cdot b^r(x) + \sum_{i,j=1}^n \partial_{ij} u(x) a_{ij}^r + o(1),$$

hence (2.4). \square

Note that (2.3) is consistent with the unweighted case $w \equiv 1$. Moreover, in Example 2.2 where $w(x, y) = (x + y)^2$, explicit calculations from (2.4) show that for any $u \in C^2(\mathbb{R}^2)$,

$$\Delta_{\mu}^{d_e} u = \frac{1}{6}(\Delta u + \partial_{xy} u) \quad \text{on } \{x = -y\}.$$

In particular, for $u(x, y) = x^2 - 3xy + y^2$ we get $\Delta_{\mu}^{d_e} u = 1/6$ on $\{x = -y\}$, hence u is not AMV harmonic.

2.3. The Lebesgue measure with a Dirac mass in \mathbb{R}^n

Let us consider $(X, d, \mu) := (\mathbb{R}^n, d_e, \mathcal{L}^n + \delta_o)$ where δ_o is a Dirac measure at the origin o of \mathbb{R}^n . Take $u \in L_{\text{loc}}^1(\mathbb{R}^n, \mathcal{L}^n)$ and $x \in \mathbb{R}^n$. If $x \neq o$, we trivially get $\Delta_{\mu}^{d_e} u(x) = \Delta_{\mathcal{L}^n}^{d_e} u(x)$. Consider therefore the case $x = o$. Since $\mu(B_r(o)) = 1 + \omega_n r^n$ (where $\omega_n = \mathcal{L}^n(B_1)$) for any $r > 0$,

$$\begin{aligned}\Delta_{\mu,r}^{d_e} u(o) &= \frac{1}{r^2(1 + \omega_n r^n)} \left(\int_{B_r(o)} u(y) - u(x) dy + \underbrace{\int_{B_r(o)} u(y) - u(o) d\delta_o(y)}_{=u(o) - u(o)=0} \right) \\ &= \frac{\mathcal{L}^n(B_r(o))}{r^2(1 + \omega_n r^n)} \int_{B_r(o)} u(y) - u(o) dy.\end{aligned}$$

Since $1/(1 + \omega_n r^n) = 1 + O(r^n)$, we get

$$\Delta_{\mu,r}^{d_e} u(o) = \omega_n r^{n-2} (1 + O(r^n)) \int_{B_r(o)} u(y) - u(o) dy.$$

This simple computation leads to the following:

Observation 2.4. Assume that o is a Lebesgue point of u with respect to \mathcal{L}^n and denote by $u^*(o)$ the unique number $a \in \mathbb{R}$ such that $\int_{B_r(x)} |u(y) - a| dy \rightarrow 0$ when $r \rightarrow 0^+$. If $n = 1$, assume additionally that $r^{-1} \int_{B_r(o)} u(y) - u(o) dy$ converges to some constant b as $r \rightarrow 0^+$. Then:

$$\Delta_{\mu}^{d_e} u(o) = \begin{cases} 0 & \text{when } n \geq 3, \\ \pi(u^*(o) - u(o)) & \text{when } n = 2, \\ 2b & \text{when } n = 1. \end{cases}$$

Note that in case $n = 1$, we obviously have $u^*(o) = u(o)$. Moreover, u differentiable at o is enough to imply convergence of $\Delta_{\mu,r}^{d_e} u(o)$ to 0 as $r \rightarrow 0^+$.

It is also worth mentioning that for the Poisson problem $\Delta_{\mu}^d u = v$, when $n \geq 3$, a necessary condition for existence of a solution is $v(o) = 0$.

This last example is a special case of a stratified measure that we discuss in the next section.

3. AMV Laplacian for stratified measures

In this section, we introduce the notion of a stratified measure and study the AMV Laplacian at the intersection of strata of such a measure. We then apply our results to the case of intersecting submanifolds in \mathbb{R}^n .

Recall that for $Q \geq 0$, a Borel measure μ on a metric space (X, d) is called Ahlfors Q -regular if it satisfies $cr^Q \leq \mu(B) \leq Cr^Q$ for any metric ball $B \subset X$ with radius $r > 0$, where $c, C > 0$ are independent of the radius and the ball. If a measure μ is Q -Ahlfors regular, we say that μ has an Ahlfors dimension equal to Q .

Definition 3.1 (*Stratified measures*). Let (X, d) be a metric space. We call μ a *stratified measure* on (X, d) if

$$\mu = \mu_1 + \dots + \mu_k,$$

where for any $1 \leq j \leq k$,

- (i) μ_j is a measure supported on a closed set $Y_j \subset X$,
- (ii) μ_j is Q_j -Ahlfors regular on $(Y_j, d|_{Y_j})$,
- (iii) $Q_1 < \dots < Q_k$.

A particular example of a stratified measure is $\mathcal{H}^{m_1} \llcorner M_1 + \dots + \mathcal{H}^{m_k} \llcorner M_k$ where M_1, \dots, M_k are submanifolds of \mathbb{R}^n of dimensions $m_1 < \dots < m_k$ respectively. See also Example 3.9.

The next theorem states that the lowest dimensional stratum determines the AMV Laplacian of a stratified measure.

Theorem 3.2 (*AMV Laplacian on intersections of strata of a stratified measure*). Let (X, d) be a metric space equipped with a stratified measure μ and $u \in L^1_{loc}(X, \mu)$. For any $x \in \bigcap_{i=1}^l Y_{j_i}$ where $\{j_i\}$ is an increasing subsequence of $\{1, \dots, k\}$, if $\Delta_{\mu_{j_1}}^d u(x)$ exists and $r^{Q_{j_i} - Q_{j_1}} |\Delta_{\mu_{j_i}, r}^d u(x)| \rightarrow 0$ as $r \rightarrow 0^+$ for all $2 \leq i \leq l$, then $\Delta_{\mu}^d u(x)$ exists and

$$\Delta_{\mu}^d u(x) = \Delta_{\mu_{j_1}}^d u(x).$$

Proof. We can without loss of generality assume that $\{j_i\} = \{1, \dots, l\}$: indeed, since Y_j^c is open for any $1 \leq j \leq k$, there exists $r > 0$ small enough such that $\mu \llcorner B_r(x) = \mu_{j_1} \llcorner B_r(x) + \dots + \mu_{j_l} \llcorner B_r(x)$, hence the validity of the above relabeling. Let therefore $x \in \bigcap_{j=1}^l Y_j$ and consider $\Delta_{\mu,r}^d u(x)$. Then

$$\begin{aligned}\Delta_{\mu,r}^d u(x) &= \frac{1}{r^2 \mu(B_r(x))} \int_{B_r(x)} u(y) - u(x) \, d\mu(y) \\ &= \frac{1}{r^2 \mu(B_r(x))} \sum_{j=1}^l \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y).\end{aligned}$$

