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Analysis of the optimal exercise boundary of American put options with delivery lags [☆]

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ABSTRACT

A make-your-mind-up option is an American derivative with delivery lags. We show that its put option can be decomposed as a European put and a new type of American-style derivative. The latter is an option for which the investor receives the Greek Theta of the corresponding European option as the running payoff, and decides an optimal stopping time to terminate the contract. Based on this decomposition and using free boundary techniques, we show that the associated optimal exercise boundary exists and is a strictly increasing and smooth curve, and analyze the asymptotic behavior of the value function and the optimal exercise boundary for both large maturity and small time lag.

1. Introduction

With a few exceptions, models of optimal stopping time problems assume that the player is able to terminate the underlying stochastic dynamics immediately after the decision to stop, or to bring a new project online without any delays after the decision to invest. In fact, both stopping stochastic dynamics and initiating a new project take time.

In this paper, we consider a class of optimal stopping problems where there exists a time lag between the player's decision time and the time that the payoff is delivered. In particular, we study American put options with delivery lags in details. In practice, there may exist a time lag between the time that the option holder decides to exercise the option and the time that the payoff is delivered. Such delivery lags may be specified in financial contracts, where the decision to exercise must be made before the exercise takes place.

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They are called *make-your-mind-up options* (see Chapter 6 of [18] and Chapter 9 of [22]). For example, the option holder must give a notice period before she exercises, and she cannot change her mind. On the other hand, even for a standard American derivative, the option holder may not be able to exercise it immediately, when there exist liquidation constraints in financial markets.

Let W be a one-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the augmented filtration generated by W. Let a constant $T > 0$ represent the maturity and another constant $\delta \in [0, T)$ represent the time lag. The player aims to choose an optimal stopping time $\tau^{0,*} \in \mathcal{R}_t^0$ in order to maximize the discounted expected payoff

$$Y_t^\delta = \operatorname*{ess\,sup}_{\tau^0 \in \mathcal{R}_t^0} \mathbf{E}\Big[e^{-r(\tau^0+\delta-t)}(K - X_{\tau^0+\delta})^+ \mathbf{1}_{\{\tau^0+\delta<T\}} + e^{-r(T-t)}(K - X_T)^+ \mathbf{1}_{\{\tau^0+\delta\geq T\}} | \mathcal{F}_t\Big], \quad t \in [0, T], \tag{1}$$

where Y^δ represents the value process, the $\mathbb{F}$-adapted process X models the stock price, the constant $K > 0$ denotes the strike price, and

$$\mathcal{R}_t^0 := \{\tau^0 : \Omega \to [t, T], \text{ and } \{\tau^0 \leq s\} \in \mathcal{F}_s \text{ for any } s \in [t, T]\}.$$

Note that $\delta = 0$ corresponds to the classical optimal stopping problem for American options (see, for example, [13] and [21]), so we mainly focus on the problem in the case of $\delta > 0$ in this paper. For $\delta > 0$, if the player decides to stop at some stopping time τ^0, then the payoff will be delivered at $\tau^0 + \delta$ rather than τ^0, so there is a time lag of the delivery of the payoff. We also observe that the problem (1) is trivial for $t \in (T - \delta, T]$ for, in this situation, the expected payoff is independent of choice of τ^0, and the player may simply choose the optimal stopping time as the maturity T. *Thus, we focus on the case $t \in [0, T - \delta]$ throughout the paper.*

Although this type of optimal stopping problems with delivery lags has been well studied in the literature (see [1] and [19] with more references therein), little is known about the corresponding optimal exercise boundaries and their asymptotic behavior for small time lag δ and large maturity T. Intuitively, both the value function and the corresponding optimal exercise boundary (if exists) will converge to the solution for the case without delivery lags when $\delta \downarrow 0$, and to the solution for the perpetual case when $T \uparrow \infty$. It is the aim of this paper to prove the above asymptotic behavior using free boundary techniques.

To be more specific, under the geometric Brownian motion setup, we prove the following result.

Theorem 1. *Suppose that the stock price X follows*

$$dX_s/X_s = (r - q)ds + \sigma dW_s, \quad X_t = X,$$

where the interest rate $r > 0$, the dividend rate $q \in [0, r]$[1] *and the volatility $\sigma > 0$ are all constants. Then, the following assertions hold:*

(i) The value $Y_t^\delta = V^\delta(t, X_t)$ is decreasing with respect to δ and, moreover,

$$V^0(t, X) \geq V^\delta(t, X) \geq V^0(t, X) - K\left(1 - e^{-r\delta}\right), \quad t \in [0, T - \delta], \tag{2}$$

where $V^\delta(\cdot, \cdot)$ and $V^0(\cdot, \cdot)$ represent the value function for the American put with and without delivery lags, respectively. In addition, $V^\delta(t, X)$ is decreasing with respect to t.

[1] $q \leq r$ is a technique assumption, which ensures conclusion (i) in Proposition 4.

(ii) There exists an optimal exercise boundary $X^\delta(t) \in C^\infty[0, T-\delta)$ separating exercise and continuation regions (cf. (21) and (22))). Moreover, it is strictly increasing in t, with the end point

$$X^\delta(T-\delta) = \lim_{t\to(T-\delta)^-} X^\delta(t) = Ke^{\overline{X}},$$

where $\overline{X}$ is given in Proposition 4.

(iii) The optimal exercise boundary $X^\delta(t) \to Ke^{\underline{X}}$ as $T \to \infty$ with $\underline{X}$ given in (17), so $Ke^{\underline{X}}$ is the asymptotic line of the optimal exercise boundary $X^\delta(t)$. Moreover, $X^\delta(t) \to X^0(t)$ for any $t \in [0,T)$ as $\delta \to 0$, where $X^0(t)$ represents the optimal exercise boundary for the corresponding American put without delivery lags.

To prove Theorem 1, we need to first solve the optimal stopping problem (1). A basic idea is to introduce a new obstacle (payoff) process, which is the projection (conditional expectation) of the original expected payoff. For $t \in [0, T-\delta]$, define

$$\widehat{Y}_t^\delta = \mathbf{E}\left[e^{-r\delta}(K - X_{t+\delta})^+ | \mathcal{F}_t\right], \tag{3}$$

which is the time t value of the corresponding European put option with maturity $t+\delta$. Denote by $P(\cdot,\cdot)$ the value function of the European put option with maturity T. Then, the time homogeneity of (3) implies $\hat{Y}_t^\delta = P(T-\delta, X_t)$, and the tower property of conditional expectations further yields

$$Y_t^\delta = V^\delta(t, X_t) = \operatorname*{ess\,sup}_{\tau^0 \in \mathcal{R}_t^0} \mathbf{E}\left[e^{-r((\tau_0\wedge(T-\delta))-t)} P(T-\delta, X_{\tau_0\wedge(T-\delta)}) | \mathcal{F}_t\right] \tag{4}$$

with $x \wedge y = \min(x,y)$ and $t \in [0, T-\delta]$. Hence, we have transformed the original problem (1) to a standard optimal stopping problem (without delivery lags) with the European option price as the new obstacle process. The rest of the paper will therefore focus on (4) and its corresponding variational inequality (5) in section 2.

The existing literature of optimal stopping with delivery lags (see [1] and [19] for example) usually assumes that the payoff is a linear function of the underlying asset X, which certainly excludes the American payoff. A consequence of this simplified assumption is that the new obstacle $\hat{Y}^\delta$ is also linear in X, which follows from the linearity of the conditional expectation, and the obstacle function in the variational inequality is therefore also a linear function. Hence, the treatments of the optimal stopping problems with and without delivery lags are essentially the same in their models.

In our case, since the American payoff is only a piecewise linear function of the underlying asset X (with a kink point at K), this kink point propagates via the conditional expectation, resulting in a nonlinear obstacle function $P(T-\delta, \cdot)$. This differentiates our problem from the existing optimal stopping problems with delivery lags, and makes the analysis of the corresponding optimal exercise boundary much more challenging.

We first develop an early exercise premium decomposition formula for the American put option with delivery lags (see (10)). This helps us overcome the difficulty of handling the European option price as the modified payoff. We show that an American put option with delivery lags can be decomposed as a European put option and another American-style derivative as an auxiliary optimal stopping problem (see (9)). The latter is an option for which the investor receives the Greek Theta of the corresponding European option as the running payoff, and decides an optimal stopping time to terminate the contract. The decomposition formula (10) can also be regarded as a counterpart of the early exercise premium representation of standard American options, and is crucial to the analysis of the associated optimal exercise boundary.

Using free-boundary techniques, we then give a detailed analysis of the associated optimal exercise boundary. An essential difficulty herein is the non-monotonicity of the difference between the value function and the payoff with respect to the stock price (a similar phenomenon also appears in [10]). As a result, it is not even clear *ex ante* whether the optimal exercise boundary exists or not. This is in contrast to standard American options, for which the value function, subtracted by the payoff, is monotonic with respect to the stock price, so the stopping and continuation regions can be easily separated.

Thanks to the auxiliary optimal stopping problem (9) and its associated variational inequality (8), we prove that the optimal exercise boundary exists and is a strictly increasing and smooth curve, with its end point closely related to the zero crossing point of the Greek Theta of the corresponding European option. Intuitively, when Theta is positive, the running payoff of the new American-style derivative is also positive, so the investor will hold the option to receive the positive Theta continuously. In contrast, when Theta is negative, one may think that the investor would then exercise the option to stop her losses. However, we show that when Theta is negative but not too small, the investor may still hold the option and wait for Theta to rally at a later time to recover her previous losses. We further quantify such negative values of Theta by identifying the asymptotic line of the optimal exercise boundary, which turns out to be the optimal exercise boundary of the corresponding perpetual problem.

