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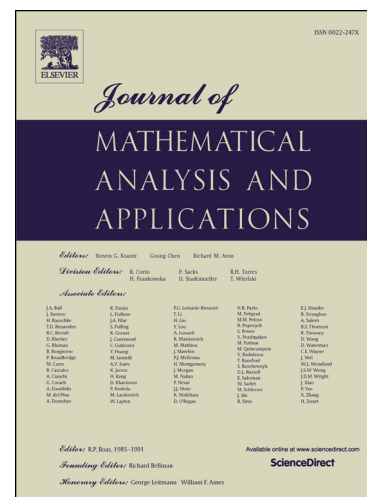
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DOMINATING SETS IN BERGMAN SPACES AND SAMPLING CONSTANTS

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ABSTRACT. We discuss sampling constants for dominating sets in Bergman spaces. Our method is based on a Remez-type inequality by Andrievskii and Ruscheweyh. We also comment on extensions of the method to other spaces such as Fock and Paley-Wiener spaces.

1. INTRODUCTION

Let $A^{p,\alpha}(\mathbb{D})$, $1 \leq p < \infty$, $\alpha > -1$, be the weighted Bergman space on the unit disk $\mathbb{D}$ in the complex plane defined by

$$A^{p,\alpha}(\mathbb{D}) = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_p^p = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < +\infty\}. \quad (1.1)$$

Here dA denotes normalized Lebesgue area measure on $\mathbb{D}$. We will also use the notation $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ (see [7] for more information on Bergman spaces). The unweighted Bergman space is denoted by $A^p = A^{p,0}$. In this paper we are interested in measurable sets $E \subset \mathbb{D}$ for which we have

$$\int_E |f(z)|^p dA_\alpha(z) \geq C^p \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z). \quad (1.2)$$

We will occasionally write $L^{p,\alpha}(F)$ for the Lebesgue space on a measurable set $F \subset \mathbb{D}$ with respect to dA_α . A set satisfying (1.2) will be called *dominating* for $A^{p,\alpha}$. We mention that dominating sets are closely related to so-called reverse Carleson measures for which we refer to the survey [4] for more information. The question of dominating sets in the Bergman space has been studied first by Luecking [11]–[13] who completely characterized these dominating sets in A^p in terms of *relative density*, which morally speaking means that every pseudohyperbolic disk of a certain minimal fixed radius (depending on E) meets the set E with uniformly positive density (precise definitions will be given below). We mention the closely related question of sampling sequences which have been characterized by Seip (see the monograph [19] for a general reference on sampling and interpolation).

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A question which arises quite naturally is whether it is possible to estimate the constant C appearing in (1.2) in terms of the underlying density. A prominent example where this question has been studied is the Paley-Wiener space. Dominating sets in this space go back to work by Logvinenko-Sereda [10] and Panejah [15]-[16]. We mention also some recent work on so-called Paley-Wiener measures (see e.g. [17, Theorem 15]). Having precise estimates on the sampling constants is important in applications when one has to decide on the trade-off between the cost of the sampling and the accuracy of the estimates. In the early 2000's Kovrijkine [9] considered the Paley-Wiener space and gave a precise and optimal estimate on C in terms of the underlying density. His method is quite clever and based on the Bernstein inequality which does not hold in Bergman or Fock spaces but for instance in model spaces where the proof was adapted in [6]. It was brought to our attention that very recently, Jaming and Speckbacher [8] considered the case of polyanalytic Fock spaces using a strong result by Brudnyi — which gives a Remez-type inequality for plurisubharmonic functions — and a clever trick allowing to switch from polyanalytic functions to analytic ones in 2 variables. Since the Fock space is a special case of the polyanalytic Fock space, they in particular get an estimate of the sampling constant in Fock spaces. It is well known that the Bernstein inequality does not hold in the Fock space. In order to turn around this obstacle, Jaming and Speckbacher define a *good* dominating set on which a kind of superharmonic inequality holds (see [8, Lemma 4.4]) and use a local reproducing formula for the short-time Fourier transform. Our method is more elementary, following Kovrijkine's original idea of defining good disks — now without appealing to the Bernstein inequality — and using holomorphy. While our result is presented for the Bergman space, for which the geometry is different, the method applies quite directly to consider the case of the (analytic) Fock space. It should also be noted that our intermediate step (Proposition 1) allows to circumvent the Bernstein inequality in a large set of situations, including of course Fock spaces but also the Paley-Wiener space. We will comment on these observations in the last section.

We need to introduce some notation. Let

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

be the pseudohyperbolic distance between two points $z, w \in \mathbb{D}$. We consider the associated pseudohyperbolic balls: $D_{phb}(z, r) = \{w \in \mathbb{D} : \rho(z, w) < r\}$, where $0 < r < 1$. A measurable set $E \subset \mathbb{D}$ is called (γ, r_0) -dense for some $\gamma > 0$ and $0 < r_0 < 1$, if for every $z \in \mathbb{D}$

$$\frac{|E \cap D_{phb}(z, r_0)|}{|D_{phb}(z, r_0)|} \geq \gamma. \quad (1.3)$$

Here $|F|$ denotes planar Lebesgue measure of a measurable set F . We will just say that the set is relatively dense if there is some $\gamma > 0$ and some $0 < r < 1$ such that the set is (γ, r) -dense. For a relatively dense set E we can have $\check{r} = \inf\{r_0 > 0 :$

$\inf_{z \in \mathbb{D}} |E \cap D_{phb}(z, r_0)| / |D_{phb}(z, r_0)| > 0\} > 0$. This means that E has “holes” of (pseudo-hyperbolic) diameter $\check{r}$, but not for $\check{r} < r_0 < 1$.

Luecking’s result on dominating sets can be stated as follows ([11], and in particular condition (2’) in [11, p.4]).

Theorem 1 (Luecking). *A (Lebesgue) measurable set E is dominating in A^p ($p > 0$) if and only if it is relatively dense.*

Looking closely at the proof proposed by Luecking — who was not interested in the magnitude of the sampling constant — it turns out that his sampling constant depends exponentially on γ : $C^p \geq \gamma e^{-c_p/\gamma}$. A more general result concerning sampling measures in a large class of spaces of analytic functions has been discussed again by Luecking in [13], and in particular in Bergman spaces [13, Theorem 1]. Even if the density appears explicitly in his proof, a compactness argument used in [13, Lemma 4] introduces an implicit dependence on the density.

Our main result is the following

Theorem 2. *Let $1 \leq p < +\infty$. There exists L such that for every measurable set $E \subset \mathbb{D}$ which is (γ, r) -dense, we have*

$$\|f\|_{L^{p,\alpha}(E)} \geq \left(\frac{\gamma}{c}\right)^L \|f\|_{A^{p,\alpha}} \quad (1.4)$$

for every $f \in A^p$.

The constants c and L depend on r . For L we can choose

$$L = c_1 \frac{1+\alpha}{p} \frac{1}{(1-r)^4} \ln \frac{1}{1-r},$$

where c_1 is some universal constant. If we understand r to be determined by the size of the maximal “holes” of E (r has to be strictly bigger than that size in order to get (γ, r) -density), then this enters in the power L of γ .

It should be noted that there is a competing relation between γ and r . The density γ can be very small for a given r (because E has holes which have pseudohyperbolic radius close to r), but it can become rather big when we choose a bigger radius (e.g. pseudohyperbolic doubling of r : $2r/(1+r^2)$). In a sense one needs to optimize $L \ln \gamma$, and L depends on r .