For a given j , let c_j, C_j be the Ahlfors constants related to μ_j . Note that

$$\mu(B_r(x)) = \sum_{j=1}^l \mu_j(B_r(x) \cap Y_j) \geq \sum_{j=1}^l c_j r^{Q_j} \geq c_1 r^{Q_1}.$$

For $j \geq 2$ we have, by assumption,

$$\begin{aligned}& \left| \frac{1}{r^2 \mu(B_r(x))} \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) \right| \\ & \leq \frac{1}{c_1 r^{Q_1+2}} \left| \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) \right| \\ & = \frac{\mu_j(B_r(x) \cap Y_j)}{c_1 r^{Q_1+2}} \left| \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) \right| \\ & \leq \frac{C_j r^{Q_j - Q_1}}{c_1} \left| \Delta_{\mu_j,r}^d u(x) \right| \rightarrow 0, \quad r \rightarrow 0^+.\end{aligned}\tag{3.1}$$

Then

$$\begin{aligned}& \left| \Delta_{\mu,r}^d u(x) - \Delta_{\mu_1}^d u(x) \right| \\ & = \left| \frac{1}{r^2 \mu(B_r(x))} \sum_{j=1}^l \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) - \Delta_{\mu_1}^d u(x) \right| \\ & = \left| \sum_{j=1}^l \frac{\mu_j(B_r(x) \cap Y_j)}{r^2 \mu(B_r(x))} \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) - \Delta_{\mu_1}^d u(x) \right| \\ & = \left| \sum_{j=1}^l \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_j,r}^d u(x) - \Delta_{\mu_1}^d u(x) \right| \\ & \leq \left| \frac{\mu_1(B_r(x) \cap Y_1)}{\mu(B_r(x))} \Delta_{\mu_1,r}^d u(x) - \Delta_{\mu_1}^d u(x) \right| + \left| \sum_{j=2}^l \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_j,r}^d u(x) \right|.\end{aligned}$$

The second term tends to zero as $r \rightarrow 0^+$ by (3.1). Moreover,

$$\begin{aligned}\frac{\mu_1(B_r(x) \cap Y_1)}{\mu(B_r(x))} &= \frac{\mu_1(B_r(x) \cap Y_1)}{\sum_{j=1}^k \mu_j(B_r(x) \cap Y_j)} \leq 1, \\ \frac{\mu_1(B_r(x) \cap Y_1)}{\mu(B_r(x))} &= \frac{\mu_1(B_r(x) \cap Y_1)}{\sum_{j=1}^k \mu_j(B_r(x) \cap Y_j)}\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1 + \sum_{j=2}^k \frac{\mu_j(B_r(x) \cap Y_j)}{\mu_1(B_r(x) \cap Y_1)}} \\
 &\geq \frac{1}{1 + \sum_{j=2}^k \frac{C_j}{c_1} r^{Q_j - Q_1}} \rightarrow 1, \quad r \rightarrow 0^+.
 \end{aligned}$$

Therefore

$$\limsup_{r \rightarrow 0^+} |\Delta_{\mu,r}^d u(x) - \Delta_{\mu_1}^d u(x)| = \limsup_{r \rightarrow 0^+} \left| \frac{\mu_1(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_1,r}^d u(x) - \Delta_{\mu_1}^d u(x) \right| = 0. \quad \square$$

Remark 3.3. Note that a sufficient condition for $r^{Q_{j_i} - Q_{j_1}} |\Delta_{\mu_{j_i},r}^d u(x)|$ to be $o(1)$ is boundedness of $|\Delta_{\mu_{j_i},r}^d u(x)|$ for all $2 \leq i \leq l$.

The previous theorem can be extended in a straightforward way to stratified measures with (iii) weakened to $Q_1 \leq Q_2 \leq \dots \leq Q_k$, if limits of the form $\lim_{r \rightarrow 0^+} \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))}$ exist as for the lowest dimensional stratum, reducing the AMV Laplacian to a convex combination of the AMV Laplacians of this stratum.

Corollary 3.4. Let (X, d, μ) and $x \in \bigcap_{i=1}^l Y_{j_i}$ be as in Theorem 3.2 except that Q_j is nondecreasing with $Q_{j_1} = Q_{j_2} = \dots = Q_{j_n} < Q_{j_{n+1}}$ and $r^{Q_{j_i} - Q_{j_1}} |\Delta_{\mu_{j_i},r}^d u(x)| \rightarrow 0$ as $r \rightarrow 0^+$ for any $n + 1 \leq i \leq l$. Assume that $\Delta_{\mu_{j_1}}^d u(x), \Delta_{\mu_{j_2}}^d u(x), \dots, \Delta_{\mu_{j_n}}^d u(x)$ exist. Assume also that $\alpha_i(x) := \lim_{r \rightarrow 0^+} \mu_j(B_r(x) \cap Y_{j_i}) / \mu(B_r(x))$ exists for any $1 \leq i \leq n$. Then $\Delta_{\mu}^d u(x)$ exists,

$$\Delta_{\mu}^d u(x) = \sum_{i=1}^n \alpha_i(x) \Delta_{\mu_{j_i}}^d u(x),$$

and $\sum_{i=1}^n \alpha_i(x) = 1$.