The paper is organized as follows. In section 2, we prove Theorem 1 (i) and introduce the early excise premium decomposition formula. We then consider the corresponding perpetual problem in section 3, and in section 4 we prove Theorem 1 (ii) and (iii). Some technical proofs about the property of the Greek Theta are provided in the appendix.

2. The variational inequality characterization

We first solve the optimal stopping problem (4) via its associated variational inequality

$$\begin{cases} (-\partial_t - \mathcal{L})V^\delta(t,X) = 0, & \text{if } V^\delta(t,X) > P(T-\delta,X), \\ & \text{for } (t,X) \in \Omega_{T-\delta}; \\ (-\partial_t - \mathcal{L})V^\delta(t,X) \geq 0, & \text{if } V^\delta(t,X) = P(T-\delta,X), \\ & \text{for } (t,X) \in \Omega_{T-\delta}; \\ V^\delta(T-\delta,X) = P(T-\delta,X), & \text{for } X \in \mathbb{R}_+, \end{cases} \tag{5}$$

with $\Omega_{T-\delta} = [0, T-\delta\,) \times \mathbb{R}_+$, and the operator $\mathcal{L}$ given by the Black-Scholes differential operator

$$\mathcal{L} = \frac{1}{2}\sigma^2 X^2 \partial_{XX} + (r-q)X\partial_X - r.$$

Note that if $\delta = 0$, $P(T-\delta, X) = (K-X)^+$, and variational inequality (5) reduces to the standard variational inequality for American put options. On the other hand, since variational inequality (5) is with smooth coefficients and obstacle, its (strong) solution $V^\delta(\cdot,\cdot)$ characterizes the value function and the optimal stopping rule for the optimal stopping problem (4).

Proposition 2. *The value function $V^\delta(\cdot,\cdot)$ of the optimal stopping problem (4) is the unique bounded strong solution to variational inequality (5), and the optimal stopping rule is given by*

$$\tau^{0,*} = \inf\{s \in [t, T-\delta] : V^\delta(s, X_s) = P(T-\delta, X_s)\}. \tag{6}$$

Moreover, $V^\delta \in W^{2,1}_{p,loc}(\Omega_{T-\delta}) \cap C(\overline{\Omega_{T-\delta}})$ for any $p \geq 1$, and $\partial_x V^\delta \in C(\overline{\Omega_{T-\delta}})$.

Herein, $W^{2,1}_{p,loc}(\Omega_{T-\delta})$ is the set of all functions whose restriction on any compact subset $\Omega^*_{T-\delta} \subset \Omega_{T-\delta}$ belong to $W^{2,1}_p(\Omega^*_{T-\delta})$, where $W^{2,1}_p(\Omega^*_{T-\delta})$ is the completion of $C^\infty(\Omega^*_{T-\delta})$ under the norm

$$||V^\delta||_{W_p^{2,1}(\Omega^*_{T-\delta})} = \left[\int\limits_{\Omega^*_{T-\delta}} (|V^\delta|^p + |\partial_t V^\delta|^p + |\partial_x V^\delta|^p + |\partial_{xx} V^\delta|^p) dxdt\right]^{\frac{1}{p}}.$$

The proof follows along the similar arguments used in Chapter 1 of [17], or more recently [24], and is thus omitted.

2.1. Proof of Theorem 1 (i)

In this subsection, we prove Theorem 1 (i).[2] Note that the arguments below do not rely on the geometric Brownian motion assumption on X, as long as its discounted price $e^{-(r-q)t}X_t$ is a martingale.

We first prove the monotone property of $V^\delta(\cdot,\cdot)$ with respect to δ. Fix $0 \le \delta_1 < \delta_2$. For any $\tau_2 \in \mathcal{R}_t^0$, take $\tau_1 = (\tau_2 + (\delta_2 - \delta_1)) \wedge T$. Since δ_1, δ_2, T are constants, we know that $\tau_1 \in \mathcal{R}_t^0$. Moreover, it is easy to check that $\{\tau_1 + \delta_1 \ge T\} = \{\tau_2 + \delta_2 \ge T\}$ and

$$e^{-r(\tau_1+\delta_1-t)}(K - X_{\tau_1+\delta_1})^+\mathbf{1}_{\{\tau_1+\delta_1<T\}} = e^{-r(\tau_2+\delta_2-t)}(K - X_{\tau_2+\delta_2})^+\mathbf{1}_{\{\tau_2+\delta_2<T\}}.$$

Hence, from (1), we know that $Y_t^{\delta_1} \ge Y_t^{\delta_2}$ and $V^\delta(\cdot,\cdot)$ is decreasing with respect to δ.

For the second inequality in (2), for any $\tau \in \mathcal{R}_t^0$, take $\widehat{\tau} = (\tau+\delta)\wedge T$. Note that $\widehat{\tau} \in \mathcal{F}_\tau$ and $\widetilde{X}_t = e^{-(r-q)t}X_t$ is a martingale. Hence,

$$\mathbf{E}\left(e^{-r\widehat{\tau}}\left(K - X_{\widehat{\tau}}\right)^+ \Big| \mathcal{F}_t\right) = \mathbf{E}\left(e^{-r\widehat{\tau}}\left(K - e^{(r-q)\widehat{\tau}}\widetilde{X}_{\widehat{\tau}}\right)^+ \Big| \mathcal{F}_t\right)$$

$$\ge \mathbf{E}\left(e^{-r\widehat{\tau}}\left(K - e^{(r-q)\widehat{\tau}}\widetilde{X}_{\tau}\right)^+ \Big| \mathcal{F}_t\right) \ge \mathbf{E}\left(\left(e^{-r\widehat{\tau}}K - e^{-q\tau}\widetilde{X}_{\tau}\right)^+ \Big| \mathcal{F}_t\right),$$

where the first inequality follows from the facts that $e^{-r\widehat{\tau}}\left(K - e^{(r-q)\widehat{\tau}}x\right)^+$ is convex with respect to x and measurable with respect to $\mathcal{F}_\tau$, so we may take conditional expectation with respect to $\mathcal{F}_\tau$ and apply Jensen's inequality. For the second inequality, we have used the facts that $q \ge 0$ and $\widehat{\tau} \ge \tau$. In turn,

$$\begin{aligned}
&\mathbf{E}\left(e^{-r\tau}\left(K - X_{\tau}\right)^+ \Big| \mathcal{F}_t\right)\\
&= \mathbf{E}\left(\left(e^{-r\widehat{\tau}}K - e^{-q\tau}\widetilde{X}_{\tau}\right)^+ + \left[e^{-r\tau}\left(K - X_\tau\right)^+ - \left(e^{-r\widehat{\tau}}K - e^{-r\tau}X_\tau\right)^+\right] \Big| \mathcal{F}_t\right)\\
&\le \mathbf{E}\left(e^{-r\widehat{\tau}}\left(K - X_{\widehat{\tau}}\right)^+ \Big| \mathcal{F}_t\right) + K\mathbf{E}\left(e^{-r\tau} - e^{-r\widehat{\tau}} \Big| \mathcal{F}_t\right)\\
&\le \mathbf{E}\left(e^{-r((\tau+\delta)\wedge T)}\left(K - X_{(\tau+\delta)\wedge T}\right)^+ \Big| \mathcal{F}_t\right) + K\left(1 - e^{-r\delta}\right),
\end{aligned}$$

where we have used the above conclusion and the fact that $(x+y)^+ - x^+ \le y^+$ in the first inequality, and $\tau \ge 0$ and $\widehat{\tau} - \tau \le \delta$ in the second inequality. Until now, we have proved that for any $\tau \in \mathcal{R}_t^0$,

$$\begin{aligned}
&\mathbf{E}\left[e^{-r(\tau-t)}(K - X_\tau)^+\mathbf{1}_{\{\tau<T\}} + e^{-r(T-t)}(K - X_T)^+\mathbf{1}_{\{\tau\ge T\}}|\mathcal{F}_t\right]\\
&\le \mathbf{E}\left[e^{-r(\tau+\delta-t)}(K - X_{\tau+\delta})^+\mathbf{1}_{\{\tau+\delta<T\}} + e^{-r(T-t)}(K - X_T)^+\mathbf{1}_{\{\tau+\delta\ge T\}}|\mathcal{F}_t\right]
\end{aligned}$$

[2] We thank the referee for outlining the current probabilistic proof for us. Note that the proof does not require the geometric Brownian motion model of the underlying asset, which is more general than our original proof based on PDE arguments.

$$+K\left(1-e^{-r\delta}\right).$$

Thus, from (1), we obtain the second inequality in (2).

Finally, we prove the following inequality (7), which is important to analyze the properties of the optimal exercise boundary later on.

$$\partial_t V^\delta \le 0 \ \text{ a.e. in } \ \Omega_{T-\delta}. \tag{7}$$

By the Markov property and time homogeneity, it is clear that

$$V^\delta(t,x) = \operatorname*{ess\,sup}_{\tau^0\in\mathcal{R}_0^0} \mathbf{E}\left[e^{-r((\tau^0+\delta)\wedge(T-t))}(K-X^{0,x}_{(\tau^0+\delta)\wedge(T-t)})^+|\mathcal{F}_t\right], \ \ (t,x)\in\Omega_{T-\delta},$$

where the notation $X^{0,x}$ means the state process X starts at the initial time 0 and position x

Let $0\le t_1 < t_2 \le T-\delta$. For any $\tau_2\in\mathcal{R}_0^0$, take $\tau_1=\tau_2\wedge(T-\delta-t_2)$. It is not difficult to check that $\tau_1\in\mathcal{R}_0^0$ and

$$(\tau_1+\delta)\wedge(T-t_1) = (\tau_2+\delta)\wedge(T-t_2).$$

Thus, we deduce that $V^\delta(t_1,x)\ge V^\delta(t_2,x)$ and V^δ is non-increasing with respect to t, which further implies (7).