Here is another observation. Though this might be obvious, it should be mentioned that there is no reason a priori why a holomorphic function for which the integral $\int_E |f|^p dA_\alpha$ is bounded for a relative dense set E should be in $A^{p,\alpha}$. Outside the class $A^{p,\alpha}$ relative density is in general not necessary for domination (see also a remark in [11, p.11]).

The discussion of the necessary condition of relative density in [11, p.5], involving testing on reproducing kernels, shows that given a sampling constant C in (1.4), then $\gamma \gtrsim C^p$, i.e. $C \lesssim \gamma^{1/p}$. We mention that for $p = 2$, using orthonormal bases of monomials in Bergman

spaces on disks, it can be shown for $E_R = \mathbb{D} \setminus D(0, R)$, $D(0, R) = \{z \in \mathbb{D} : |z| < R\}$, $0 < R < 1$, that

$$\lim_{R \rightarrow 1^-} \left(\frac{1}{\gamma_R} \inf_{0 \neq f \in A^2} \frac{\int_{E_R} |f|^2 dA}{\int |f|^2 dA} \right) = 2,$$

where γ_R is the density of E_R when testing (1.3) on disks $D_{phb}(z, r_0)$ with $r_0 = (1 + R)/2$, so that in this specific example $L = 1/2 = 1/p$ is sharp.

There are three main ingredients in the proof of Theorem 2. The first one is a Remez-type inequality, which allows to get a local estimate depending on the density. These have been studied in a large set of situations. We refer to [2] for a general source on polynomials and Remez-type inequalities, and to [3] for a survey. In our situation, we need a Remez-type inequality for planar domains, which can be found in [1]. The second ingredient is to decompose the integration domain into good and bad parts. Good meaning here that the Remez-type inequality applies. In Kovrijkine's work, as well as in many of the succeeding work using his method, the separation in good and bad parts was achieved *via* a Bernstein-type inequality. Such an inequality holds for instance in Paley-Wiener spaces and more generally in so-called model spaces. However, it is no longer true in Fock or Bergman spaces. So a different approach has to be found to get the good intervals (rather disks in Fock or Bergman spaces). The way of circumventing Bernstein's inequality given in this paper is one of the new features and applies to many other situations, including the Paley-Wiener space itself (see the last section). The third ingredient will be a translation allowing to translate the good parts to a reference situation (the "origin") where we apply the Remez-type inequality. Such translations are known to be well-behaved for instance in Fock spaces and Paley-Wiener spaces so that our construction easily carries over to those situations. Since in those spaces the result is already known ([9] for the Paley-Wiener space and [8] for the Fock space), in this paper we focus on the case of the Bergman spaces.

As a matter of fact, the method proposed here can also be seen as a new way of proving sampling results in the Bergman space (also e.g. in Fock or Paley-Wiener spaces). It essentially exploits locally the Remez-type inequalities, the proofs of which often involve heavy machinery. But once these inequalities are established they prove being powerful tools in sampling problems.

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2. REMEZ-TYPE INEQUALITIES

In this section we start recalling some results of the paper [1]. Let G be a (bounded) domain in $\mathbb{C}$. Let $0 < s < |\overline{G}|$ (Lebesgue measure of $\overline{G}$). Denoting Pol_n the space of complex polynomials of degree at most $n \in \mathbb{N}$, we introduce the set

$$P_n(\overline{G}, s) = \{p \in \text{Pol}_n : |\{z \in \overline{G} : |p(z)| \leq 1\}| \geq s\}.$$

Next, let

$$R_n(z, s) = \sup_{p \in P_n(\overline{G}, s)} |p(z)|.$$

This expression bounds the biggest possible value at z of a polynomial p of degree at most n and being at most 1 on a set of measure at least s . In particular Theorem 1 from [1] says that for $z \in \partial G$, we have

$$R_n(z, s) \leq \left(\frac{c}{s}\right)^n. \quad (2.1)$$

This result corresponds to a generalization to the two-dimensional case of the Remez inequality which is usually given in dimension 1. In what follows we will essentially consider G to be a disk or a rectangle. By the maximum modulus principle, the above constant gives an upper estimate on G for an arbitrary polynomial of degree at most n which is bounded by one on a set of measure at least s . Obviously, if this set is small (s close to 0), i.e. p is controlled by 1 on a small set, then the estimate has to get worse.

Remark 1. *Let us make another observation. If c is the constant in (2.1) associated with the unit disk $G = \mathbb{D} = D(0, 1)$, then a simple argument based on homothety shows that the corresponding constant for an arbitrary disk $D(0, r)$ is cr^2 (considering $D(0, r)$ as underlying domain, the constant c appearing in [1, Theorem 1] satisfies $c > 2 \times m_2(D(0, r))$). So, in the sequel we will use the estimate*

$$R_n(z, s) \leq \left(\frac{cr^2}{s}\right)^n, \quad (2.2)$$

where c does not depend on r .

Let us recall Lemma 1 from [9].

Lemma 1 (Kovrijkine). *Let ϕ be analytic in $D(0, 5)$ and let I be an interval of length 1 such that $0 \in I$ and let $E \subset I$ be a measurable set of positive measure. If $|\phi(0)| \geq 1$ and $M = \max_{|z| \leq 4} |\phi(z)|$ then*

$$\sup_{z \in I} |\phi(z)| \leq \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}} \sup_{x \in E} |\phi(x)|.$$

We will discuss the following counterpart for the planar case:

Lemma 2. *Let $0 < r < \rho$ be fixed. There exists a constant $\eta > 0$ such that the following holds. Let ϕ be analytic in $D(0, \rho)$, let $E \subset D(0, r)$ be a planar measurable set of positive measure, and let $z_0 \in D(0, r)$. If $|\phi(z_0)| \geq 1$ and $M = \sup_{|z| < \rho} |\phi(z)| < \infty$ then*

$$\sup_{z \in D(0, r)} |\phi(z)| \leq \left(\frac{cr^2}{|E|}\right)^{\eta \ln M} \sup_{z \in E} |\phi(z)|,$$

where c does not depend on r , and

$$\eta \leq c' \frac{\rho^4}{(\rho - r)^4} \ln \frac{\rho}{\rho - r}$$

for an absolute constant c' .

For the case of the Bergman space which is the main object in this paper we can consider $(1 - r) = \kappa(1 - \rho)$ for some $\kappa > 1$, and, of course, $\rho < 1$, so that the above estimate gives

$$\eta \leq c'' \frac{1}{(1 - r)^4} \ln \frac{1}{1 - r}, \quad (2.3)$$

where c'' is another absolute constant (one can choose $c'' = c'(\kappa - 1)/\kappa < c'$ which gives a better constant since we require a control of the sup-norm of ϕ on the smaller disk $D(0, \rho)$ and not only on $\mathbb{D} = D(0, 1)$).