Proof. As in the proof of Theorem 3.2, we have $\mu(B_r(x)) \geq c_1 r^{Q_1}$, and we can reduce to the case $\{j_i\} = \{1, \dots, n, \dots, l\}$ for which

$$\Delta_{\mu,r}^d u(x) = \frac{1}{r^2 \mu(B_r(x))} \sum_{j=1}^l \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y).$$

For $j \geq n + 1$ we have, by assumption,

$$\begin{aligned}
 &\left| \frac{1}{r^2 \mu(B_r(x))} \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) \right| \\
 &\leq \frac{1}{c_1 r^{Q_1 + 2}} \left| \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) \right| \\
 &= \frac{\mu_j(B_r(x) \cap Y_j)}{c_1 r^{Q_1 + 2}} \left| \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) \right| \\
 &\leq \frac{C_j}{c_1} r^{Q_j - Q_1} \left| \Delta_{\mu_{j_i},r}^d u(x) \right| \rightarrow 0, \quad r \rightarrow 0^+. \tag{3.2}
 \end{aligned}$$

Continuing,

$$\begin{aligned}
& \left| \Delta_{\mu,r}^d u(x) - \sum_{j=1}^n \alpha_j(x) \Delta_{\mu_j}^d u(x) \right| \\
&= \left| \frac{1}{r^2 \mu(B_r(x))} \sum_{j=1}^l \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) - \sum_{j=1}^n \alpha_j(x) \Delta_{\mu_j}^d u(x) \right| \\
&= \left| \sum_{j=1}^l \frac{\mu_j(B_r(x) \cap Y_j)}{r^2 \mu(B_r(x))} \int_{B_r(x) \cap Y_j} u(y) - u(x) \, d\mu_j(y) - \sum_{j=1}^n \alpha_j(x) \Delta_{\mu_j}^d u(x) \right| \\
&= \left| \sum_{j=1}^l \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_j,r}^d u(x) - \sum_{j=1}^n \alpha_j(x) \Delta_{\mu_j}^d u(x) \right| \\
&\leq \left| \sum_{j=1}^n \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_j,r}^d u(x) - \sum_{j=1}^n \alpha_j(x) \Delta_{\mu_j}^d u(x) \right| \\
&\quad + \left| \sum_{j=n+1}^l \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_j,r}^d u(x) \right|
\end{aligned}$$

The second term tends to zero as $r \rightarrow 0^+$ by (3.2). Also, by assumption,

$$\frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \rightarrow \alpha_j(x), \quad r \rightarrow 0^+.$$

Therefore,

$$\begin{aligned}
& \limsup_{r \rightarrow 0^+} \left| \Delta_{\mu,r}^d u(x) - \sum_{j=1}^n \alpha_j(x) \Delta_{\mu_j}^d u(x) \right| \\
&\leq \limsup_{r \rightarrow 0^+} \left| \sum_{j=1}^n \left(\frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} \Delta_{\mu_j,r}^d u(x) - \alpha_j(x) \Delta_{\mu_j}^d u(x) \right) \right| = 0.
\end{aligned}$$

The fact $\sum_{j=1}^n \alpha_j(x) = 1$ is immediate since

$$\begin{aligned}
\left| \sum_{j=1}^n \alpha_j(x) - 1 \right| &= \left| \sum_{j=1}^n \lim_{r \rightarrow 0^+} \frac{\mu_j(B_r(x) \cap Y_j)}{\mu(B_r(x))} - 1 \right| \\
&= \left| \lim_{r \rightarrow 0^+} \frac{1}{\sum_{j=1}^l \mu(B_r(x) \cap Y_j)} \sum_{j=1}^n \mu_j(B_r(x) \cap Y_j) - 1 \right| \\
&= \left| \lim_{r \rightarrow 0^+} \frac{1}{\sum_{j=1}^l \mu(B_r(x) \cap Y_j)} \left(\sum_{j=1}^n \mu_j(B_r(x) \cap Y_j) - \sum_{j=1}^l \mu_j(B_r(x) \cap Y_j) \right) \right| \\
&= \left| \lim_{r \rightarrow 0^+} \frac{1}{O(r^{Q_1})} O(r^{Q_{n+1}}) \right| \\
&= 0. \quad \square
\end{aligned}$$

Example 3.5. An interesting example is $(\mathbb{R}^2, d_e, \mu := \mu_1 + \dots + \mu_l)$ where $\mu_i = \mathcal{H}^1 \llcorner S_i$ for any $1 \leq i \leq l$ and S_i is the image of a smooth curve c_i emanating from $o = (0, 0)$ with direction τ_i . Here \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure. A direct computation shows that for any $u \in C^2(\mathbb{R}^2)$,

$$\Delta_{\mu}^{d_e} u(o) \in \mathbb{R} \iff \sum_{i=1}^l \partial_{\tau_i} u(o) = 0,$$

where $\partial_{\tau_i} u$ is the directional derivative of u along τ_i . This is the well-known Kirchhoff condition (compare for instance with [3, Section 4]). If this condition holds, Corollary 3.4 implies that

$$\Delta_{\mu}^{d_e} u(o) = \frac{1}{l} \sum_{i=1}^l \Delta_{\mu_i}^{d_e} u(o).$$

Let us apply the previous results to the case of intersecting submanifolds in \mathbb{R}^n . Recall that a Riemannian submanifold of \mathbb{R}^n is equipped with the Riemannian metric inherited from the Euclidean metric of \mathbb{R}^n . If (M, g) is a smooth m -dimensional Riemannian submanifold of \mathbb{R}^n , then the topological metric d_g induced by g in the usual way (i.e. minimizing the length of curves joining two points) satisfies $d_g \geq d_e$, and the canonically associated Riemannian volume measure vol_g on M coincides with the m -dimensional Hausdorff measure \mathcal{H}^m .

Proposition 3.6. *Let (M, g) be a smooth m -dimensional Riemannian submanifold of \mathbb{R}^n , and consider $(\mathbb{R}^n, d_e, \mathcal{H}^m \llcorner M)$. Then, for any $u \in C^2(\mathbb{R}^n)$ and any interior point x in M ,*

$$\Delta_{\mathcal{H}^m \llcorner M}^{d_e} u(x) = \frac{\Delta_g u(x)}{2(m+2)},$$

where Δ_g is the Laplace–Beltrami operator on (M, g) .