2.2. An early exercise premium decomposition formula

We derive a decomposition formula for the American put option with delivery lags. Such a decomposition formula is crucial to the analysis of the optimal exercise boundary in sections 3 and 4. Let $U^\delta(t,X) = Y^\delta(t,X) - P(T-\delta,X)$. Then, we deduce that $U^\delta(t,X)$ satisfies the variational inequality

$$\begin{cases} (-\partial_t-\mathcal{L})U^\delta(t,X) = \Theta^\delta(X), & \text{if } U^\delta(t,X)>0, \ \text{for } (t,X)\in\Omega_{T-\delta}; \\ (-\partial_t-\mathcal{L})U^\delta(t,X) \ge \Theta^\delta(X), & \text{if } U^\delta(t,X)=0, \ \text{for } (t,X)\in\Omega_{T-\delta}; \\ U^\delta(T-\delta,X)=0, & \text{for } X\in\mathbb{R}_+, \end{cases} \tag{8}$$

where $\Theta^\delta(\cdot)$ is the Greek Theta of the European option:

$$\Theta^\delta(X) = -\partial_t P(T-\delta,X).$$

Interestingly, we observe that the above variational inequality (8) also corresponds to an auxiliary optimal stopping problem

$$U^\delta(t,X_t) = \operatorname*{ess\,sup}_{\tau^0\in\mathcal{R}_t^0} \mathbf{E}\left[\int_t^{\tau^0} e^{-r(s-t)}\Theta^\delta(X_s)ds|\mathcal{F}_t\right], \tag{9}$$

with its optimal stopping time $\tau^{0,*}$ given in (6). In turn, we obtain a decomposition formula for the American put option with delivery lags

$$Y^\delta(t,X_t) = P(T-\delta,X_t) + U^\delta(t,X_t), \tag{10}$$

Remark 3. One advantage of the optimal stopping formulation (9) is that it does not have final payoff but only has running payoff, and this will facilitate our analysis of the associated optimal exercise boundary. In the rest of the paper, we shall focus our analysis on the optimal stopping problem (9) and its associated variational inequality (8).

To solve (8), introduce the transformation[3]

$$x = \ln X - \ln K, \quad \tau = T - \delta - t, \quad u(\tau, x) = U^{\delta}(t, X), \quad \theta(x) = \Theta^{\delta}(X). \tag{11}$$

Consequently, (8) reduces to

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})u(t,x) = \theta(x), & \text{if } u(\tau,x) > 0, \text{ for } (\tau,x) \in \mathcal{N}_{T-\delta}; \\ (\partial_\tau - \widetilde{\mathcal{L}})u(t,x) \geq \theta(x), & \text{if } u(\tau,x) = 0, \text{ for } (\tau,x) \in \mathcal{N}_{T-\delta}; \\ u(0,x) = 0, & \text{for } x \in \mathbb{R}, \end{cases} \tag{12}$$

where $\mathcal{N}_{T-\delta} = (0, T-\delta\,] \times \mathbb{R}$, and

$$\widetilde{\mathcal{L}} = \frac{\sigma^2}{2}\partial_{xx} + \left(r - q - \frac{\sigma^2}{2}\right)\partial_x - r.$$

Moreover, it follows from Proposition 2 that $u \in W^{2,1}_{p,loc}(\mathcal{N}_{T-\delta}) \cap C(\overline{\mathcal{N}_{T-\delta}})$ for $p \geq 1$ and $\partial_x u \in C(\overline{\mathcal{N}_{T-\delta}})$.

For the latter use, we present some basic properties of the Greek $\Theta^{\delta}(X)$ whose proof is given in Appendix A.

Proposition 4. *Let $\theta(x) = \Theta^{\delta}(X)$ with $x = \ln X - \ln K$. Then, the following assertions hold:*

(i) There exists a unique zero crossing point $\overline{X} \in \mathbb{R}$ such that $\theta(\overline{X}) = 0$. In addition, $\theta(x) < 0$ for any $x < \overline{X}$, $\theta(x) > 0$ for any $x > \overline{X}$, and $\theta'(\overline{X}) > 0$.

(ii) For any $x < \overline{X}$, $\theta(x) \to qKe^x - rK$ as $\delta \to 0^+$.

3. The perpetual case and its optimal exercise boundary

We consider the perpetual version of the optimal stopping problem (9), whose solution admits explicit expressions (cf. (19) and (20) below). The perpetual problem is also closely related to the asymptotic analysis of the optimal exercise boundary in section 4.

For any $\mathbb{F}$-stopping time $\tau^0 \geq t$, we consider the perpetual version of (9), i.e.

$$U^{\delta}_{\infty}(X_t) = \operatorname*{ess\,sup}_{\tau^0 \geq t} \mathbf{E}\left[\int_t^{\tau^0} e^{-r(s-t)}\Theta^{\delta}(X_s)ds|\mathcal{F}_t\right]. \tag{13}$$

Using the similar arguments as in section 2, we obtain that $U^{\delta}_{\infty}(X) = u_{\infty}(x)$, where $x = \ln X - \ln K$, and $u_{\infty}(\cdot)$ is the unique bounded strong solution to the stationary variational inequality

$$\begin{cases} -\widetilde{\mathcal{L}}\,u_{\infty}(x) = \theta(x), & \text{if } u_{\infty}(x) > 0, \text{ for } x \in \mathbb{R}; \\ -\widetilde{\mathcal{L}}\,u_{\infty}(x) \geq \theta(x), & \text{if } u_{\infty}(x) = 0, \text{ for } x \in \mathbb{R}, \end{cases} \tag{14}$$

[3] For notation simplicity, we suppress the superscript δ in u^{δ} and θ^{δ}, and use u and θ instead. The same convention applies to the optimal exercise boundary $x(\tau)$ in section 4.

with $u_\infty \in W^2_{p,loc}(\mathbb{R})$ for any $p \geq 1$ and $(u_\infty)' \in C(\mathbb{R})$.

From Proposition 4, we know that $\{\theta(x) \geq 0\} = \{x \geq \overline{X}\}$. In this domain, we consider the following PDE,

$$-\widetilde{\mathcal{L}}\, v_\infty(x) = \theta(x) > 0,\ x \in (\overline{X}, +\infty) \qquad v_\infty(\overline{X}) = 0. \tag{15}$$

The above PDE has a unique classical solution $v_\infty \in C^2(\overline{X}, +\infty) \cap C[\overline{X}, +\infty)$. The strong maximum principle (see [15]) implies that $v_\infty > 0$ in $(\overline{X}, +\infty)$.

Moreover, it is clear that u_∞ satisfies

$$-\widetilde{\mathcal{L}}\, u_\infty(x) \geq \theta(x),\ x \in (\overline{X}, +\infty) \qquad u_\infty(\overline{X}) \geq 0.$$

Using the comparison principle (see [17] or [23]) for the strong solution of PDE in $(\overline{X}, +\infty)$, we deduce that $u_\infty \geq v_\infty > 0$ in $(\overline{X}, +\infty)$. So, it follows that

$$\{x > \overline{X}\} \subseteq \{u_\infty(x) > 0\} \ \text{ and } \ \{x \leq \overline{X}\} \supseteq \{u_\infty(x) = 0\}. \tag{16}$$

We can then define the *optimal exercise boundary* $\underline{X}$ as[4]

$$\underline{X} = \inf\{x \in \mathbb{R} : u_\infty(x) > 0\}. \tag{17}$$

The continuity of $u_\infty(\cdot)$ implies that $u_\infty(x) = 0$ for $x \leq \underline{X}$ and, therefore, the player will exercise the option in $(-\infty, \underline{X}]$. Moreover, it follows from (16) and (17) that $\underline{X} \leq \overline{X}$.

The next proposition relates variational inequality (14) to a free-boundary problem, which in turn provides the explicit expressions for $u_\infty(\cdot)$ and $\underline{X}$.

Proposition 5. *For $x > \underline{X}$, it holds that $u_\infty(x) > 0$. Moreover, $(u_\infty(\cdot), \underline{X})$ is the unique bounded solution to the free-boundary problem*

$$\begin{cases} -\widetilde{\mathcal{L}}\, u_\infty(x) = \theta(x), & \textit{for } x > \underline{X}; \\ u_\infty(x) = 0, & \textit{for } x \leq \underline{X}; \\ (u_\infty)'(\underline{X}) = 0, & \textit{(smooth-pasting condition)}, \end{cases} \tag{18}$$

and satisfies $\overline{X} > \underline{X} > -\infty$.

Proof. *Step 1.* We prove that $(u_\infty(\cdot), \underline{X})$ satisfies the free-boundary problem (18). To this end, we first show what $u_\infty(x) > 0$ for $x > \underline{X}$. Since $u_\infty(x) > 0$ for $x > \overline{X}$, we only need to show that $u_\infty > 0$ on $(\underline{X}, \overline{X}]$. If not, let $x_1, x_2 \in [\,\underline{X}, \overline{X}\,]$ be such that

$$x_1 < x_2,\ u_\infty(x_1) = u_\infty(x_2) = 0,\ \text{ and } u_\infty(x) > 0 \text{ for any } x \in (x_1, x_2).$$

Using variational inequality (14) and Proposition 4, we obtain that

$$\begin{cases} -\widetilde{\mathcal{L}}\, u_\infty(x) = \theta(x) \leq 0, & \text{for } x \in (x_1, x_2); \\ u_\infty(x_1) = u_\infty(x_2) = 0. \end{cases}$$

[4] Note that from the definition of $\underline{X}$, it may be possible that $\underline{X} = -\infty$. We will however exclude such a situation in Proposition 5.