Kovrijkine's proof, based on Jensen's inequality and the Remez inequality in the one-dimensional case, carries almost verbatim over to the two-dimensional case. Still, since our geometrical setting is not exactly the same and we need to take into account the location of z_0 , and moreover we wish to keep track of the underlying constants, we reproduce here the adapted version of the proof. Also, in our proof we make use of the pseudohyperbolic metric which changes a bit the setting and which could be of interest in other situations. For $a \in \mathbb{D}$, define the usual disk automorphism by

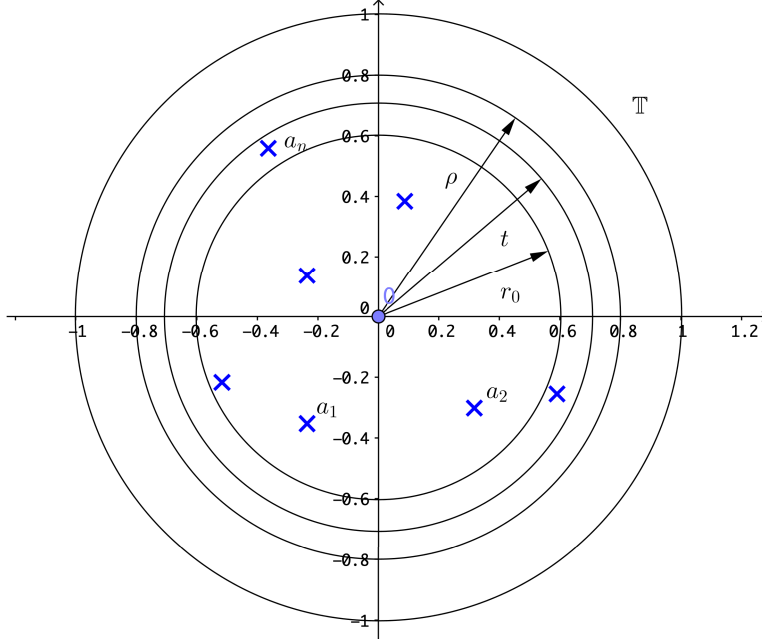
$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}. \quad (2.4)$$

It is well known that $\varphi_a(\varphi_a(z)) = z$.

Proof. We first observe that by a rescaling argument we can assume $\rho = 1$. Let

$$\psi(z) = \phi \circ \varphi_{z_0}(z) = \phi\left(\frac{z_0 - z}{1 - \bar{z}_0 z}\right),$$

which translates z_0 to 0. Also, letting $r_0 = 2r/(1 + r^2)$ (which corresponds to a pseudohyperbolic doubling of r), we get $E \subset D_{phb}(z_0, r_0)$. Define $\tilde{E} = \varphi_{z_0}(E)$ so that $\tilde{E} \subset D(0, r_0)$, and $\psi|_{\tilde{E}} = \phi \circ \varphi_{z_0}|_{\tilde{E}} = \phi|_E$. From now on, we will consider ψ on $D(0, r_0)$ which thus contains $\tilde{E}$, and $|\psi(0)| = |\phi(z_0)| \geq 1$. Clearly, ϕ and ψ are bounded by the same constant M .



We let $t = 2r_0/(1 + r_0^2)$ be another pseudohyperbolic doubling, now of r_0 . Consider $a_1, \dots, a_n$, the zeros of ψ in $D(0, t)$, and B the Blaschke product on the disk $D(0, t)$ associated with these zeros. Then we can divide by $B = P/Q$ to obtain a function $g = \psi/B$ zero-free in $D(0, t)$. Here $P = \prod_{k=1}^n t(a_k - z)$ and $Q = \prod_{k=1}^n (t^2 - \bar{a}_k z)$ are polynomials of degree n . Since $|B| = 1$ on $\partial D(0, t)$ and $|B| \leq 1$ in $D(0, t)$, we have $|g(0)| \geq |\psi(0)| = |\phi(z_0)| \geq 1$ and $\max_{|z| \leq t} |g(z)| \leq M$. Applying Harnack's inequality to the positive harmonic function $u = \ln M - \ln |g|$ in $D(0, t)$ we get for every z in the compact set $\overline{D(0, r_0)} \subset D(0, t)$ that

$$\frac{t - r_0}{t + r_0} u(0) \leq u(z) \leq \frac{t + r_0}{t - r_0} u(0) \leq \frac{t + r_0}{t - r_0} \ln M.$$

Set $\alpha = (t + r_0)/(t - r_0)$ to simplify notation. We deduce that for every $z \in \overline{D(0, r_0)}$, $|g(z)| \geq M^{1-\alpha}$. Hence

$$\frac{\max_{|z| \leq r_0} |g(z)|}{\min_{|z| \leq r_0} |g(z)|} \leq M/M^{1-\alpha} = M^\alpha.$$

Let us estimate Q :

$$\frac{\max_{|z| \leq r_0} |Q(z)|}{\min_{|z| \leq r_0} |Q(z)|} \leq \frac{\max_{|z| \leq r_0} \prod_{k=1}^n |t^2 - \bar{a}_k z|}{\min_{|z| \leq r_0} \prod_{k=1}^n |t^2 - \bar{a}_k z|} \leq \left(\frac{t + r_0}{t - r_0} \right)^n$$

We now use the Andrievskii-Ruscheweyh estimate. Given a polynomial P of degree at most n . Let $m = \sup_{z \in \tilde{E}} |P(z)|$. Set $P_1 = P/m$ and $s = |\tilde{E}|$. Then $P_1 \in P_n(\overline{D(0, r_0)}, s)$. By (2.2) we get

$$|P_1(z)| \leq \left(\frac{cr_0^2}{|\tilde{E}|} \right)^n,$$

and hence

$$\sup_{|z| \leq r_0} |P(z)| \leq \left(\frac{cr_0^2}{|\tilde{E}|} \right)^n \sup_{z \in \tilde{E}} |P(z)|.$$

The exact same estimates as Kovrijkine's lead to

$$\begin{aligned} \sup_{|z| \leq r_0} |\psi(z)| &= \sup_{|z| \leq r_0} |g(z)P(z)/Q(z)| \leq M^\alpha \left(\frac{t+r_0}{t-r_0} \right)^n \times \left(\frac{cr_0^2}{|\tilde{E}|} \right)^n \times \sup_{z \in \tilde{E}} |\psi(z)| \\ &\leq M^{\frac{t+r_0}{t-r_0}} \left(\frac{t+r_0}{t-r_0} \right)^n \times \left(\frac{cr_0^2}{|\tilde{E}|} \right)^n \times \sup_{z \in \tilde{E}} |\psi(z)| \\ &\leq e^{\frac{t+r_0}{t-r_0} \ln M + n \ln \frac{t+r_0}{t-r_0}} \left(\frac{cr_0^2}{|\tilde{E}|} \right)^n \sup_{z \in \tilde{E}} |\psi(z)|. \end{aligned}$$

Since $\tilde{E} = \varphi_{z_0}(E)$ and $|z_0| < r_0$, we have, by a change of variables, $|\tilde{E}| \geq (1-r_0)^2|E|/4$, so that

$$\sup_{|z| \leq r_0} |\psi(z)| \leq e^{\frac{t+r_0}{t-r_0} \ln M + n \ln \frac{t+r_0}{t-r_0} - 2n \ln(1-r_0) + n \ln 4} \left(\frac{cr_0^2}{|E|} \right)^n \sup_{z \in \tilde{E}} |\psi(z)|.$$

Assuming $e \leq cr_0^2/|E|$ (note that this inequality means that the density of E is less than a fixed constant; if the density is bigger than some fixed constant, say $1/2$, then the estimate is not really of interest). This yields

$$\sup_{|z| \leq r_0} |\psi(z)| \leq \left(\frac{cr_0^2}{|E|} \right)^{\frac{t+r_0}{t-r_0} \ln M + n \ln \frac{t+r_0}{t-r_0} - 2n \ln(1-r_0) + n(1+\ln 4)} \sup_{z \in \tilde{E}} |\psi(z)|$$