Proof. Since $d_g \geq d_e$, the geodesic ball $B_r^g(x)$ is included in the subset $B_r(x) \cap M$ of \mathbb{R}^n . Therefore

$$\begin{aligned} \Delta_{\mathcal{H}^m \llcorner M, r}^{d_e} u(x) &= \frac{1}{r^2} \int_{B_r(x) \cap M} u(y) - u(x) \, d\mathcal{H}^m(y) \\ &= \frac{1}{r^2 \mathcal{H}^m(B_r(x) \cap M)} \left(\int_{(B_r(x) \cap M) \setminus B_r^g(x)} u(y) - u(x) \, d\mathcal{H}^m(y) \right. \\ &\quad \left. + \int_{B_r^g(x)} u(y) - u(x) \, dV_g(y) \right). \end{aligned}$$

For the first term, by a first order Taylor expansion of u around x ,

$$\begin{aligned} &\left| \frac{1}{r^2 \mathcal{H}^m(B_r(x) \cap M)} \int_{(B_r(x) \cap M) \setminus B_r^g(x)} u(y) - u(x) \, d\mathcal{H}^m(y) \right| \\ &\leq \frac{O(r)}{r^2 \mathcal{H}^m(B_r(x) \cap M)} (\mathcal{H}^m(B_r(x) \cap M) - \mathcal{H}^m(B_r^g(x))). \end{aligned}$$

Now, from the works of Karp and Pinsky [11], the volume of the extrinsic ball for small r is given by

$$\mathcal{H}^m(B_r(x) \cap M) = \omega_m r^m \left(1 + \frac{2\|II(x)\| - \|H(x)\|}{8(m+2)} r^2 + O(r^3) \right), \tag{3.3}$$

where II denotes the second fundamental form of (M, g) and $H = \text{Tr } II$ the mean curvature. Furthermore, the volume of the intrinsic ball has been calculated by Gray [10] for small r as

$$(\mathcal{H}^m(B_r^g(x)) =) \text{vol}_g(B_r^g(x)) = \omega_m r^m \left(1 - \frac{R(x)}{6(m+2)} r^2 + O(r^4) \right),$$

where R is the scalar curvature of (M, g) . Hence

$$\mathcal{H}^m(B_r(x) \cap M) - \mathcal{H}^m(B_r^g(x)) = O(r^{m+2}).$$

Therefore

$$\frac{O(r)}{r^2 \mathcal{H}^m(B_r(x) \cap M)} (\mathcal{H}^m(B_r(x) \cap M) - \mathcal{H}^m(B_r^g(x))) = \frac{O(r)O(r^{m+2})}{O(r^{m+2})} = O(r),$$

so the first term tends to zero with r . The second term tends to $\frac{\Delta_g u(x)}{2(m+2)}$ since

$$\frac{\mathcal{H}^m(B_r^g(x))}{\mathcal{H}^m(B_r(x) \cap M)} = \frac{\omega_m r^m (1 + O(r^2))}{\omega_m r^m (1 + O(r^2))} \rightarrow 1, \quad r \rightarrow 0^+,$$

and by the result of Gray and Willmore [19],

$$\frac{1}{r^2} \int_{B_r^g(x)} u(y) - u(x) \, dV_g \rightarrow \frac{\Delta_g u(x)}{2(m+2)}.$$

Consequently,

$$\begin{aligned} \frac{1}{r^2 \mathcal{H}^m(B_r(x) \cap M)} \int_{B_r^g(x)} u(y) - u(x) \, dV_g &= \frac{\mathcal{H}^m(B_r^g(x))}{r^2 \mathcal{H}^m(B_r(x) \cap M)} \int_{B_r^g(x)} u(y) - u(x) \, dV_g \\ &\rightarrow \frac{\Delta_g u(x)}{2(m+2)}, \quad \text{as } r \rightarrow 0^+. \quad \square \end{aligned}$$

In this smooth context, we can prove the following refinement of Corollary 3.4 where we get a mean value of the AMV Laplacians of the lowest stratum.

Corollary 3.7. *Let $(X, d, \mu) = (\mathbb{R}^n, d_e, \mathcal{H}^{m_1} \llcorner M_1 + \dots + \mathcal{H}^{m_k} \llcorner M_k)$ where $\{m_j\}$ is a non-decreasing sequence of integers and (M_j, g_j) is a smooth m_j -dimensional Riemannian submanifold of \mathbb{R}^n without boundary for any $1 \leq j \leq k$. Take $u \in L^1_{loc}(X, \mu)$ and let $x \in \bigcap_{i=1}^l M_{j_i}$ for a subsequence $\{j_i\}$ of $\{1, \dots, k\}$ with $m_{j_1} = \dots = m_{j_t}$. If $\Delta^d_{\mathcal{H}^{m_{j_i}} \llcorner M_{j_i}} u(x)$ exists for all $i \in \{1, \dots, t\}$ and $r^{m_{j_i} - m_{j_1}} |\Delta^d_{\mu_{j_i}, r} u(x)| \rightarrow 0$ as $r \rightarrow 0^+$ for all $i \in \{t+1, \dots, k\}$, then $\Delta^d_\mu u(x)$ exists and*

$$\Delta^d_\mu u(x) = \frac{1}{t} \sum_{i=1}^t \Delta^d_{\mathcal{H}^{m_{j_i}} \llcorner M_{j_i}} u(x).$$

Furthermore, if $u \in C^2(\mathbb{R}^n)$, we have

$$\Delta^d_\mu u(x) = \frac{1}{2(m_{j_1} + 2)t} \sum_{i=1}^t \Delta_{g_i} u(x).$$

Proof. The first part follows from Corollary 3.4 if we can show that the limits $\alpha_i(x) := \lim_{r \rightarrow 0^+} \frac{\mu_{j_i}(B_r(x) \cap M_{j_i})}{\mu(B_r(x))}$ for $1 \leq i \leq t$ exist and are all equal to $\frac{1}{t}$. This is true since for any such i , applying (3.3) with $m := m_{j_1}$ gives

$$\begin{aligned} \lim_{r \rightarrow 0^+} \frac{\mu_{j_i}(B_r(x) \cap M_{j_i})}{\mu(B_r(x))} &= \lim_{r \rightarrow 0^+} \frac{\omega_m r^m (1 + O(r^2))}{\sum_{k=1}^l \omega_m r^m (1 + O(r^2))} \\ &= \lim_{r \rightarrow 0^+} \frac{\omega_m r^m (1 + O(r^2))}{\sum_{k=1}^t \omega_m r^m (1 + O(r^2)) + O(r^{m+1})} \\ &= \lim_{r \rightarrow 0^+} \frac{\omega_m r^m (1 + O(r^2))}{t \omega_m r^m (1 + O(r^2))} = \frac{1}{t}. \end{aligned}$$