The comparison principle then implies that $u_\infty(x) \le 0$ for $x \in (x_1, x_2)$, which is a contradiction.

To prove the smooth-pasting condition, we observe that $(u_\infty)'$ is continuous, and that $u_\infty(x) = 0$ for $x \le \underline{X}$. Therefore, $(u_\infty)'(\underline{X}+0) = (u_\infty)'(\underline{X}-0) = 0$, and $(u_\infty(\cdot), \underline{X})$ indeed satisfies the free boundary problem (18).

Step 2. We prove that $(u_\infty(\cdot), \underline{X})$ is actually the unique solution to (18). To this end, we first show that if $(u_{\infty,1}(\cdot), \underline{X}_1)$ is any solution solving (18), then it is necessary that $\underline{X}_1 < \overline{X}$. If not, by (18) and Proposition 4, we have

$$\begin{cases} -\widetilde{\mathcal{L}}\, u_{\infty,1}(x) = \theta(x) > 0, & \text{for } x > \underline{X}_1 \ge \overline{X}; \\ u_{\infty,1}(\underline{X}_1) = (u_{\infty,1})'(\underline{X}_1) = 0. \end{cases}$$

The strong comparison principle (see [15]) then implies that $u_{\infty,1}(x) > 0$ for $x > \underline{X}_1$.

Next we compare $u_{\infty,1}(x)$ with an auxiliary function

$$\underline{w}(x) = u_{\infty,1}(\underline{X}_1 + 1) w(x; \underline{X}_1, \underline{X}_1 + 1)$$

in the interval $(\underline{X}_1, \underline{X}_1 + 1)$, where

$$w(x; a, b) = \frac{e^{\lambda^+(x-a)} - e^{\lambda^-(x-a)}}{e^{\lambda^+(b-a)} - e^{\lambda^-(b-a)}},$$

with λ^+ and λ^- being, respectively, the positive and negative characteristic roots of $\widetilde{\mathcal{L}}$:

$$\frac{\sigma^2}{2}\lambda^2 + \left(r - q - \frac{\sigma^2}{2}\right)\lambda - r = 0.$$

It is clear that

$$w(a; a, b) = 0, \quad w(b; a, b) = 1, \quad w'(a; a, b) > 0, \quad -\widetilde{\mathcal{L}} w = 0 \text{ in } (a, b).$$

In turn,

$$\begin{cases} -\widetilde{\mathcal{L}}\underline{w}(x) = 0 < -\widetilde{\mathcal{L}} u_{\infty,1}(x), & \text{for } x \in (\underline{X}_1, \underline{X}_1 + 1); \\ u_{\infty,1}(\underline{X}_1) = \underline{w}(\underline{X}_1), & u_{\infty,1}(\underline{X}_1 + 1) = \underline{w}(\underline{X}_1 + 1). \end{cases}$$

Hence, the comparison principle implies that $u_{\infty,1}(x) \ge \underline{w}(x)$ for $x \in (\underline{X}_1, \underline{X}_1 + 1)$. In turn, $(u_{\infty,1})'(\underline{X}_1) \ge \underline{w}'(\underline{X}_1) > 0$, which contradicts the smooth-pasting condition $(u_{\infty,1})'(\underline{X}_1) = 0$.

Now we show that $(u_\infty(\cdot), \underline{X})$ is the unique solution to (18). If not, let $(u_{\infty,1}, \underline{X}_1)$ be another solution of the free-boundary problem (18). Without loss of generality, we may assume that $\underline{X}_1 < \underline{X} < \overline{X}$. It is immediate to check that

$$\begin{cases} -\widetilde{\mathcal{L}} u_{\infty,1}(x) = \theta(x) \\ \qquad\qquad \le \theta(x) I_{\{x > \underline{X}\}} = -\widetilde{\mathcal{L}} u_\infty(x), \ \text{for } x \in (\underline{X}_1, \infty); \\ u_{\infty,1}(\underline{X}_1) = u_\infty(\underline{X}_1) = 0; \\ (u_{\infty,1})'(\underline{X}_1) = (u_\infty)'(\underline{X}_1) = 0, \end{cases}$$

where we have used the fact $\theta(x) < 0$ for any $x \le \underline{X} < \overline{X}$. The comparison principle then implies that $u_{\infty,1}(x) \le u_\infty(x)$ and, in particular, $u_{\infty,1}(x) \le u_\infty(x) = 0$ for $x \in [\underline{X}_1, \underline{X}]$.

On the other hand, applying Taylor's expansion to $u_{\infty,1}(x)$ yields

$$u_{\infty,1}(x) = \frac{1}{2}u''_{\infty,1}(\underline{X}_1+0)(x-\underline{X}_1)^2(1+o(1)) = \frac{-\theta(\underline{X}_1)}{\sigma^2}(x-\underline{X}_1)^2(1+o(1)),$$

which further implies that $u_{\infty,1}(x) > 0$ if x is close enough to $\underline{X}_1$. Thus, we obtain a contradiction.

Step 3. We prove that $\overline{X} > \underline{X} > -\infty$. Since we have already showed that $\underline{X} < \overline{X}$ in Step 2, it is sufficient to prove that $\underline{X} > -\infty$.

In fact, using the free-boundary formulation (18), we further obtain that its solution must have the form

$$u_\infty(x) = CKe^{\lambda^- x} - p(x), \text{ for } x > \underline{X},$$

where the constant C is to be determined, λ^- is the negative root of the characteristic equation for $\widetilde{\mathcal{L}}$, and $p(x) = p(T-\delta, x)$ is the price of the European put option (cf. (32) with $t = T - \delta$).

In order to fix the constant C and the optimal exercise boundary $\underline{X}$, we make use of the boundary and smooth-pasting conditions in (18), and obtain that

$$\begin{cases} CKe^{\lambda^-\underline{X}} = p(\underline{X}\,) = \left[\,Ke^{-r\delta}N(-\underline{d}_{\,2}) - Ke^{\underline{X}-q\delta}N(-\underline{d}_{\,1})\right]; \\ CK\lambda^- e^{\lambda^-\underline{X}} = p'(\underline{X}\,) = -Ke^{\underline{X}-q\delta}N(-\underline{d}_{\,1}), \end{cases}$$

where $\underline{d}_{\,1}$ and $\underline{d}_{\,2}$ are the same as d_1 and d_2 in (31) except that x is replaced by $\underline{X}$ (see Appendix A for the notations). Thus, we obtain that

$$u_\infty(x) = \begin{cases} p(\underline{X})\,e^{\lambda^-(x-\underline{X}\,)} - p(x), & \text{for } x > \underline{X}\,; \\ 0 & \text{for } x \le \underline{X}\,, \end{cases} \tag{19}$$

and $\underline{X}$ is the zero crossing point of the algebraic equation

$$l(x) = \lambda^- e^{-r\delta}N(-d_2) + (1-\lambda^-)e^{x-q\delta}N(-d_1) = 0. \tag{20}$$

Next, we prove that the zero crossing point of $l(x) = 0$ exists and is unique. It is clear that, when $x \to -\infty$,

$$d_1,\, d_2 \to -\infty, \;\; N(-d_1),\, N(-d_2) \to 1, \;\; l(x) \to \lambda^- e^{-r\delta} + o(1) < 0.$$

Hence, $l(x)$ is negative provided x is small enough. On the other hand, by (34) and (35), we have

$$d_1,\, d_2 \to +\infty, \;\; N(-d_1) = \frac{N'(-d_1)}{d_1}\Big(1+o(1)\Big), \;\; N(-d_2) = \frac{N'(-d_2)}{d_2}\Big(1+o(1)\Big),$$

as $x \to +\infty$, and therefore,

$$\begin{aligned} \frac{l(x)e^{r\delta}}{N'(-d_2)} &= \frac{\lambda^-}{d_2}\Big(1+o(1)\Big) + \frac{1-\lambda^-}{d_1}\Big(1+o(1)\Big) \\ &= \frac{d_2 + \lambda^-(d_1-d_2)}{d_1\,d_2}\Big(1+o(1)\Big) = \frac{1}{d_1}\Big(1+o(1)\Big). \end{aligned}$$

Hence, $l(x)$ is positive provided x is large enough. Thus, we deduce that there exists at least one zero crossing point of $l(x) = 0$. Thanks to the uniqueness of the solution to the free-boundary problem (18), we know that the zero crossing point of the algebraic equation (20) is also unique, from which we conclude that $\underline{X} > -\infty$. □

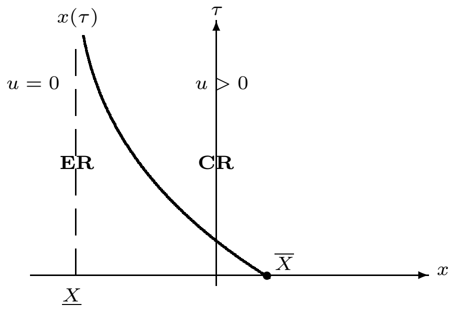

Fig. 1. Optimal exercise boundary $x(\tau)$ under the coordinates (τ, x).

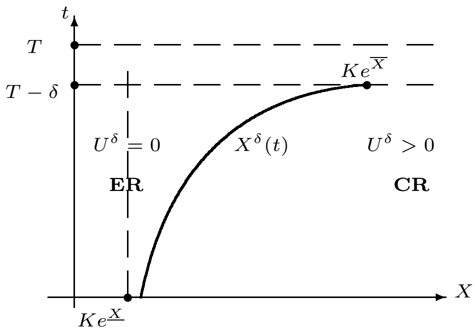

Fig. 2. Optimal exercise boundary $X^\delta(t)$ under the coordinates (t, X).