Applying Jensen's formula in a similar way as did Kovrijkine we obtain $n \leq \frac{\ln M}{\ln(1/t)}$ (recall that ρ was supposed to be 1). Since $\ln 4 < 2$, setting

$$\eta = \frac{t+r_0}{t-r_0} + \left(\ln \frac{t+r_0}{t-r_0} - 2 \ln(1-r_0) + 3 \right) / \ln(1/t),$$

and retranslating to ϕ we get

$$\sup_{|z| \leq r} |\phi(z)| \leq \sup_{|z| \leq r_0} |\psi(z)| \leq \left(\frac{cr_0^2}{|E|} \right)^{\eta \ln M} \sup_{z \in \tilde{E}} |\psi(z)| \leq \left(\frac{cr_0^2}{|E|} \right)^{\eta \ln M} \sup_{z \in E} |\phi(z)|$$

Let us finish with the discussion of the estimate for η . Note first that $t = 2r_0/(1+r_0^2)$, so that

$$\frac{t+r_0}{t-r_0} = \frac{3+r_0^2}{1-r_0^2} \leq \frac{4}{1-r_0^2}.$$

Also $\ln(1/t) \sim (1 - r_0)^2/(2r_0)$, and since $r_0 = 2r/(1 + r^2)$ we have $1 - r_0 \sim (1 - r)^2/2$. Hence

$$\begin{aligned} \eta &\lesssim \frac{4}{1 - r_0^2} + \left(\ln \frac{4}{1 - r_0^2} + \ln \frac{1}{(1 - r_0)^2} + 3 \right) \times \frac{2}{(1 - r_0)^2} \leq \frac{c}{(1 - r_0)^2} \ln \frac{1}{1 - r_0^2} \\ &\sim \frac{c'}{(1 - r)^4} \ln \frac{1}{1 - r} \end{aligned}$$

where c and c' are absolute constants.

Getting back to arbitrary $\rho > 0$, the above becomes

$$\eta \leq \frac{c' \rho^4}{(\rho - r)^4} \ln \frac{\rho}{\rho - r}$$

□

The corresponding case for p -norms is deduced exactly as in Kovrijkine's work.

Corollary 1. *Let $0 < r < \rho$ be fixed. There exists a constant $\eta > 0$ such that following holds. Let ϕ be analytic in $D(0, \rho)$, let $E \subset D(0, r)$ be a planar measurable set of positive measure, and let $z_0 \in D(0, r)$. If $|\phi(z_0)| \geq 1$ and $M = \sup_{|z| < \rho} |\phi(z)|$ then for $p \in [1, +\infty)$ we have*

$$\|\phi\|_{L^p(D(0, r))} \leq \left(\frac{cr^2}{|E|} \right)^{\eta \ln M + \frac{1}{p}} \|\phi\|_{L^p(E)}.$$

The estimates on η are the same as in the lemma. The constant c does not depend on r .

3. PROOF OF THEOREM 2

In the proof of Theorem 2 we will use a change of variable formula. Recall the definition of the disk-automorphism from (2.4). Then we can introduce the change of variable formula

$$T_a : A^{p, \alpha} \longrightarrow A^{p, \alpha}, \quad (T_a f)(z) = (f \circ \varphi_a)(z) \varphi'_a(z)^{(2+\alpha)/p}.$$

Recall that

$$\varphi'_a(z) = -\frac{1 - |a|^2}{(1 - \bar{a}z)^2}.$$

Clearly T_a is an isomorphism of the Bergman space $A^{p, \alpha}$. We will also consider restrictions of T_a from pseudohyperbolic disks to other corresponding pseudohyperbolic disks.

We now enter into the main part of the proof which gives a simple way of avoiding the Bernstein inequality. We need to introduce some preliminary notation. For $n \in \mathbb{N}$ and $k = 0, 1, \dots, 2^n - 1$, let $z_{n,k} = (1 - 2^{-n})e^{i2\pi k/2^n}$ and define

$$D_{n,k}^r = D_{phb}(z_{n,k}, r),$$

where $r \in (0, 1)$. It can be checked that for a sufficiently big choice of r these disks cover $\mathbb{D}$. Let r_0 be such a radius. We will henceforth set $D_{n,k} = D_{n,k}^{r_0}$.

We need a finite covering property. Denote by χ_F the characteristic function of a measurable set F in $\mathbb{D}$.

Lemma 3. *For every $r \in (0, 1)$, there exists a constant N such that*

$$\sum_{n,k} \chi_{D_{n,k}^r} \leq N.$$

Moreover there is some universal overlapping constant c_{ov} such that

$$N \leq c_{ov} \frac{1}{(1-r)^2} \ln \frac{1}{1-r}.$$

In this lemma we did not require $D_{n,k}$ to cover the whole disk $\mathbb{D}$. If we want this also to hold we additionally have to impose $r \in (r_0, 1)$.

Obviously, the constant N is at least equal to 1. In the estimates below N will enter only logarithmically so that the power in $(1-r)$ as well as the logarithmic term are not very important.

Proof of Lemma. This is certainly a well know fact, but we include a proof for completeness and to give an idea of the dependence of N on r .

The result is equivalent to say that a disk $D_{phb}(z, r)$, $z \in \mathbb{D}$, contains at most N points $z_{n,k}$. Since the problem is rotation invariant, we can assume $z = x \in [0, 1)$.

Recall from [5, p.3] that $D_{phb}(x, r)$ is a euclidean disk with diameter

$$[\alpha, \beta] = \left[\frac{x-r}{1-rx}, \frac{x+r}{1+rx} \right]$$

Thus $D_{phb}(x, r)$ is included in an annulus with radii α and β . We estimate the distance of α and β to the boundary $\mathbb{T}$:

$$\begin{aligned} 1 - \frac{x-r}{1-rx} &= \frac{1-rx-x+r}{1-rx} = \frac{(1+r)(1-x)}{1-rx} \leq d_1(1-x), \\ 1 - \frac{x+r}{1+rx} &= \frac{1+rx-x-r}{1+rx} = \frac{(1-r)(1-x)}{1+rx} \geq d_2(1-x), \end{aligned}$$

where d_1 and d_2 are strictly positive constants (e.g. one can choose $d_1 = 2/(1-r)$ and $d_2 = 1/d_1 = (1-r)/2$). In particular for $z_{n,k} \in D_{phb}(x, r)$ it is necessary that $d_2(1-x) \leq 2^{-n} \leq d_1(1-x)$ which may happen at most $(\ln(d_2(1-x))^{-1} - \ln(d_1(1-x))^{-1})/\ln 2$ times. With the above choice of d_1 and d_2 we can pick $d_1/d_2 \leq 4/(1-r)^2$ so that the number of possible n is bounded by

$$\ln \frac{d_1}{d_2} / \ln 2 \leq \ln \frac{4}{(1-r)^2} / \ln 2. \quad (3.1)$$

We also need to check that the number of k such that $z_{n,k} \in D_{phb}(x, r)$ is uniformly bounded. Note that r is fixed. We will distinguish two cases, $x \leq 2r$ and $2r < x < 1$ (in case $r \geq 1/2$ the latter case does not occur).