For $u \in C^2(\mathbb{R}^n)$,

$$\frac{1}{t} \sum_{i=1}^t \Delta_{\mathcal{H}^{m_{j_i}} \llcorner M_{j_i}}^d u(x) = \frac{1}{2(m+2)t} \sum_{i=1}^t \Delta_{g_i} u(x)$$

by Proposition 3.6. \square

Remark 3.8. Let us point out that Theorem 3.2 and Corollary 3.4 also hold true if the Ahlfors regularity assumption on the measures μ_i is replaced by a pointwise version, namely

$$c r^{Q_{j_i}(x)} \leq \mu_{j_i}(B_r(x)) \leq C r^{Q_{j_i}(x)}$$

for any $r > 0$, any i and any $x \in \cap_i C_{j_i}$, where $\{j_i\} \subseteq \{1, \dots, k\}$ and the constants $c, C > 0$ might depend on j_i and x .

Let us apply this remark in the following example.

Example 3.9. Set $L := [0, 1] \times \{0\}$, $S := [-1, 0] \times [-1/2, 1/2]$, and consider (\mathbb{R}^2, d_e) equipped with the measures μ^1, μ^2, μ^3 , where $\mu^i := \mu_1^i + \mu_2$ and

$$\mu_2 := dx \, dy \llcorner S, \quad \mu_1^i := x^{i-1} dx \llcorner L$$

for any $i \in \{1, 2, 3\}$. Then μ_2 is 2-Ahlfors regular, while at the intersection point $o = (0, 0)$, for any $r > 0$, one has

$$\mu_1^1(B_r(o)) = r, \quad \mu_1^2(B_r(o)) = \frac{r^2}{2}, \quad \mu_1^3(B_r(o)) = \frac{r^3}{3}.$$

Let us focus on μ^1 . An immediate computation shows that $\mu^1(B_r(o)) = (r + \pi r^2)/2$ holds for any $r > 0$ small enough. Take $u \in C^2(\mathbb{R}^2)$. The second order Taylor expansion with Laplace remainder of $u(\cdot, 0)$ at 0 implies

$$\Delta_{\mu_1^1, r}^d u(o) = \frac{2}{r^3(1 + \pi r)} \left(\frac{r^2}{2} \partial_x u(o) + \int_0^r \partial_{xx}^2 u(t) \frac{(r-t)^2}{2} dt \right).$$

On the other hand, applying Taylor's theorem to $u(\cdot, \cdot)$ at o , we know that for some $d > 0$ and $C > 0$,

$$|u(x, y) - u(o) - \nabla u(o) \cdot (x, y)| \leq C \|(x, y)\|^2$$

holds for all $(x, y) \in B_d(o)$. Therefore, for any $0 < r < d$, we get

$$\begin{aligned}
|\Delta_{\mu_2, r}^d u(o)| &\leq \frac{2}{r^3(1+\pi r)} \left(\iint_{S \cap B_r(o)} |\nabla u(o)| \|(x, y)\| \, dx \, dy + C \iint_{S \cap B_r(o)} \|(x, y)\|^2 \, dx \, dy \right) \\
&\leq \frac{2}{r^3(1+\pi r)} (r|\nabla u(o)| + Cr^2) \underbrace{\mathcal{H}^2(S \cap B_r(o))}_{=\pi r^2/2} \leq \frac{\pi(|\nabla u(o)| + Cr)}{1+\pi r}.
\end{aligned}$$

In particular, $r \mapsto |\Delta_{\mu_2, r}^d u(o)|$ is bounded in a neighborhood of 0.

We thus deduce that $\Delta_{\mu_1}^d u(o)$ exists if and only if

$$\partial_x u(o) = 0 \quad \text{and} \quad a := \lim_{r \rightarrow 0^+} r^{-3} \int_0^r \partial_{xx}^2 u(t)(r-t)^2/2 \, dt \text{ exists in } \mathbb{R},$$

in which case $\Delta_{\mu^1}^d u(o) = \Delta_{\mu_1}^d u(o) = 2a$.

Performing similar calculations for μ^3 shows that $\Delta_{\mu^3}^d u(o)$ exists if and only if $\partial_x u(o) = 0$, in which case we have $\Delta_{\mu^3}^d u(o) = \Delta_{\mu_2}^d u(o)$. Note that in this case the main contribution to the AMV Laplacian comes from the 2-dimensional piece S , while in the previous case it was coming from the 1-dimensional piece L . In fact, this example shows that the Hausdorff dimension of a piece does not matter when one computes the AMV Laplacian at an intersection point: what really matters is the so-to-say Ahlfors regular dimension of the measures.

Finally in the case of μ^2 , explicit computations show that $\Delta_{\mu^2}^d u(o)$ exists if and only if $\partial_x u(o) = 0$, in which case

$$\Delta_{\mu^2}^d u(o) = (1+\pi)^{-1} \Delta_{\mu_1}^d u(o) + \pi(1+\pi)^{-1} \Delta_{\mu_2}^d u(o),$$

as expected from Corollary 3.4.

4. Maximum and comparison principles

In this section we introduce the notion of AMV sub- and superharmonic functions, and show that an upper semicontinuous AMV subharmonic function attains its maximum at the boundary. We recall that a metric space is called proper whenever all closed subsets are compact, in which case any u.s.c. function defined on the closure of a bounded domain attains its maximum.

Definition 4.1 (*Pointwise upper and lower AMV Laplacian*). Let (X, \mathbf{d}, μ) be a metric measure space and $u \in L_{\text{loc}}^1(X, \mu)$. Then we define the *upper AMV Laplacian* $\bar{\Delta}_\mu^d u$ and *lower AMV Laplacian* $\underline{\Delta}_\mu^d u$ respectively as

$$\begin{aligned}
\bar{\Delta}_\mu^d u(x) &:= \overline{\lim}_{r \rightarrow 0^+} \frac{1}{r^2} \int_{B_r(x)} u(y) - u(x) \, d\mu(y), \\
\underline{\Delta}_\mu^d u(x) &:= \underline{\lim}_{r \rightarrow 0^+} \frac{1}{r^2} \int_{B_r(x)} u(y) - u(x) \, d\mu(y),
\end{aligned}$$

for μ -a.e. $x \in X$.