4. The optimal exercise boundary and its asymptotic analysis

With all the preparations, we are ready to prove Theorem 1 (ii) and (iii). For illustration purpose, we first demonstrate the optimal exercise boundary through Figs. 1 and 2.

Fig. 1 is under the coordinates (τ, x), and Fig. 2 is under the coordinates (t, X), where $\tau = T - \delta - t$ and $x = \ln X - \ln K$ (cf. the transformation (11)). Fig. 2 illustrates that the whole region $\Omega_{T-\delta}$ is divided by a curve $X^\delta(t)$ into two parts. In the left region, the investor will exercise the option (with time lag δ), and in the right region the investor will hold the option. Hence, $X^\delta(t)$ is called the *optimal exercise boundary*. If we denote by $x(\cdot)$ the optimal exercise boundaries under the coordinates (τ, x), as shown in Fig. 1, then we have the relationship

$$X^\delta(t) = K \exp\{x(T - \delta - t)\}. \tag{21}$$

4.1. Proof of Theorem 1 (ii)

Due to Remark 3, we will mainly work with variational inequality (12) for $u(\cdot,\cdot)$. Recall $\mathcal{N}_{T-\delta} = (0, T - \delta\,] \times \mathbb{R}$. Define the exercise domain **ER** and the continuation domain **CR** as

$$\mathbf{ER} = \{(\tau, x) \in \mathcal{N}_{T-\delta} : u(\tau, x) = 0\};$$
$$\mathbf{CR} = \{(\tau, x) \in \mathcal{N}_{T-\delta} : u(\tau, x) > 0\}.$$

Lemma 6. *Let $\overline{X}$ and $\underline{X}$ be given in Proposition 4 and (17), respectively. Then, it holds that*

$$\{\, x \le \overline{X} \,\} \supseteq \mathbf{ER} \supseteq \{\, x \le \underline{X} \,\} \quad \textit{and} \quad \{\, x > \overline{X} \,\} \subseteq \mathbf{CR} \subseteq \{\, x > \underline{X} \,\}.$$

Proof. In order to prove that $\mathbf{ER} \supseteq \{ x \leq \underline{X} \}$, we compare $u(\cdot,\cdot)$ and $u_\infty(\cdot)$, the latter of which is the solution to variational inequality (14). Note that

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})u_\infty(x) = \theta(x), & \text{if } u_\infty(x) > 0, \text{ for } (\tau, x) \in \mathcal{N}_{T-\delta}; \\ (\partial_\tau - \widetilde{\mathcal{L}})u_\infty(x) \geq \theta(x), & \text{if } u_\infty(x) = 0, \text{ for } (\tau, x) \in \mathcal{N}_{T-\delta}; \\ u_\infty(x) \geq 0 = u(0, x), & \text{for } x \in \mathbb{R}. \end{cases}$$

The comparison principle for variational inequality (12) in the domain $\mathcal{N}_{T-\delta}$ then implies that $u(\tau, x) \leq u_\infty(x)$. But if $x \leq \underline{X}$, according to the free-boundary problem (18), $u_\infty(x) = 0$. In turn, $u(\tau, x) = 0$. This proves that $\{x \leq \underline{X}\} \subseteq \mathbf{ER}$.

Repeating the above argument to compare u and v_∞ in the domain $\{x \geq \underline{X}\}$, where v_∞ is the classical solution of PDE (15), we obtain $u \geq v_\infty > 0$ in the domain $\{x > \underline{X}\}$, it follows that $\{x > \overline{X}\} \subseteq \mathbf{CR}$. $\square$

Intuitively, when $\theta(x)$ is positive (i.e. $x > \overline{X}$), the running payoff in (9) is also positive, so the investor will hold the option. In the contrary, when $\theta(x)$ is non-positive (i.e. $x \leq \overline{X}$), one may think that the investor would then exercise the option to stop her losses. However, the above lemma shows that for $x \leq \underline{X}$, the investor may still hold the option, and wait for the running payoff to rally at a later time to recover her previous losses.

Next, we define the *optimal exercise boundary* $x(\tau)$ as

$$x(\tau) = \inf\{x \in \mathbb{R} : u(\tau, x) > 0\}, \tag{22}$$

for any $\tau \in (0, T-\delta]$. It follows from Lemma 6 that $x(\tau) \in [\underline{X}, \overline{X}]$, and by the continuity of $u(\cdot,\cdot)$, $u(\tau, x) = 0$ for $x \leq x(\tau)$.

Lemma 7. *For $\tau \in (0, T-\delta]$, let*

$$x_1(\tau) = \sup\{x \in \mathbb{R} : u(\tau, x) = 0\}.$$

Then, $x(\tau) = x_1(\tau)$. Hence, $x(\tau)$ is the unique curve separating $\mathcal{N}_{T-\delta}$ such that $u(\tau, x) = 0$ for $x \leq x(\tau)$ and $u(\tau, x) > 0$ for $x \geq x(\tau)$.

Proof. The definition of $x_1(\tau)$ implies that $x(\tau) \leq x_1(\tau)$ and $u(\tau, x) > 0$ for $x \geq x_1(\tau)$. Moreover, it follows from Lemma 6 that $x_1(\tau) \in [\underline{X}, \overline{X}]$.

Suppose $x(\tau^*) < x_1(\tau^*)$ for some $\tau^* \in (0, T-\delta]$. The continuity of u implies that $u(\tau^*, x(\tau^*)) = u(\tau^*, x_1(\tau^*)) = 0$. Let x^* be a maximum point of $u(\tau^*, \cdot)$ in the interval $[x(\tau^*), x_1(\tau^*)]$. Suppose that $u(\tau^*, x^*) > 0$; otherwise $u(\tau^*, x) \equiv 0$ in the interval $[x(\tau^*), x_1(\tau^*)]$, which contradicts the definition of $x(\tau)$. Since $u(\tau^*, x^*) > 0$, $\partial_x u(\tau^*, x^*) = 0$ and $\partial_{xx} u(\tau^*, x^*) \leq 0$, we have

$$-\widetilde{\mathcal{L}}u(\tau^*, x^*) = -\frac{\sigma^2}{2}\,\partial_{xx}u(\tau^*, x^*) - \left(r - q - \frac{\sigma^2}{2}\right)\partial_x u(\tau^*, x^*) + ru(\tau^*, x^*) > 0.$$

On the other hand, by the continuity of u, there exists a neighborhood of (τ^*, x^*) such that $u > 0$, so $\partial_\tau u - \widetilde{\mathcal{L}}u = \theta$. In turn,

$$-\widetilde{\mathcal{L}}u(\tau^*, x^*) = \theta(x^*) - \partial_\tau u(\tau^*, x^*).$$

Since

$$\partial_\tau u(\tau,x) = -\partial_t U^\delta(t,X) = -\partial_t V^\delta(t,X) \geq 0, \tag{23}$$

where we have used the transformation (11) and the decomposition (10) in the first two equalities, and (7) in the last inequality, we further get

$$-\widetilde{\mathcal{L}}u(\tau^*,x^*) \leq \theta(x^*) < 0.$$

This is a contradiction. Thus, we must have $x(\tau^*) = x_1(\tau^*)$. $\square$

From the above lemma, we deduce that the exercise region and the continuation region are equivalent to

$$\mathbf{ER} = \{(\tau,x) \in \mathcal{N}_{T-\delta} : x \leq x(\tau)\};$$
$$\mathbf{CR} = \{(\tau,x) \in \mathcal{N}_{T-\delta} : x > x(\tau)\}.$$

We return to the proof of Theorem 1 (ii). Note that it is equivalent to the following proposition in terms of $x(\cdot)$.

Proposition 8. *Let $x(\tau)$ be the optimal exercise boundary given in (22). Then, the following assertions hold:*
(i) Monotonicity: $x(\tau)$ is strictly decreasing in τ[5]*;*
(ii) Position: $x(\tau)$ is with the starting point $x(0) = \lim_{\tau\to 0^+} x(\tau) = \overline{X}$;
(iii) Regularity: $x(\cdot) \in C^\infty(0,T-\delta\,]$ and $u(\cdot,\cdot) \in C^\infty(\{x \geq x(\tau) : \tau \in (0,T-\delta\,]\})$.

Proof. (i) We first show that $x(\tau)$ is non-increasing. For any $0 \leq \tau_1 < \tau_2 \leq T-\delta$, we then have $0 = u(\tau_2,x(\tau_2)) \geq u(\tau_1,x(\tau_2)) \geq 0$. Thus, $u(\tau_2,x(\tau_2)) = u(\tau_1,x(\tau_2)) = 0$, and together with Lemma 7, we deduce that $x(\tau_1) \geq x(\tau_2)$, i.e. $x(\tau)$ is non-increasing.

If $x(\tau)$ is not strictly decreasing, then there exist $x_1 \in [\underline{X},\overline{X}]$ and $0 \leq \tau_1 < \tau_2 \leq T-\delta$ such that $x(\tau) = x_1$ for any $\tau \in [\tau_1,\tau_2]$. See Fig. 3 below. Note that $\partial_x u(\tau,x_1) = 0$ and, moreover, $\partial_\tau\partial_x u(\tau,x_1) = 0$ for any $\tau \in [\tau_1,\tau_2]$.