Case $x \leq 2r$. In this case we know that $D_{phb}(x, r) \subset D(0, 3r/(1+2r^2))$. In order that $z_{n,k} \in D(0, 3r/(1+2r^2))$ it is necessary that $1-2^{-n} \in D(0, 3r/(1+2r^2))$ and thus that $1-2^{-n} \leq 3r/(1+2r^2)$ which happens when $n \lesssim \ln(1-r)^{-1}$ (the constant appearing here does not depend on r). In order to get all $z_{n,k} \in D(0, 3r/(1+2r^2))$, it remains to sum

$\sum_{n=0}^{\ln(1-r)^{-1}} 2^n \simeq (1-r)^{-1}$. So for $x \leq 2r$ we have $N \lesssim (1-r)^{-1}$ which is neglectible with respect to the estimate required in the statement of the lemma.

Case $2r < x < 1$. Then

$$\frac{x-r}{1-rx} \geq \frac{2r-r}{1} = r. \quad (3.2)$$

Observe that in order that $z_{n,k} \in D_{phb}(x, r)$ we need $1 - |z_{n,k}| = 2^{-n} \in [d_2(1-x), d_1(1-x)]$ (in (3.1) we had already obtained the number of n such that this is satisfied). As before, for these n , we have to estimate the number of k such that $z_{n,k} \in D_{phb}(x, r)$. We will again use the fact from [5, p.3] that $D_{phb}(x, r)$ is a euclidean disque $D(c, R)$ for which we will now more conveniently use the explicit expression of the center and the radius:

$$c = x \frac{1-r^2}{1-r^2x^2}, \quad R = r \frac{1-x^2}{1-r^2x^2}.$$

Hence the argument of a point $w \in D_{phb}(x, r)$ can be estimated in the following way

$$\begin{aligned} |\arg w| &\leq \arcsin \frac{R}{c} = \arcsin \left(\frac{r}{x} \frac{1-x^2}{1-r^2} \right) \leq \arcsin \frac{1-x^2}{2(1-r^2)} \simeq \frac{1-x^2}{2(1-r^2)} \\ &\leq \frac{1-x}{1-r^2} \leq \frac{1}{1-r^2} \frac{1}{d_2} 2^{-n} \\ &\leq \frac{2}{(1-r)^2} 2^{-n}. \end{aligned}$$

Hence, in order that $z_{n,k} = (1-2^{-n})e^{2\pi i k/2^n} \in D_{phb}(x, r)$ we need

$$2\pi k \leq \frac{2}{(1-r)^2}. \quad (3.3)$$

We conclude that the number of k is also uniformly controlled by r .

The estimates (3.1) and (3.3) yield the required bound on N . $\square$

Fix $r_0 < s < t < 1$. For $K > 1$ the set

$$I_f^{K-good} = \{(n, k) : \|f\|_{L^{p,\alpha}(D_{n,k}^t)} \leq K \|f\|_{L^{p,\alpha}(D_{n,k}^s)}\}$$

will be called the set of K -good disks for (t, s) (in order to keep notation light we will not include s and t as indices). This set depends on f .

The key-result of this paper is the following proposition which allows to obtain the set of good disks in a very simple way. This might have some independent interest.

Proposition 1. *Let $r_0 \leq s < t < 1$. For every constant $c \in (0, 1)$, there exists K such that for every $f \in A^{p,\alpha}$ we have*

$$\sum_{(n,k) \in I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p \geq c \|f\|_{A^{p,\alpha}}^p.$$

It will be clear from the proof that one can pick $K^p = N/(1 - c)$ where N corresponds to the overlapping constant from Lemma 3 for the (pseudohyperbolic) radius t .

Proof of proposition. Since $r_0 \leq s < t < 1$, we have $\bigcup_{n,k} D_{n,k}^s = \mathbb{D}$, and by Lemma 3 we have a finite overlap property:

$$\sum_{n,k} \chi_{D_{n,k}^t} \leq N.$$

Hence

$$\|f\|_{A^{p,\alpha}}^p \leq \sum_{(n,k)} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p \leq \sum_{(n,k)} \|f\|_{L^{p,\alpha}(D_{n,k}^t)}^p \leq N \|f\|_{A^{p,\alpha}}^p.$$

Now pick a $c \in (0, 1)$. Then

$$\begin{aligned} N \|f\|_{A^{p,\alpha}}^p &\geq \sum_{(n,k) \notin I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^t)}^p > K^p \sum_{(n,k) \notin I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p \\ &= K^p \sum_{(n,k)} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p - K^p \sum_{(n,k) \in I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p \\ &\geq K^p \|f\|_{A^{p,\alpha}}^p - K^p \sum_{(n,k) \in I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p. \end{aligned}$$

For $K^p > N$ we thus get

$$\sum_{(n,k) \in I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^s)}^p \geq \frac{K^p - N}{K^p} \|f\|_{A^{p,\alpha}}^p$$

Hence, setting $c = (K^p - N)/K^p \in (0, 1)$ we see that $K^p = N/(1 - c)$ is suitable. $\square$

We are now in a position to prove the theorem.

Proof of Theorem 2. Let $r_1 = \max(r, r_0)$, where r_0 ensures the covering of $\mathbb{D}$ by $D_{n,k}$. It is not hard to see that there exists a universal constant $\eta > 0$ such that if E is (γ, r) -dense then it is $(\eta\gamma, r_1)$ -dense (if $r \geq r_0$ then $\eta = 1$, if $r < r_0$ we can cover a least portion of an arbitrary disk $D_{phb}(z, r_0)$ by disjoint disks with radius r). Since the multiplicative constant η does not change the estimate claimed in the theorem, and in order to not overload notation, in all what follows we will use γ instead of $\eta\gamma$.

Now fix $c \in (0, 1)$. We will choose r_1 for s in Proposition 1 and t with $s = r_1 < t < 1$ to be fixed later. Let also K be a corresponding choice from Proposition 1 (which thus depends on t since $N \leq c \ln(1/(1 - t))/(1 - t)^2$).

Pick $f \in A^{p,\alpha}$, $f \neq 0$. Let I_f^{K-good} be the set of K -good disks for (r_1, t) . Then by Proposition 1

$$\|f\|_{A^{p,\alpha}}^p \leq \frac{1}{c} \sum_{(n,k) \in I_f^{K-good}} \|f\|_{L^{p,\alpha}(D_{n,k}^{r_1})}^p.$$

In all what follows we suppose $(n, k) \in I_f^{K-good}$. Recall that $z_{n,k}$ is the center of $D_{n,k}^{r_1}$. Clearly $D(0, r_1) = \varphi_{z_{n,k}}(D_{n,k}^{r_1})$ which allows to translate the situation on $D_{n,k}^{r_1}$ to a euclidean

disk at the origin (since pseudohyperbolic and euclidean disks are the same when centered at 0, we will not use the index “phb” in this situation). Then the change of variable operator $T_{z_{n,k}}$ restricted to $D_{n,k}^{r_1}$ gives

$$\|f\|_{L^{p,\alpha}(D_{n,k}^{r_1})}^p = \int_{D_{n,k}^{r_1}} |f(z)|^p dA_\alpha(z) = \int_{D(0,r_1)} |(f \circ \varphi_{z_{n,k}})(u)(\varphi'_{z_{n,k}}(u))^{(2+\alpha)/p}|^p dA_\alpha(u).$$