Accordingly, for $\Omega \subseteq X$, the function $u \in L_{\text{loc}}^1(X, \mu)$ is called pointwise

- upper AMV subharmonic in Ω if $\bar{\Delta}_\mu^d u(x) \geq 0$ holds for all $x \in \Omega$,

- lower AMV subharmonic in Ω if $\underline{\Delta}_\mu^d u(x) \geq 0$ holds for all $x \in \Omega$,
- upper AMV superharmonic in Ω if $\overline{\Delta}_\mu^d u(x) \leq 0$ holds for all $x \in \Omega$,
- lower AMV superharmonic in Ω if $\underline{\Delta}_\mu^d u(x) \leq 0$ holds for all $x \in \Omega$.

We also add the word *strictly* whenever the inequalities involved are strict.

Lemma 4.2 (*Maximum principle for strictly upper AMV subharmonic functions*). *Let (X, d, μ) be a proper metric measure space and $u \in L^1_{loc}(\Omega, \mu)$ an u.s.c. function in $\overline{\Omega} \subseteq X$ such that $\overline{\Delta}_\mu^d u > 0$ in Ω . Then*

$$\max_{\partial\Omega} u = \max_{\overline{\Omega}} u,$$

and u does not attain its maximum in Ω .

Proof. Let $C := \{x \in \Omega : u(x) = \max_{\overline{\Omega}} u\}$. If C is empty we are done by the upper semicontinuity of u . Assume therefore that C is nonempty and let $\max_{\overline{\Omega}} u = u(x)$ for some $x \in C$. Note that $u(x) < \infty$ since otherwise $\Delta_{\mu,r}^d u(x) \equiv -\infty$ for all $r > 0$, which would contradict $\overline{\Delta}_\mu^d u(x) > 0$. Also, since $u \in L^1_{loc}(\Omega, \mu)$, $u(x) > -\infty$. Therefore $u(x)$ is finite and, for some r small enough, we get the following contradiction,

$$0 < \Delta_{\mu,r}^d u(x) = \frac{1}{r^2} \int_{B_r(x)} \underbrace{u(y) - u(x)}_{\leq 0} d\mu(y) \leq 0.$$

Therefore C has to be empty. \square

From this lemma we can deduce the weak maximum principle for pointwise upper AMV subharmonic functions given the existence of a strictly lower AMV subharmonic function.

Theorem 4.3 (*Weak maximum principle for upper AMV subharmonic functions*). *Let (X, d, μ) be a proper metric measure space and $u \in L^1_{loc}(\Omega, \mu)$ be an u.s.c. function in $\overline{\Omega} \subseteq X$ such that $\overline{\Delta}_\mu^d u \geq 0$ in Ω . Assume that there exists a bounded function ϕ which is u.s.c. in $\overline{\Omega}$ and such that $\underline{\Delta}_\mu^d \phi > 0$ in Ω . Then $\max_{\partial\Omega} u = \max_{\overline{\Omega}} u$.*

Proof. Assume that $\max_{\partial\Omega} u < \max_{\overline{\Omega}} u = u(x)$ for some $x \in \Omega$. By the same argument as in Lemma 4.2, $u(x)$ is finite. Let $M := \|\phi\|_{L^\infty(\overline{\Omega})}$ and take $\epsilon > 0$ such that $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u + 2\epsilon M$. In particular, this implies that

$$\max_{\overline{\Omega}} u + \epsilon \inf_{\overline{\Omega}} \phi > \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} \phi. \tag{4.1}$$

Now define $u_\epsilon := u + \epsilon\phi$. Then, pointwise in Ω ,

$$\overline{\Delta}_\mu^d u_\epsilon \geq \overline{\Delta}_\mu^d u + \epsilon \underline{\Delta}_\mu^d \phi > 0,$$

and since u_ϵ is u.s.c. in $\overline{\Omega}$, Lemma 4.2 implies that $\max_{\partial\Omega} u_\epsilon = \max_{\overline{\Omega}} u_\epsilon$. However, by (4.1),

$$\begin{aligned} \max_{\overline{\Omega}} u_\epsilon &\geq u_\epsilon(x) = \max_{\overline{\Omega}} u + \epsilon\phi(x) \\ &\geq \max_{\overline{\Omega}} u + \epsilon \inf_{\overline{\Omega}} \phi \\ &> \max_{\partial\Omega} u + \epsilon \max_{\overline{\Omega}} \phi \\ &\geq \max_{\partial\Omega} u_\epsilon, \end{aligned}$$

a contradiction. Hence $\max_{\partial\Omega} u = \max_{\overline{\Omega}} u$. \square

Remark 4.4. As an example, in the Heisenberg group, we could for instance choose $\phi(x)$ as the graded coordinate function x_1^2 .

Remark 4.5. By considering the sign function $\operatorname{sgn} x$ defined as zero for $x = 0$, the function $u(x) = \operatorname{sgn} x - \operatorname{sgn}(x-1)$ in $\Omega := (-1, 2)$ is AMV harmonic everywhere in Ω but the maximum principle is violated, showing that upper semicontinuity is a necessary condition.²

Corollary 4.6 (*Weak minimum principle for lower AMV superharmonic functions*). Let (X, d, μ) be a proper metric measure space and $u \in L_{loc}^1(\Omega, \mu)$ be a l.s.c. function in $\overline{\Omega} \subseteq X$ such that $\underline{\Delta}_\mu^d u \leq 0$ in Ω . Assume that there exists a bounded function ϕ which is u.s.c. in $\overline{\Omega}$ and such that $\underline{\Delta}_\mu^d \phi > 0$. Then $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$.

Proof. Let $v := -u$. Then v is u.s.c. in $\overline{\Omega}$ and $\overline{\Delta}_\mu^d v = \overline{\Delta}_\mu^d(-u) = -\underline{\Delta}_\mu^d u \geq 0$. Hence we can conclude by applying Theorem 4.3 to v . \square

From Theorem 4.3 and Corollary 4.6 we can deduce that continuous AMV harmonic functions attain extremal values at the boundary.