On the other hand, we observe that in the domain $[\tau_1,\tau_2] \times (x_1,x_1+1)$, $u(\cdot,\cdot)$ satisfies

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})u(\tau,x) = \theta(x), & \text{for } (\tau,x) \in [\tau_1,\tau_2] \times (x_1,x_1+1); \\ u(\tau,x_1) = 0, & \text{for } \tau \in [\tau_1,\tau_2]. \end{cases}$$

In turn, $\partial_\tau u(\cdot,\cdot)$ satisfies

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})\partial_\tau u(\tau,x) = \partial_\tau\theta(x) = 0, & \text{for } (\tau,x) \in [\tau_1,\tau_2] \times (x_1,x_1+1); \\ \partial_\tau u(\tau,x_1) = 0, & \text{for } \tau \in [\tau_1,\tau_2]. \end{cases}$$

For any $x_2 > \overline{X}$, since $(\tau_2,x_2) \in \mathbf{CR}$, we have $u(\tau_2,x_2) > 0$, and $u(0,x_2) = 0$. Hence, there exists $\tau \in (0,\tau_2)$ such that $\partial_\tau u(\tau,x_2) > 0$. Note, however, that $\partial_\tau u \geq 0$ (cf. (23)) and, therefore, the strong maximum principle (see [15]) implies that $\partial_\tau u > 0$ in $\mathbf{CR}$.

Together with $\partial_\tau u(\tau,x_1) = 0$ for any $\tau \in [\tau_1,\tau_2]$, we deduce that $\partial_x\partial_\tau u(\tau,x_1) > 0$ using Hopf lemma (see [15]). But this is a contradiction to $\partial_\tau\partial_x u(\tau,x_1) = 0$ for any $\tau \in [\tau_1,\tau_2]$.

(ii) It is obvious that $x(0) \leq \overline{X}$ from Lemma 6, so it is sufficient to prove that $x(0) \geq \overline{X}$. If not, in the domain $(0,T-\delta] \times (x(0),\overline{X}) \subset \mathbf{CR}$, we consider

[5] Recently, [11] provides a new probabilistic argument to prove that the free boundary is strictly monotonic. We thank the referee for pointing out [11].

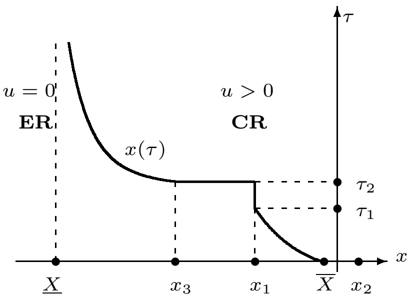

Fig. 3. Non-strictly decreasing and discontinuous free boundary $x(\tau)$.

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})u(\tau,x) = \theta(x) < 0, & \text{for } (\tau,x) \in (0, T-\delta] \times (x(0), \overline{X}); \\ u(0,x) = 0, & \text{for } x \in (x(0), \overline{X}). \end{cases}$$

Then, $\partial_\tau u(0,x) = \widetilde{\mathcal{L}}u(0,x) + \theta(x) = \theta(x) < 0$, which is a contradiction to $\partial_\tau u \geq 0$ in (23).

(iii) We first prove that $x(\tau)$ is continuous. If not, then there exists $\tau_2 \in (0, T-\delta)$ and $\underline{X} \leq x_3 < x_1 \leq \overline{X}$ such that $x(\tau_2 + 0) = x_3$ and $x(\tau_2 - 0) = x_1$. See Fig. 3.

In the domain $(\tau_2, T-\delta] \times (x_3, x_1) \subset \mathbf{CR}$, we consider

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})u(\tau,x) = \theta(x) < 0, & \text{for } (\tau,x) \in [\tau_2, T-\delta] \times (x_3, x_1); \\ u(\tau_2,x) = 0, & \text{for } x \in (x_3, x_1). \end{cases}$$

Then, $\partial_\tau u(\tau_2,x) = \widetilde{\mathcal{L}}u(\tau_2,x) + \theta(x) = \theta(x) < 0$, which is a contradiction to $\partial_\tau u \geq 0$ in (23).

Finally, since $\partial_\tau u \geq 0$, the smoothness of both the optimal exercise boundary $x(\tau)$ and the value function $u(\cdot,\cdot)$ in the continuation region follow along the similar arguments used in [16]. $\square$

4.2. Proof of Theorem 1 (iii): asymptotic behavior for large maturity

We study the asymptotic behavior of the optimal exercise boundary $x(\tau)$ and the value function $u(\tau,x)$ as $\tau \to \infty$, which in turn proves Theorem 1 (iii) for the asymptotic behavior of $X^\delta(t)$ when $T \to \infty$.

To this end, we consider the auxiliary optimal stopping time problem perturbed by $r\epsilon$,

$$U^\epsilon_\infty(X_t) = \operatorname*{ess\,sup}_{\tau^0 \geq t} \mathbf{E}\left[\int_t^{\tau^0} e^{-r(s-t)}(\Theta(X_s) - r\epsilon)ds|\mathcal{F}_t\right], \tag{24}$$

for any $\mathbb{F}$-stopping time $\tau^0 \geq t$ and any $\epsilon \geq 0$. This will help us to achieve the lower bound and, therefore, the asymptotic behavior of the optimal exercise boundary $x(\tau)$.

Following along the similar arguments used in section 3, we obtain that $u^\epsilon_\infty(x) = U^\epsilon_\infty(X)$, where $x = \ln X - \ln K$, and $u^\epsilon(\cdot)$ is the unique strong solution to the stationary variational inequality

$$\begin{cases} -\widetilde{\mathcal{L}}\, u^\epsilon_\infty(x) = \theta(x) - r\epsilon, & \text{if } u^\epsilon_\infty(x) > 0, \text{ for } x \in \mathbb{R}; \\ -\widetilde{\mathcal{L}}\, u^\epsilon_\infty(x) \geq \theta(x) - r\epsilon, & \text{if } u^\epsilon_\infty(x) = 0, \text{ for } x \in \mathbb{R}, \end{cases} \tag{25}$$

with $u^\epsilon_\infty \in W^2_{p,loc}(\mathbb{R})$ for $p \geq 1$ and $(u^\epsilon_\infty)' \in C(\mathbb{R})$.

In contrast to variational inequality (14), it is not clear how to reduce variational inequality (25) to a free-boundary problem, and to obtain its explicit solution. Nevertheless, we are able to derive a local version

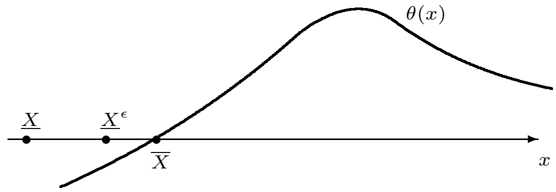

Fig. 4. The graph of $\underline{X}$, $\underline{X}^\epsilon$ and $\overline{X}$.

of the free-boundary problem with $\epsilon > 0$ small enough, which is sufficient to obtain the asymptotic behavior of the optimal exercise boundary later on.

Lemma 9. *For $\epsilon > 0$ small enough, it holds that $u_\infty(x) \geq u^\epsilon_\infty(x) \geq u_\infty(x) - \epsilon$. Define $\underline{X}^\epsilon$ as*

$$\underline{X}^\epsilon = \inf\{x \in (-\infty, \overline{X}] : u^\epsilon_\infty(x) > 0\}.$$

Then $\underline{X} \leq \underline{X}^\epsilon < \overline{X}$, and $u^\epsilon_\infty(x) > 0$ for any $x \in (\underline{X}^\epsilon, \overline{X})$, where $\underline{X}$ and $\overline{X}$ are given in (17) and Proposition 4, respectively. Moreover, $\underline{X}^\epsilon \to \underline{X}$ as $\epsilon \to 0^+$. See Fig. 4.

Proof. Note that the running payoff in the optimal stopping problem (24) satisfies

$$\begin{aligned}\int_t^{\tau^0} e^{-r(s-t)}\Theta(X_s)ds &\geq \int_t^{\tau^0} e^{-r(s-t)}(\Theta(X_s) - r\epsilon)ds \\ &= \int_t^{\tau^0} e^{-r(s-t)}\Theta(X_s)ds + \epsilon e^{-r(\tau^0-t)} - \epsilon \\ &\geq \int_t^{\tau^0} e^{-r(s-t)}\Theta(X_s)ds - \epsilon,\end{aligned}$$

for any $\mathbb{F}$-stopping time $\tau^0 \geq t$. It follows that $u_\infty(x) \geq u^\epsilon_\infty(x) \geq u_\infty(x) - \epsilon$.

Since $u_\infty(x) > 0$ for $x > \underline{X}$, and $\overline{X} > \underline{X}$ by Proposition 5, it holds that $u_\infty(\overline{X}) > 0$. Let $\epsilon > 0$ be small enough such that $\epsilon < u_\infty(\overline{X})$. Using the inequality $u^\epsilon_\infty(x) \geq u_\infty(x) - \epsilon$, we obtain that

$$u^\epsilon_\infty(\overline{X}) \geq u_\infty(\overline{X}) - \epsilon > 0.$$

In turn, the definition of $\underline{X}^\epsilon$ and the continuity of $u^\epsilon_\infty(\cdot)$ imply that $\underline{X}^\epsilon < \overline{X}$.

Repeating the similar arguments used in Proposition 5, we obtain that $u^\epsilon_\infty(x) > 0$ for $x \in (\underline{X}^\epsilon, \overline{X})$. Furthermore, the inequality $u_\infty(x) \geq u^\epsilon_\infty(x)$ and Proposition 5 imply that

$$0 = u_\infty(\underline{X}) \geq u^\epsilon_\infty(\underline{X}).$$

In turn, the definition of $\underline{X}^\epsilon$ implies that $\underline{X}^\epsilon \geq \underline{X}$.