Denote $g = T_{z_{n,k}}f$ and $D_0 = D(0, r_1)$. Clearly $\int_{D_0} |g|^p dA > 0$ so that we can set

$$h = g \times \left(\frac{\pi r_1^2}{\int_{D_0} |g|^p dA} \right)^{1/p}.$$

In particular $\int_{D_0} |h|^p dA = \text{Area}(D_0)$, which implies that there is $z_0 \in D_0$ with $|h(z_0)| \geq 1$. In order to use Corollary 1 we have to estimate h on a bigger disk. Let $\rho = \frac{2r_1}{1+r_1^2}$ be a pseudohyperbolic doubling of the radius r_1 , and $t = \frac{2\rho}{1+\rho^2}$ the pseudohyperbolic doubling of ρ . Consider now h as a function in the restricted Bergman space $A^{p,\alpha}|D(0,t)$, we have for $z \in D(0,\rho)$,

$$|h(z)|^p \leq \frac{C}{(1-\rho)^{(2+\alpha)}} \int_{D(0,t)} |h(w)|^p (1-|w|^2)^\alpha dA(w),$$

where C is an absolute constant (the proof is done as in the Bergman space on $\mathbb{D}$ and essentially based on subharmonicity of $|h|^p$ and $t - \rho \sim 1 - \rho$). Since h is a constant multiple of g we can replace in the above h by g . Since ρ is a pseudohyperbolic doubling of r_1 we have $1 - \rho \sim (1 - r_1)^2$.

Observe that

$$\int_{D(0,t)} |g(w)|^p dA_\alpha(w) = \int_{D_{n,k}^t} |f(z)|^p dA_\alpha(z),$$

and

$$\int_{D_0} |g|^p dA(z) \geq \int_{D_0} |g|^p dA_\alpha(z) = \int_{D_{n,k}^{r_1}} |f|^p dA_\alpha(z).$$

Now, putting all the pieces together,

$$\begin{aligned} \sup_{z \in D(0,\rho)} |h(z)| &= \left(\frac{\pi r_1^2}{\int_{D_0} |g|^p dA(z)} \right)^{1/p} \sup_{z \in D(0,\rho)} |g(z)| \\ &\leq \left(\frac{\pi}{\int_{D_{n,k}^{r_1}} |f|^p dA_\alpha(z)} \right)^{1/p} \times \frac{C^{1/p}}{(1-\rho)^{(2+\alpha)/p}} \left(\int_{D_{n,k}^t} |f(z)|^p dA_\alpha(z) \right)^{1/p}. \end{aligned}$$

Since $(n, k) \in I_f^{K-good}$ for (r_1, t) we get

$$M = \sup_{z \in D(0,\rho)} |h(z)| \leq \frac{DK}{(1-r_1^2)^{2 \times (2+\alpha)/p}}, \quad (3.4)$$

where D is some universal constant, and, since $(1-t) \simeq (1-\rho)^2 \simeq (1-r_1)^4$,

$$K^p = \frac{N}{1-c} \leq \frac{c_{ov}}{1-c} \frac{1}{(1-t)^2} \ln \frac{1}{1-t} \simeq \frac{1}{(1-r_1)^8} \ln \frac{1}{(1-r_1)^4}.$$

In what follows it will be enough to use the estimate

$$K \lesssim \frac{1}{(1-r_1)^{9/p}}. \quad (3.5)$$

We can now apply Corollary 1 to h and $\tilde{E} = \varphi_{z_{n,k}}(E \cap D_{n,k})$. Observe that the density of $\tilde{E}$ in D_0 is the same as that of E in $D_{n,k}$ which is thus bounded below by γ . Since we are in a disk of fixed radius we can plug in the weight $(1-|z|^2)^\alpha \leq 1$, to the prize of an additional constant in the last line:

$$\begin{aligned} \int_{D(0,r_1)} |h(z)|^p dA_\alpha(z) &\leq \int_{D(0,r_1)} |h(z)|^p dA(z) \\ &\leq \left(\frac{cr_1^2}{|\tilde{E}|} \right)^{p\eta \ln M + 1} \int_{\tilde{E}} |h(z)|^p dA(z). \\ &\leq \left(\frac{c}{\gamma} \right)^{p\eta \ln M + 1} \frac{1}{(1-r_1^2)^\alpha} \int_{\tilde{E}} |h(z)|^p dA_\alpha(z). \end{aligned}$$

By homogeneity we can replace in the above the function h by g . Thus, changing back variables, we get for every K -good disk,

$$\int_{D_{n,k}^{r_1}} |f|^p dA_\alpha \leq \left(\frac{c}{\gamma} \right)^{p\eta \ln M + 1} \frac{1}{(1-r_1^2)^\alpha} \int_{E \cap D_{n,k}^{r_1}} |f|^p dA_\alpha,$$

Since integration over K -good disks allows to recover the norm we get the desired result. $\square$

A word on the exponent appearing in the estimate. Recall from (2.3) that

$$\eta \leq c'' \frac{1}{(1-r_1)^4} \ln \frac{1}{1-r_1},$$

Also from (3.4) and (3.5),

$$\ln M \lesssim \frac{13+2\alpha}{p} \ln \frac{1}{1-r_1}$$

(the constants involved here do not depend on r_1) so that with $13+2\alpha \lesssim 1+\alpha$, we get

$$\eta \ln M \lesssim \frac{1+\alpha}{p} \times \frac{1}{(1-r_1)^4} \ln \frac{1}{1-r_1^2}.$$

The additional term $+1$ appearing in the exponent at the end of the proof above can be absorbed in the expression $\eta \ln M$ since we need to impose $r \geq r_0 > 0$, where r_0 ensures the overlapping property.

4. COMMENTS ON OTHER SPACES

4.1. Fock spaces. In Fock spaces (see [20]), the situation follows as a special case of the recent work by [8]. Still, since our method is quite elementary and universal, we feel that presenting the application of the above techniques is of interest to our readers. Let us give the necessary indications on how to obtain the estimates in the Fock space defined by

$$\mathcal{F}^{p,\alpha} = \{f \in \text{Hol}(\mathbb{C}) : \|f\|_{\mathcal{F}^{p,\alpha}}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2/2} dA(z) < +\infty\}.$$

Let us also use $dA_{\alpha,p}(z) = e^{-p\alpha|z|^2/2} dA(z)$. In the Fock space the computations are easier since we do not need to work with pseudohyperbolic distance but with euclidean distance. In this situation pick $z_{n,k} = n + ik$, $n, k \in \mathbb{Z}$. Replace the pseudohyperbolic disks by euclidean ones $D_{n,k}^r = D(z_{n,k}, r)$ where now $r > \sqrt{2}/2$ will ensure the covering property. For every fixed $r > \sqrt{2}/2$ it is clear that the finite overlap property holds (we are in euclidean geometry). One easily sees that $N \simeq r^2$. Also there is an isometric translation (change of variable) operator $T_a f(z) = e^{\alpha \bar{a} z - \alpha |a|^2/2} f(z - a)$ which will allow to translate the situation from an arbitrary disk $D_{n,k}^r$ to $D(0, r)$.

As in the Bergman space, we introduce good disks. Fix $\sqrt{2}/2 < s < t$. For $K > 1$ and a function $f \in \mathcal{F}^{p,\alpha}$ the set

$$I_f^{K\text{-good}} = \{(n, k) : \|f\|_{L^{p,\alpha}(D_{n,k}^t)} \leq K \|f\|_{L^{p,\alpha}(D_{n,k}^s)}\}$$

will be called the set of K -good disks for (t, s) , where we integrate with respect to the measure $dA_{\alpha,p}$. In the argument below we pick $s = r$ and $t = 4r$.