Corollary 4.7 (*Weak max and minimum principle for AMV harmonic functions*). Let (X, d, μ) be a proper metric measure space and $u \in L_{loc}^1(\Omega, \mu)$ be a continuous function in $\overline{\Omega} \subseteq X$ such that $\Delta_\mu^d u = 0$ in Ω . Assume that there exists a bounded function ϕ which is u.s.c. in $\overline{\Omega}$ and such that $\underline{\Delta}_\mu^d \phi > 0$. Then $\max_{\partial\Omega} u = \max_{\overline{\Omega}} u$ and $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$.

Because of the superadditivity of $\overline{\Delta}_\mu^d$ we also get the following comparison principle.

Corollary 4.8 (*Weak comparison principle*). Let (X, d, μ) be a proper metric measure space and $u, v \in L_{loc}^1(\Omega, \mu)$ be l.s.c. and u.s.c. functions respectively in $\overline{\Omega} \subseteq X$ such that $u \geq v$ on $\partial\Omega$, and either

- (i) $\underline{\Delta}_\mu^d u \leq 0$ and $\underline{\Delta}_\mu^d v \geq 0$ in Ω , or
- (ii) $\overline{\Delta}_\mu^d u \leq 0$ and $\overline{\Delta}_\mu^d v \geq 0$ in Ω .

Assume also that there exists a bounded function ϕ which is u.s.c. in $\overline{\Omega}$ and such that $\underline{\Delta}_\mu^d \phi > 0$. Then $u \geq v$ in $\overline{\Omega}$.

Proof. We assume (i) and note that $w := u - v$ is l.s.c. in $\overline{\Omega}$ and $\underline{\Delta}_\mu^d w \leq \underline{\Delta}_\mu^d u - \underline{\Delta}_\mu^d v \leq 0$. Therefore $\min_{\overline{\Omega}} w = \min_{\partial\Omega} w \geq 0$ by Corollary 4.6, hence $u \geq v$ in $\overline{\Omega}$.

If we instead assume (ii), we get $\underline{\Delta}_\mu^d w \leq \overline{\Delta}_\mu^d u - \overline{\Delta}_\mu^d v \leq 0$, and Corollary 4.6 again gives the conclusion. \square

Remark 4.9. In particular, if u and v are continuous AMV harmonic functions such that $u \geq v$ on the boundary $\partial\Omega$, and ϕ as described above exists, then $u \geq v$ in $\overline{\Omega}$.

² This example was proposed to us by A. Kijowski.

5. Green-type identity and weak AMV Laplacian

A metric measure space (X, d, μ) being given, we set

$$T_r u := \int_{B_r(x)} u \, d\mu$$

for any $u \in L^1_{\text{loc}}(X, \mu)$, $r > 0$ and $x \in X$. Let us provide a preliminary result.

Lemma 5.1. *Let (X, d, μ) be a metric measure space. Set $w(x) := \int_{B_r(x)} \frac{d\mu(y)}{\mu(B_r(y))}$ for any $x \in X$. Then for any $1 \leq p \leq +\infty$ and $r > 0$,*

1. *if $u \in L^p(X, w\mu)$ then $T_r u \in L^p(X, \mu)$ and*

$$\|T_r u\|_{L^p(X, \mu)} \leq \|u\|_{L^p(X, w\mu)},$$

2. *if $u \in L^p(X, \mu) \cap L^p(X, w\mu)$ then $\Delta_{\mu,r}^d u \in L^p(X, \mu)$ and*

$$\|\Delta_{\mu,r}^d u\|_{L^p(X, \mu)} \leq \frac{1}{r^2} (\|u\|_{L^p(X, \mu)} + \|u\|_{L^p(X, w\mu)}).$$

Proof. Let us start by proving 1. In case $p = +\infty$ the result is immediate. For $1 \leq p < +\infty$, if $u \in L^p(X, w\mu)$, Jensen’s inequality with the convex function $t \mapsto |t|^p$ implies

$$\begin{aligned} \|T_r u\|_{L^p(X, \mu)}^p &\leq \int_X \int_{B_r(x)} |u(y)|^p \, d\mu(y) \, d\mu(x) \\ &= \int_X \int_X \frac{\chi_{B_r(x)}(y)}{\mu(B_r(x))} |u(y)|^p \, d\mu(y) \, d\mu(x). \end{aligned}$$

Applying the Fubini–Tonelli theorem and the simple observation that $\chi_{B_r(x)}(y) = \chi_{B_r(y)}(x)$, we get

$$\begin{aligned} \|T_r u\|_{L^p(X, \mu)}^p &\leq \int_X |u(y)|^p \int_X \frac{\chi_{B_r(y)}(x)}{\mu(B_r(x))} \, d\mu(x) \, d\mu(y) \\ &= \int_X |u(y)|^p \int_{B_r(y)} \frac{1}{\mu(B_r(x))} \, d\mu(x) \, d\mu(y) \\ &= \int_X |u(y)|^p w(y) \, d\mu(y) = \|u\|_{L^p(X, w\mu)}^p. \end{aligned}$$

This shows 1. That 2. follows is straightforward: for any $1 \leq p \leq +\infty$, if $u \in L^p(X, \mu) \cap L^p(X, w\mu)$, then

$$\begin{aligned} \|\Delta_{\mu,r}^d u\|_{L^p(X, \mu)} &\leq \frac{1}{r^2} \|T_r u - u\|_{L^p(X, \mu)} \leq \frac{1}{r^2} (\|T_r u\|_{L^p(X, \mu)} + \|u\|_{L^p(X, \mu)}) \\ &\leq \frac{1}{r^2} (\|u\|_{L^p(X, w\mu)} + \|u\|_{L^p(X, \mu)}) \end{aligned}$$

due to 1. \square

Remark 5.2. Lemma 5.1 applies especially when μ is a doubling measure, in which case the condition $u \in L^p(X, w\mu)$ is superfluous. Explicit computations show that if C_μ is the doubling constant of μ , then $\|T_r\|_{p \rightarrow p} \leq C_\mu^2$ and $\|\Delta_{\mu,r}^d\|_{p \rightarrow p} \leq (1 + C_\mu^2)r^{-2}$. For the particular case of a Q -Ahlfors regular measure, one has $\|T_r\|_{p \rightarrow p} \leq C/c$ and $\|\Delta_{\mu,r}^d\|_{p \rightarrow p} \leq (1 + C/c)r^{-2}$. Finally, in case μ is uniform, meaning that there exists $\omega > 0$ and $Q \geq 0$ such that $\mu(B) = \omega r^Q$ for any ball B with radius r , one has $\|T_r\|_{p \rightarrow p} \leq 1$ and $\|\Delta_{\mu,r}^d\|_{p \rightarrow p} \leq 2r^{-2}$.