Next, we prove that $\underline{X}^\epsilon \to \underline{X}$ as $\epsilon \to 0^+$. In fact, from the definition of $\underline{X}^\epsilon$ and the continuity of u^ϵ_∞, we know that $u^\epsilon_\infty(\underline{X}^\epsilon) = 0$. Using the inequality $u^\epsilon_\infty(x) \geq u_\infty(x) - \epsilon$ again, we obtain $u_\infty(\underline{X}^\epsilon) \leq \epsilon$.

On the other hand, applying Taylor's expansion to $u_\infty(x)$ yields

$$u_\infty(x) = \frac{1}{2}u''_\infty(\underline{X} + 0)(x - \underline{X})^2(1 + o(1)) = \frac{-\theta(\underline{X})}{\sigma^2}(x - \underline{X})^2(1 + o(1)),$$

which further implies that $u_\infty(x) > \kappa(x - \underline{X})^2$ with some positive constant κ if x is close enough to $\underline{X}$. Moreover, since $u_\infty(x) > 0$ in the interval $(\underline{X}, \overline{X}\,]$ and is continuous, we deduce that if ϵ is small enough, then $\underline{X}^\epsilon \le \underline{X} + \sqrt{\epsilon/\kappa}$. Recalling $\underline{X}^\epsilon \ge \underline{X}$, we conclude that $\underline{X}^\epsilon \to \underline{X}$ as $\epsilon \to 0^+$. □

We return to the proof of Theorem 1 (iii), which is equivalent to the following proposition.

Proposition 10. *Let $u(\cdot,\cdot)$ and $x(\tau)$ be the solution to variational inequality (12) and its associated optimal exercise boundary (22), respectively. Then,*

$$u(\tau, \cdot) \to u_\infty(\cdot) \quad \text{and} \quad x(\tau) \to \underline{X},$$

as $\tau \to \infty$, where $u_\infty(\cdot)$ and $\underline{X}$ are the solution of the stationary variational inequality (14) and its associated optimal exercise boundary (17), respectively,

Proof. From the optimal stopping problems (9) and (13), it is immediate that $u(\cdot,\cdot) \le u_\infty(\cdot)$. For $t \le (T-\delta)/2$, define

$$u^t(\tau, x) = u_\infty^{\exp\{-rt\}}(x) - e^{-r(\tau - t)} + e^{-rt},$$

where $u_\infty^{\exp\{-rt\}}(\cdot)$ is the solution of variational inequality (25) with $\epsilon = \exp\{-rt\}$. It is routine to check that $u^t \in W^{2,1}_{p,loc}(\mathcal{N}_{2t}) \cap C(\overline{\mathcal{N}_{2t}}\,)$, and satisfies

$$\begin{cases} (\partial_\tau - \widetilde{\mathcal{L}})u^t(\tau,x) = \theta(x), & \text{if } u^t(\tau,x) > -e^{-r(\tau-t)} + e^{-rt}, \text{ for } (\tau,x) \in \mathcal{N}_{2t}; \\ (\partial_\tau - \widetilde{\mathcal{L}})u^t(\tau,x) \ge \theta(x), & \text{if } u^t(\tau,x) = -e^{-r(\tau-t)} + e^{-rt}, \text{ for } (\tau,x) \in \mathcal{N}_{2t}; \\ u^t(0,x) = u_\infty^{\exp\{-rt\}}(x) - e^{rt} + e^{-rt} < 0, & \text{for } x \in \mathbb{R}, \end{cases}$$

provided that t and T are large enough. Since the obstacle $-e^{-r(\tau-t)} + e^{-rt} \le 0$ in the domain $\mathcal{N}_{2t}$, using the comparison principle (see [17] or [23]) for variational inequality (12) in the domain $\mathcal{N}_{2t}$, we deduce that $u(\tau,x) \ge u^t(\tau,x)$ for $(\tau,x) \in \mathcal{N}_{2t}$. In turn, Lemma 9 implies that

$$u(2t,\cdot) \ge u^t(2t,\cdot) = u_\infty^{\exp\{-rt\}}(\cdot) \ge u_\infty(\cdot) - e^{-rt}. \tag{26}$$

Together with $u(2t,\cdot) \le u_\infty(\cdot)$, we obtain that $u(2t,\cdot) \to u_\infty(\cdot)$ as $t \to \infty$.

To prove the convergence of the optimal exercise boundary $x(\tau)$ to $\underline{X}$, we choose t large enough such that $\underline{X}^{\exp\{-rt\}} + \exp\{-rt\} < \overline{X}$. Then, (26) yields that

$$u\left(2t, \underline{X}^{\exp\{-rt\}} + \exp\{-rt\}\right) \ge u_\infty^{\exp\{-rt\}}\left(\underline{X}^{\exp\{-rt\}} + \exp\{-rt\}\right) > 0,$$

where we have used $u_\infty^{\exp\{-rt\}}(x) > 0$ for $x \in (\underline{X}^{\exp\{-rt\}}, \overline{X})$ (cf. Lemma 9) in the second inequality. It then follows from the definition of $x(\tau)$ in (22) that

$$x(2t) \le \underline{X}^{\exp\{-rt\}} + \exp\{-rt\}.$$

By Lemma 6, we also have $x(\tau) \ge \underline{X}$ for any $\tau \in [\,0, T-\delta\,]$. Hence, we have proved that

$$\underline{X} \le x(2t) \le \underline{X}^{\exp\{-rt\}} + \exp\{-rt\}.$$

Finally, we send $t \to \infty$ in the above inequalities, and conclude the convergence of $x(2t)$ to $\underline{X}$ by Lemma 9. □

Remark 11. Under the original coordinates (t, X), it follows from the relationship (21) and Proposition 10 that $X^\delta(t) \to Ke^{\underline{X}}$ as $T \to \infty$, so $Ke^{\underline{X}}$ is the asymptotic line of the optimal exercise boundary $X^\delta(t)$.

Proposition 10 also establishes the connection between the optimal stopping problems (9) and (13): $U^\delta(t, X) \to U^\delta_\infty(X)$ uniformly in $X \in \mathbb{R}_+$ as $T \to \infty$. Moreover, it follows from the decomposition formula (10) that the value function of the American put option with time lag δ has the long maturity limit: $V^\delta(t, X) \to P(T-\delta, X) + U^\delta_\infty(X)$ uniformly in $X \in \mathbb{R}_+$ as $T \to \infty$.

4.3. Proof of Theorem 1 (iii): asymptotic behavior for small time lag

Finally, we prove Theorem 1 (iii) for the asymptotic behavior of $X^\delta(t)$ when $\delta \to 0$. Recall that $X^0(t)$ denotes the optimal exercise boundary of the corresponding standard American put option. It is well known that $X^0(t)$ is a strictly increasing and smooth function with $X^0(T) = K$. We refer to [6] and [18] for its proof.

We first extend variational inequality (5) from $\Omega_{T-\delta}$ to Ω_T by defining $V^\delta(t, X) = P(t, X)$ for $(t, X) \in [T-\delta, T] \times \mathbb{R}_+$, and rewrite (5) as

$$\begin{aligned}(-\partial_t - \mathcal{L})V^\delta(t, X) &= I_{\{V^\delta = P(T-\delta, X)\}}(-\partial_t - \mathcal{L})P(T-\delta, X) \\ &= -I_{\{V^\delta = P(T-\delta, X)\}}\Theta(X), \end{aligned} \tag{27}$$

for $(t, X) \in \Omega_T$, and $V^\delta(T, X) = (K - X)^+$ for $X \in \mathbb{R}^+$.

Denote $\mathcal{N}^n_T := (0, T] \times \mathcal{N}^n$ and $\mathcal{N}^n := (-n, K - \frac{1}{n})$. Then, we apply the $W^{2,1}_p$-estimates (see Lemma A.4 in [23] for example) to the above PDE (27) for $V^\delta(\cdot, \cdot)$, and obtain that for any $n \in \mathbb{N}$,

$$\|V^\delta\|_{W^{2,1}_p(\mathcal{N}^n_T)} \le C\Big(\|V^\delta\|_{L^p(\mathcal{N}^{2n}_T)} + \|\Theta\|_{L^p(\mathcal{N}^{2n})} + \|K - X\|_{W^{2,1}_p(\mathcal{N}^{2n})}\Big). \tag{28}$$

Note that the right hand side of the above inequality is independent of δ due to the fact that $V^0(t, X) - K(1 - e^{-r\delta}) \le V^\delta(t, X) \le V^0(t, X)$ (cf. (2)), and the formula (30) for $\Theta(X)$.

From Theorem 1 (i), V^δ converges to V^0 in $C(\overline{\Omega_T})$ as $\delta \to 0$. Hence, the above estimate (28) implies that V^δ also converges weakly to V^0 in $W^{2,1}_p(\mathcal{N}^n_T)$ and

$$-I_{\{V^\delta = P(T-\delta, X)\}}\Theta(X) = (-\partial_t - \mathcal{L})V^\delta(t, X) \rightharpoonup (-\partial_t - \mathcal{L})V^0(t, X)$$

weakly in $L^p(\mathcal{N}^n_T)$ as $\delta \to 0$. But note that

$$(-\partial_t - \mathcal{L})V^0(t, X) = I_{\{V^0 = K - X\}}(rK - qX).$$

In turn,

$$-I_{\{V^\delta = P(T-\delta, X)\}}\Theta(X) \rightharpoonup I_{\{V^0 = K - X\}}(qX - rK) \tag{29}$$

weakly in $L^p(\mathcal{N}^n_T)$.

Now suppose that $X^\delta(t)$ does not converge to $X^0(t)$. Then there exist $t_0 \in [0, T)$ and a sequence $\{X^{\delta_m}\}_{m=1}^\infty$ such that when $\delta_m \to 0$, $X^{\delta_m}(t_0)$ does not converge to $X^0(t_0)$.