Now, given f with $\|f\|_{\mathcal{F}^{p,\alpha}} = 1$, let, as in the Bergman space, $g = T_{-z_{n,k}} f$ for $(n, k) \in I_f^{K\text{-good}}$, and set

$$h = c_0 g, \quad c_0 = \left(\frac{\pi r^2}{\int_{D(0,r)} |g|^p dA(z)} \right)^{1/p}.$$

Again there is $z_0 \in D(0, r)$ with $|h(z_0)| \geq 1$.

Set $\rho = 2r$. We have to estimate the maximum modulus of h on $D(0, \rho)$ in terms of a local integral of h . To that purpose, we can assume $h \in A^p(D(0, 4r))$ which justifies the first of the estimates below. Hence

$$\begin{aligned} \max_{z \in D(0, \rho)} |h(z)|^p &\leq \frac{C}{r^2} \int_{D(0, 4r)} |h|^p dA(z) \\ &= \frac{C}{r^2} \times \frac{\pi r^2}{\int_{D(0, r)} |g|^p dA(z)} \int_{D(0, 4r)} |g|^p dA(z) \\ &\leq \frac{C e^{8\alpha p r^2}}{r^2} \frac{\pi r^2}{\int_{D(0, r)} |g|^p dA_{\alpha,p}(z)} \int_{D(0, 4r)} |g|^p dA_{\alpha,p}(z) \end{aligned}$$

(in the last estimate, in order to switch from dA to $dA_{\alpha,p}$ we had to introduce an additional factor $e^{\alpha p (4r)^2/2}$). Since (n, k) is K -good for $(r, 4r)$, we see that the last expression in the above inequalities is bounded by a constant (depending on r) times K^p :

$M^p = \max_{z \in D(0, \rho)} |h(z)|^p \leq c_r K^p$, where $c_r = C\pi e^{8\alpha r^2}$. Exactly as in Proposition 1 we see that $K^p \simeq N \lesssim r^2$, so that

$$M \lesssim r^{2/p} e^{8\alpha r^2}. \quad (4.1)$$

Hence, setting $\tilde{E} = (E \cap D_{n,k}^r) - z_{n,k}$ we get using Corollary 1 applied to h :

$$\int_{D(0,r)} |h(z)|^p dA(z) \leq \left(\frac{cr^2}{|\tilde{E}|} \right)^{p\eta \ln M + 1} \int_{\tilde{E}} |h(z)|^p dA(z)$$

The factor r^2 appearing inside the brackets is a rescaling factor (see Remark 1). Again, by homogeneity we can replace in the above inequality h by g . Note also that $\pi r^2/|\tilde{E}|$ is controlled by $1/\gamma$. This yields

$$\begin{aligned} \int_{D_{n,k}^r} |f|^p dA_{\alpha,p}(z) &= \int_{D(0,r)} |g(z)|^p dA_{\alpha,p}(z) \leq \int_{D(0,r)} |g(z)|^p dA \\ &\leq \left(\frac{cr^2}{|\tilde{E}|} \right)^{p\eta \ln M + 1} \int_{\tilde{E}} |g(z)|^p dA(z) \\ &\leq e^{\alpha r^2/2} \left(\frac{c_1}{\gamma} \right)^{p\eta \ln M + 1} \int_{\tilde{E}} |g(z)|^p e^{-p\alpha|z|^2/2} dA(z) \\ &= e^{\alpha r^2/2} \left(\frac{c_1}{\gamma} \right)^{p\eta \ln M + 1} \int_{E \cap D_{n,k}^r} |f(z)|^p dA_{\alpha,p}, \end{aligned}$$

where c_1 is an absolute constant.

Summing over all K -good (n, k) we obtain the required result

$$\|f\|_{\mathcal{F}^{p,\alpha}} \lesssim e^{\alpha r^2/2} \left(\frac{c_1}{\gamma} \right)^{\eta \ln M + 1/p} \|f\|_{L^p(E, dA_{\alpha,p})}$$

where

$$\ln M \lesssim 8\alpha r^2 + \frac{2}{p} \ln r,$$

and, in view of (2.3),

$$\eta = c'' \times 2^4 \ln 2.$$

Remark 2. In the above reasoning we had fixed the centers to be $z_{n,k} = n + ik$ which makes them independent of r . Still, one could replace these by $\tilde{z}_{n,k} = rn + irk$ (and consider the disks with these $\tilde{z}_{n,k}$ and radius $(\sqrt{2}/2 + \epsilon)r$), which will give a better control on the overlapping constant N (actually $N \leq 4$ in this case). But this enters logarithmically into $\ln M$ and is thus neglectable with respect to $8\alpha r^2$.

4.2. Paley-Wiener spaces. The above arguments can also be arranged for the Paley-Wiener spaces, in particular to find the good intervals. We do not claim that our proof is better than Kovrijkine's nor that we get better constants (also, Kovrijkine's key lemma is still used). We would just like to point out that also in this case Bernstein's inequality is not required to get estimates on sampling constants. Let us recall the definition of Paley-Wiener spaces:

$$PW_b^p = \{f \in \text{Hol}(\mathbb{C}) : f \text{ is of exponential type at most } b, \|f\|_{PW_b^p}^p = \frac{1}{\pi} \int_{\mathbb{R}} |f(x)|^p dx < \infty\}$$

where $b > 0$ is the bandwidth (often chosen to be π).

We will now use Proposition 1 to find the good intervals. Let us give some indications for this case. We first observe that an equivalent norm in the Paley-Wiener space can be given by integrating over a strip $S^h = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, |y| < h/2\}$ for some fixed width $h > 0$:

$$\frac{1}{c_h^p} \int_{S^h} |f(z)|^p dA(z) \leq \|f\|_{PW_b^p}^p \leq c_h^p \int_{S^h} |f(z)|^p dA(z),$$

where the constant c_h depends on b , p and h . This can be seen from the Plancherel-Polya estimate (see [19, p.95]) which also shows that the constant c_h depends exponentially on h : we essentially have $c_h \simeq e^{bh/(2p)}$ (at least when $h > h_0 > 0$). As the reader will observe in the discussion below, the size of the “holes” — which determines r in the (γ, r) -density — enters into the width h of the strip S^h which might thus introduce an exponential dependence on the size of the holes (it is possible that a more precise analysis of Kovrijkine's lemma could remove this obstacle). We will restrict ourselves in this subsection to the dependence of the sampling constant on γ without discussing the size of these “holes”, and we will thus assume that the r in the (γ, r) -density is 1 and essentially look at the dependence of the sampling constants on γ .

Suppose now E satisfies

$$\frac{|I \cap E|}{|I|} \geq \gamma$$

for every interval I with fixed length $|I| = 1$.