Theorem 5.3 (Green-type identity). *Let (X, \mathbf{d}, μ) be a metric measure space, and w be as in Lemma 5.1. Then for any $u, v \in L^2(X, w\mu) \cap L^2(X, \mu)$ and $r > 0$,*

$$\begin{aligned} \int_X v \Delta_{\mu,r}^d u - u \Delta_{\mu,r}^d v \, d\mu &= \frac{1}{r^2} \int_X u(x) \int_{B_r(x)} v(y) \left(\frac{\mu(B_r(x))}{\mu(B_r(y))} - 1 \right) \, d\mu(y) \, d\mu(x) \\ &= \frac{1}{r^2} \int_X u(x) \int_{B_r(x)} v(y) \left(\frac{1}{\mu(B_r(y))} - \frac{1}{\mu(B_r(x))} \right) \, d\mu(y) \, d\mu(x). \end{aligned} \quad (5.1)$$

Proof. By 2. in Lemma 5.1, the integral on the left hand side exists. Moreover,

$$r^2 \int_X v \Delta_{\mu,r}^d u - u \Delta_{\mu,r}^d v \, d\mu = \int_X \int_{B_r(x)} u(y)v(x) - v(y)u(x) \, d\mu(y) \, d\mu(x). \quad (5.2)$$

Again by Fubini–Tonelli and the fact that $\chi_{B_r(x)}(y) = \chi_{B_r(y)}(x)$,

$$\begin{aligned} \int_X \int_{B_r(x)} u(y)v(x) \, d\mu(y) \, d\mu(x) &= \int_X \int_X \frac{\chi_{B_r(x)}(y)}{\mu(B_r(x))} u(y)v(x) \, d\mu(y) \, d\mu(x) \\ &= \int_X \int_X \frac{\chi_{B_r(x)}(y)}{\mu(B_r(x))} u(y)v(x) \, d\mu(x) \, d\mu(y) \\ &= \int_X \int_X \frac{\chi_{B_r(y)}(x)}{\mu(B_r(x))} u(y)v(x) \, d\mu(x) \, d\mu(y). \end{aligned}$$

By relabeling x to y and vice versa in the final expression above, we find that

$$\begin{aligned} \int_X \int_X \frac{\chi_{B_r(y)}(x)}{\mu(B_r(x))} u(y)v(x) \, d\mu(x) \, d\mu(y) &= \int_X \int_X \frac{\chi_{B_r(x)}(y)}{\mu(B_r(y))} u(x)v(y) \, d\mu(y) \, d\mu(x) \\ &= \int_X \int_{B_r(x)} \frac{\mu(B_r(x))}{\mu(B_r(y))} u(x)v(y) \, d\mu(y) \, d\mu(x). \end{aligned}$$

By putting this in (5.2), we conclude that

$$\begin{aligned} r^2 \int_X v \Delta_{\mu,r}^d u - u \Delta_{\mu,r}^d v \, d\mu &= \int_X \int_{B_r(x)} u(x)v(y) \left(\frac{\mu(B_r(x))}{\mu(B_r(y))} - 1 \right) \, d\mu(y) \, d\mu(x) \\ &= \int_X \int_{B_r(x)} u(x)v(y) \left(\frac{1}{\mu(B_r(y))} - \frac{1}{\mu(B_r(x))} \right) \, d\mu(y) \, d\mu(x). \quad \square \end{aligned}$$

Remark 5.4. The theorem shows that $\Delta_{\mu,r}^d$ in general is not self-adjoint on $L^2(X, \mu)$. However for measures such that $\mu(B_r(x))/\mu(B_r(y)) = 1 + o(r^2)$ uniformly, in particular uniform measures, the right-hand side of (5.1) is zero. This can be compared to the result of Burago et al. [6] where they show that $\Delta_{\mu,r}^d$ is self-adjoint on $L^2(X, \phi\mu)$, where $\phi(x) := r^2\mu(B_r(x))$.

If one seeks for a weak definition of the AMV Laplacian, Theorem 5.3 suggests to avoid “differentiation” of the test functions as one would naturally do. Indeed, by (5.1), there is no a priori reason to get $\int_X v\Delta_{\mu,r}^d u - u\Delta_{\mu,r}^d v \, d\mu \rightarrow 0$ when $r \rightarrow 0^+$. Moreover, the low regularity of test functions in metric spaces do not guarantee the existence of the AMV Laplacian in any case. Therefore we propose the following definition.

Definition 5.5 (Weak AMV Laplacian). Let (X, d, μ) be a metric measure space. We say that a Borel measure ν on X is the *weak AMV Laplacian* of a μ -measurable function $u : X \rightarrow \overline{\mathbb{R}}$, and we denote it by $\Delta_{\mu}^d u \stackrel{w}{=} \nu$, if

$$\lim_{r \rightarrow 0^+} \int_X \phi(x)\Delta_{\mu,r}^d u(x) \, d\mu(x) = \int_X \phi(x) \, d\nu(x)$$

holds for all $\phi \in C_c(X)$. When $\Delta_{\mu}^d u \stackrel{w}{=} 0$ we say that u is *weakly AMV harmonic*.

Note that a function which is pointwise AMV harmonic might not be weakly AMV harmonic (see also the works of Zalcman [20, Section 5]). For instance, for $(X, d, \mu) = (\mathbb{R}, d_e, \mathcal{L}^1)$, the sign function $u(x) = \operatorname{sgn} x$ defined as zero at the origin satisfies $\Delta_{\mu}^d u = 0$ everywhere in \mathbb{R} , but a straightforward computation shows that

$$\Delta_{\mu}^d u \stackrel{w}{=} \frac{2\delta'}{6},$$

which coincides with the distributional Laplacian of u divided by the dimensional constant $2(n + 2)$ for $n = 1$.

Declaration of competing interest

None.

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