Since $X^0(t)$ is continuous and strictly increasing with $X^0(T) = K$, we may assume there exists $\epsilon > 0$ and an integer M such that $X^0(t_0) + 2\epsilon < \min\{X^{\delta_m}(t_0), K\}$ for any $m \ge M$. See Fig. 5 below. Other cases can be treated in a similar way.

By the continuity and strictly increasing property of both $X^0(t)$ and $X^\delta(t)$, we can find $\eta > 0$ such that the compact set $[t_0, t_0 + \eta] \times [X^0(t_0) + \epsilon, X^0(t_0) + 2\epsilon]$ is in the exercise region of V^{δ_m} and the continuation region of V^0. Therefore, in this compact set, $V^{\delta_m}(t, X) = P(T - \delta_m, X)$, $V^0(t, X) > K - X$, and

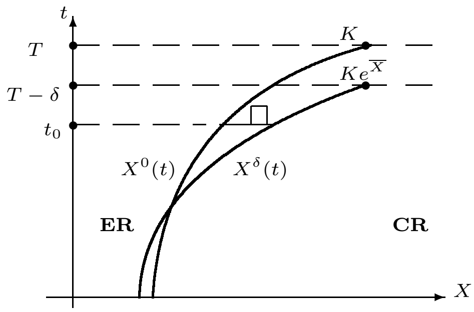

Fig. 5. Non-convergence of the free boundaries $X^\delta(t)$ to $X^0(t)$ as $\delta \to 0$.

$$
\begin{aligned}
&- I_{\{V^{\delta_m}=P(T-\delta_m,X)\}}\Theta(T-\delta_m,X) - I_{\{V^0=K-X\}}(qX-rK)\\
=&-\Theta(T-\delta_m,X),
\end{aligned}
$$

where we use the notation $\Theta(T-\delta_m,\cdot)$ to emphasize its dependence on $T-\delta_m$. However, from Proposition 4, it is immediate to check that

$$
\lim_{\delta_m\to 0}\Theta(T-\delta_m,X) = qX - rK < 0, \text{ for } X < K,
$$

which is a contradiction to (29).

5. Conclusion

This paper studies the asymptotic behavior of the value function and the optimal exercise boundary of American put options with delivery lags through free boundary techniques. On one hand, it would be interesting to carry out the free boundary analysis to the real option setup such as reversible investment ([2], [3], [9]), impulse control ([4], [5], [20]), and recursive optimal stopping ([8], [12]). On the other hand, it might be possible to prove the convexity of the optimal exercise boundary (as in [7] and [14] for the standard American put case). Such extensions are left for the future research.

Appendix A. Proof of Proposition 4

(i) We first show that the function $\theta(x)$ (or equivalently, $\Theta(X)$ with $X = Ke^x$) has the explicit form

$$
\theta(x) = qKe^{x-q\delta}N(-d_1) + \frac{\sigma K}{2\sqrt{\delta}}\, e^{-r\delta}N'(-d_2) - rKe^{-r\delta}N(-d_2), \tag{30}
$$

where $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{\frac{-\xi^2}{2}}d\xi$, $N'(d) = \frac{1}{\sqrt{2\pi}}e^{\frac{-d^2}{2}}$, and

$$
d_1 = \frac{x}{\sigma\sqrt{\delta}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{\delta}, \qquad d_2 = d_1 - \sigma\sqrt{\delta}. \tag{31}
$$

To this end, let $x = \ln X - \ln K$ and $p(t,x) = P(t,X)$. It is well known that $p(t,x)$ has the explicit expression (see [18] for example)

$$
p(t,x) = Ke^{-r(T-t)}N(-d_2^t) - Ke^{x-q(T-t)}N(-d_1^t), \tag{32}
$$

where d_1^t and d_t^2 are the same as d_1 and d_2 in (31) except that δ is replaced $T-t$:

$$d_1^t = \frac{x}{\sigma\sqrt{T-t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T-t}, \quad d_2^t = d_1^t - \sigma\sqrt{T-t}.$$

Differentiating $p(t,x)$ against t yields that

$$\begin{aligned}\partial_t p(t,x) &= rKe^{-r(T-t)}N(-d_2^t) - qKe^{x-q(T-t)}N(-d_1^t) \\ &\quad - Ke^{-r(T-t)}N'(-d_2^t)\left(\partial_t d_1^t + \frac{\sigma}{2\sqrt{T-t}}\right) + Ke^{x-q(T-t)}N'(-d_1^t)\,\partial_t d_1^t \\ &= rKe^{-r(T-t)}N(-d_2^t) - qKe^{x-q(T-t)}N(-d_1^t) \\ &\quad - \frac{\sigma K}{2\sqrt{T-t}}\,e^{-r(T-t)}N'(-d_2^t),\end{aligned}$$

where we have used the fact that

$$e^{-r(T-t)}N'(-d_2^t) = e^{x-q(T-t)}N'(-d_1^t). \tag{33}$$

Thus we have proved (30).

To prove Proposition 4 (i), we use the following two elementary inequalities: For $d \geq 0$,

$$N(-d) < \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d} e^{-\frac{\xi^2}{2}}\frac{\xi}{-d}d\xi = \frac{1}{\sqrt{2\pi}d}e^{-\frac{d^2}{2}} = \frac{1}{d}N'(-d); \tag{34}$$

$$N(-d) > \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d} e^{-\frac{\xi^2}{2}}\frac{1+\frac{1}{\xi^2}}{1+\frac{1}{d^2}}d\xi = \frac{1}{\sqrt{2\pi}(d+\frac{1}{d})}e^{-\frac{d^2}{2}} = \frac{1}{d+\frac{1}{d}}N'(-d). \tag{35}$$

We first show that there exists $\overline{X}$ such that $\theta(\overline{X}) = 0$. Note that this is equivalent to show that $\theta(\overline{X})/N'(-\overline{d}_2) = 0$, where $\overline{d}_2$ is the same as d_2 in (31) except that x is replaced by $\overline{X}$.

For x large enough such that $d_1, d_2 \geq 0$ (cf. (31)), we have

$$\begin{aligned}\frac{\theta(x)}{N'(-d_2)} &= qKe^{x-q\delta}\frac{N(-d_1)}{N'(-d_2)} + \frac{\sigma K}{2\sqrt{\delta}}e^{-r\delta} - rKe^{-r\delta}\frac{N(-d_2)}{N'(-d_2)} \\ &\geq qKe^{x-q\delta}\frac{1}{d_1+\frac{1}{d_1}}\frac{N'(-d_1)}{N'(-d_2)} + \frac{\sigma K}{2\sqrt{\delta}}e^{-r\delta} - rKe^{-r\delta}\frac{1}{d_2}\frac{N'(-d_2)}{N'(-d_2)},\end{aligned}$$

by using the inequalities (34) and (35). From (33), we further obtain that

$$\frac{\theta(x)}{N'(-d_2)} \geq \frac{qK}{d_1+\frac{1}{d_1}}e^{-r\delta} + \frac{\sigma K}{2\sqrt{\delta}}e^{-r\delta} - rKe^{-r\delta}\frac{1}{d_2} > 0,$$

provided that $d_2 \geq 2r\sqrt{\delta}/\sigma$, so $\frac{\theta(x)}{N'(-d_2)} > 0$ for large enough x.

On the other hand, when $x \to -\infty$, we have that $d_1, d_2 \to -\infty$ and, therefore,

$$N(-d_1),\ N(-d_2) \to 1, \text{ and } N'(-d_2) \to 0.$$

Hence, $\theta(x) \to -rKe^{-r\delta} < 0$. This means that $\theta(x)$ is negative provided x is small enough, so $\frac{\theta(x)}{N'(-d_2)} < 0$ for small enough x. Since $\frac{\theta(x)}{N'(-d_2)}$ is obviously continuous in x, we conclude that there exists $\overline{X} \in \mathbb{R}$ such that $\theta(\overline{X})/N'(-\overline{d}_2) = 0$.

Next, we show that $\frac{\theta(x)}{N'(-d_2)}$ is strictly increasing in x, so its zero crossing point $\overline{X}$ is unique. Indeed, note that

$$\left(\frac{\theta(x)}{N'(-d_2)}\right)' = \frac{Ke^{-r\delta}}{\sigma\sqrt{\delta}}\left[r - q + q\frac{N(-d_1)d_1}{N'(-d_1)} - r\frac{N(-d_2)d_2}{N'(-d_2)}\right].$$

Let $h(d) := \frac{N(-d)d}{N'(-d)}$. Then, we calculate its derivative against d as

$$h'(d) = -d + \frac{N(-d)}{N'(-d)} + \frac{N(-d)}{N'(-d)}d^2.$$

It is obvious that $h'(d) > 0$ when $d \le 0$. For $d > 0$, by using the inequalities (34) and (35), we obtain that

$$h'(d) > -d + \frac{1}{d + \frac{1}{d}} + \frac{1}{d + \frac{1}{d}}d^2 = 0.$$

In turn, $h(d_2) < h(d_1)$, which yields that

$$\left(\frac{\theta(x)}{N'(-d_2)}\right)' > \frac{Ke^{-r\delta}}{\sigma\sqrt{\delta}}\left[r - q + (q - r)h(d_1)\right] \ge 0$$

by noting that $h(d_2) < \lim_{d\to\infty} h(d) = 1$ and $r > q$.

(ii) For any $x < 0$, since $\delta \to 0^+$, $d_1, d_2 \to -\infty$, and, therefore,

$$N(-d_1),\ N(-d_2) \to 1,\ \text{and } N'(-d_2) \to 0,\ \theta(x) \to qKe^x - rK.$$

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