For general $r > 0$, let us introduce $R_{x_n, r} = [x_n - r/2, x_n + r/2] \times [-r/2, r/2]$ where we will from now on fix $x_n = n$, $n \in \mathbb{Z}$, and observe that $R_{x_n, 1}$ form a partition of S^1 (up to a set of zero planar Lebesgue measure). We start defining $R_f^{K\text{-good}}$ to be K -good rectangles for $0 < r = 1 < t$, $K > 1$, satisfying

$$\int_{R_{x_n, t}} |f|^p dA \leq K^p \int_{R_{x_n, 1}} |f|^p dA,$$

where the parameter t will be determined later. In this case, $R_{x_n, t}$ cover S^t with finite overlap $\sum_{n \in \mathbb{Z}} \chi_{R_{x_n, t}} \leq t/1 = t$. Arguing as in Proposition 1:

$$\|f\|_{L^p(\mathbb{R})}^p \geq \frac{1}{c_t^p} \|f\|_{L^p(S^t)}^p \geq \frac{1}{t c_t^p} \sum_{k \notin R_f^{K\text{-good}}} \|f\|_{L^p(R_{x_k, t})}^p \geq \frac{K^p}{t c_t^p} \sum_{k \notin R_f^{K\text{-good}}} \|f\|_{L^p(R_{x_k, 1})}^p.$$

Writing the last sum as the sum over all k minus the sum over the K -good k 's, and using the fact that the sum over all k of $\|f\|_{L^p(R_{x_n,1})}^p$ yields $\|f\|_{L^p(S^1)}^p$, which is bounded from below by $(1/c_1^p)\|f\|_{L^p(\mathbb{R})}^p$, we can reach as in Proposition 1

$$\sum_{k \in R_f^{K\text{-good}}} \|f\|_{L^p(R_{k,1})}^p \geq \frac{K^p/(tc_1^p c_t^p) - 1}{K^p/tc_t^p} \|f\|_{L^p(\mathbb{R})}^p =: \kappa \|f\|_{L^p(\mathbb{R})}^p. \quad (4.2)$$

(Observe that $\kappa < 1$.) So, when K is sufficiently big, the norm of $f \in PW_b^p$ can be recovered from the local norms on K -good rectangles. On these we will be able to control the relation between the maximum of the function whilst having a point z_0 in a smaller disk where the modulus of the function is bounded from below by 1.

We will need Kovrijkine's lemma in its following adaption.

Lemma 4 (Kovrijkine-bis). *Fix $r > 0$. Let ϕ be analytic in $D(0, 10r)$ and let $I = [-\frac{r}{2}, \frac{r}{2}]$. Let $z_0 \in D(0, r)$, and $E \subset I$ a measurable set of positive measure. If $|\phi(z_0)| \geq 1$ and $M = \max_{|z| \leq 9r} |\phi(z)|$ then*

$$\sup_{z \in I^2} |\phi(z)| \leq \left(\frac{C}{|E|} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in E} |\phi(x)|.$$

By rescaling we can assume that $r = 1$. Observe that we consider the supremum of ϕ on $I^2 = R_{x_0,1}$ on the left hand side. The idea is that this supremum does not vary too much when replacing I by $I + iy$, $|y| \leq 1/2$. Let us make some comments. The proof is essentially as in Lemma 2. The estimates for g and Q given in that proof when writing $\phi = gP/Q$ are the same (we got uniform estimates on disks, e.g. with radius 1; observe that $R_{x_0,1} \subset D(0, 1)$). The main difference is the control on the polynomial P . In Lemma 2 we used an estimate by Andrievskii-Ruscheweyh to take into account planar sets. Here we will proceed as Kovrijkine using the original Remez inequality which allows to get the estimate on I , and it remains to estimate uniformly P on $I^2 = R_{x_0,1}$ in terms of $\|P\|_{L^\infty(I)}$. For that we can use for instance [2, Exercice 17, p.239], which implies that $|p(z)| \leq 4^n \|p\|_{L^\infty(I)}$ when z is in the disk $D(0, \sqrt{2}/2)$ which obviously contains $R_{x_0,1}$. The expression 4^n produces an additional factor 4 in the constant C appearing in the estimate of Lemma 4 (see also [9, p.3040]).

Again, there is also an L^p -version of this result (note that integration in the direction of iy , $|y| \leq 1/2$, does not affect the right hand side):

Corollary 2. *Under the conditions of the lemma, and let $1 \leq p < +\infty$, we have*

$$\|\phi\|_{L^p(R_{0,1})} \leq \left(\frac{C}{|E|} \right)^{\frac{\ln M}{\ln 2} + \frac{1}{p}} \|\phi\|_{L^p(E)}.$$

Now recall that the set E was supposed $(\gamma, 1)$ -dense. We fix $t = 20$. As explained above we find a set of K -good rectangles satisfying now $\int_{R_{x_n,20}} |f|^p dA \leq K^p \int_{R_{x_n,1}} |f|^p dA$ (in view of (4.2) the constant K can be fixed independently of γ). In order to apply Lemma 4 and its corollary we use the same kind of estimates as before to control the maximum on

$D(x_n, 9)$ in a uniform manner depending only on the local norm. More precisely, given $f \in PW_b^p$ and $n \in R_f^{K-good}$, we will use Lemma 4 and its corollary translated to x_n and with $r = 1$ applied to

$$h = \left(\frac{\pi}{\int_{D(x_n, 1)} |f|^p dA} \right)^{1/p} f.$$

Then there is $z_0 \in D(x_n, 1)$ with $|h(z_0)| \geq 1$. Observe that $R_{x_n, 1} \subset D(x_n, 1)$ and $D(x_n, 10) \subset R_{x_n, 20}$. With this in mind, and using for instance subharmonicity of $|h|^p$ in the first inequality below, we get

$$\begin{aligned} \max_{D(x_n, 9)} |h(z)|^p &\leq c \int_{D(x_n, 10)} |h|^p dA = c\pi \int_{D(x_n, 10)} |f|^p dA / \int_{D(x_n, 1)} |f|^p dA \\ &\leq c\pi \int_{R_{x_n, 20}} |f|^p dA / \int_{R_{x_n, 1}} |f|^p dA \leq c\pi K^p. \end{aligned}$$

From here on, the rest follows as in Kovrijkine's argument using Lemma 4, its version for L^p and the fact that $R_{n, 1}$, $n \in R_f^{K-good}$, allow to recover the norm.

4.3. Dirichlet and Besov spaces. It is not completely clear how to define dominating sets for Dirichlet spaces. Recall that (weighted) Dirichlet spaces, or more general Besov spaces, can be defined by

$$\mathcal{B}_\alpha^p = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_{\mathcal{B}_\alpha^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha dm < \infty\}.$$

Defining a positive, finite measure μ on $\mathbb{D}$ as reverse Carleson measure by asking $\int_{\mathbb{D}} |f|^p d\mu \geq C \|f\|_{\mathcal{B}_\alpha^p}^p$, thus generalizing (1.2) to arbitrary positive measures, leads to an empty result in general. Indeed, at least for $p = 2$ and $\alpha = 0$ it was mentioned after [4, Theorem 8.3] that such measures do simply not exist. If, instead, one defines dominating sets as measurable sets $E \subset \mathbb{D}$ containing 0 such that

$$C^p \|f\|_{\mathcal{B}_\alpha^p}^p \leq |f(0)|^p + \int_E |f'(z)|^p (1 - |z|^2)^\alpha dm$$

then the key observation is that $f \in \mathcal{B}_\alpha^p$ if and only if $f' \in A^{p, \alpha}$ and use the results found in the Bergman space to get the same sampling constant estimates (which will thus give the constant when $0 \in E$; otherwise one could add 0 or an arbitrarily small neighborhood of 0 which does not change the density).

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