Closed Geodesics with Lipschitz Obstacle

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Geodesics on a general subset M of $\mathbb{R}^n$ are considered. When M is the closure of an open subset with Lipschitz boundary and suitable topological conditions are fulfilled, the existence of a nonconstant closed geodesic is proved.   © 1999 Academic Press

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1. INTRODUCTION

The study of geodesics on a Riemannian manifold M without boundary is a classical topic in differential geometry and in global analysis. One of the first results in this area, due to Ljusternik and Fet (see, e.g., [17]), asserts that there exists a nonconstant closed geodesic on each compact Riemannian manifold without boundary.

This result was extended more recently to Riemannian manifolds with boundary in [21]. Actually, in those years, several facts concerning geodesics were generalized to Riemannian manifolds with boundary; see [1, 18, 20,

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22, 26]. A common feature is that, even if the manifold with boundary M is smooth, irregularities of various kind appear. For instance, there is no uniqueness for the Cauchy problem and the natural domain of the energy functional is not a smooth set. Geodesics themselves are differentiable curves with locally Lipschitzian derivative, but they are not, in general, of class C^2. Accordingly, the geodesic equation is satisfied only pointwise almost everywhere.

A natural development was then provided in [2–4], where geodesics on certain nonsmooth sets, called p-convex sets (see Definition 4.2 below), are considered. Each C^2-submanifold of $\mathbb{R}^n$, possibly with boundary, is a p-convex subset of $\mathbb{R}^n$. However, these last sets may also have corners of a certain type and are not, in general, topological manifolds (although they are absolute neighborhood retracts). For instance,

$$M = \left\{ x \in \mathbb{R}^n \colon \max_{1 \le j \le n} |x_j| \le 1 \le \sum_{j=1}^n x_j^2 \right\}$$

is a compact, p-convex subset of $\mathbb{R}^n$. In spite of the lack of regularity in the set M, the results about geodesics on p-convex sets are similar to those on Riemannian manifolds with boundary. In particular, the regularity of geodesics and the interpretation of the geodesic equation are essentially the same as in the previous case.

Our purpose is to consider a different kind of nonsmooth set, obtained by taking as M the closure of an open subset of $\mathbb{R}^n$ with Lipschitz boundary. It is clear that, in such a case, we cannot expect a geodesic to be more than Lipschitzian. Consequently, the notion itself of geodesic deserves a reformulation. We do this in Section 3, where we propose a new definition of geodesic on a general subset M of $\mathbb{R}^n$ and we show the basic property that each geodesic γ is Lipschitzian with $|\gamma'|$ constant almost everywhere. Our definition is related to the nonsmooth critical point theory developed in [9, 13], which we briefly recall in the next section. In Section 4, we show that our notion of geodesic agrees with that of [2], when M is locally closed and p-convex. A fortiori, the same fact holds when M is a C^2-submanifold of $\mathbb{R}^n$, possibly with boundary.

The last sections are devoted to the proof of our main result, which is

THEOREM 1.1.   *Let* $g \colon \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitzian function such that*

$$\forall x \in \mathbb{R}^n \colon \quad g(x) = 0 \Rightarrow 0 \notin \partial g(x)$$

(∂g *denotes Clarke's subdifferential* [7]) *and let*

$$M = \{ x \in \mathbb{R}^n \colon g(x) \le 0 \}.$$

Assume that M is compact, connected, and noncontractible in itself. Then there exists at least one nonconstant closed geodesic on M.

This is the analogue of the Ljusternik–Fet theorem in our setting. To prove it, we consider, as in [3, 21], the set

$$X = \left\{ \gamma \in W^{1,2}(0,1;\mathbb{R}^n) \colon \gamma(0) = \gamma(1), \, \gamma(s) \in M \quad \forall s \right\}$$

and the lower semicontinuous function $f \colon L^2(0,1;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(\gamma) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 |\gamma'(s)|^2 \, ds & \text{if } \gamma \in X, \\ +\infty & \text{otherwise}. \end{cases}$$

Since we prefer to deal with a continuous functional, we introduce, according to the general device of [10], the metric space

$$\mathrm{epi}(f) = \left\{ (\gamma, \lambda) \in L^2(0,1;\mathbb{R}^n) \times \mathbb{R} \colon f(\gamma) \leq \lambda \right\}$$

and the continuous function $\mathcal{G}_f \colon \mathrm{epi}(f) \to \mathbb{R}$ defined by $\mathcal{G}_f(\gamma, \lambda) = \lambda$.

In Sections 5 and 6, we show that $\mathrm{epi}(f)$ is homotopically equivalent to the free loop space of M and that each critical point (γ, λ) of $\mathcal{G}_f$ is of the form $(\gamma, f(\gamma))$ with γ a critical point of f. Finally, in Section 7, we apply the nonsmooth critical point theory of [9, 13] to prove our main result.

By our methods, it is also possible to treat the case where M is the closure of an open subset with Lipschitz boundary in a smooth Riemannian manifold without boundary. However, we have preferred to consider a more particular situation, not to add further technicalities.

On the contrary, it is an open problem whether Theorem 1.1 holds when M is, say, a C^1-submanifold of $\mathbb{R}^n$ without boundary. More generally, it would be interesting to consider geodesics on LNRs (Lipschitz Neighborhood Retracts; see Definition 5.1 below). All the sets M we have considered fall, up to isometry, into this large class (for p-convex sets, see Theorem 4.3 below). On the other hand, several steps in our proof are valid for LNRs, but in Section 6 we exploit the fact that M is just the closure of an open subset.

Some results of this paper were announced in [11].

2. SOME ELEMENTS OF NONSMOOTH ANALYSIS

Let X be a metric space endowed with the metric d. In the following, $\mathrm{B}_r(u)$ will denote the open ball of center u and radius r. More generally, if

$Y \subseteq X$, $B_r(Y)$ will denote the open r-neighborhood of Y (we agree that $B_r(\varnothing) = \varnothing$). Finally, $\text{int}(Y)$ and $\bar{Y}$ will denote the interior and the closure of Y, respectively.

The next notion has been independently introduced in [9, 13, 16], while a variant can be found in [15].

DEFINITION 2.1. Let $f: X \to \mathbb{R}$ be a continuous function. For every $u \in X$, we denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathscr{H}: B_\delta(u) \times [0, \delta] \to X$ such that

$$\forall v \in B_\delta(u), \quad \forall t \in [0, \delta]: \quad d(\mathscr{H}(v, t), v) \leq t,$$

$$\forall v \in B_\delta(u), \quad \forall t \in [0, \delta]: \quad f(\mathscr{H}(v, t)) \leq f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called the *weak slope of f at u*.

It is easy to see that the function $|df|: X \to [0, +\infty]$ is lower semicontinuous.

Now consider a function $f: X \to \mathbb{R} \cup \{+\infty\}$. Set

$$\mathscr{D}(f) = \{u \in X: f(u) < +\infty\},$$

$$\forall b \in \mathbb{R} \cup \{+\infty\}: \quad f^b = \{u \in \mathscr{D}(f): f(u) \leq b\},$$

$$\text{epi}(f) = \{(u, \lambda) \in X \times \mathbb{R}: f(u) \leq \lambda\},$$

and define a function $\mathscr{G}_f: \text{epi}(f) \to \mathbb{R}$ by $\mathscr{G}_f(u, \lambda) = \lambda$. The set $X \times \mathbb{R}$ will be endowed with the metric

$$d((u, \lambda), (v, \mu)) = \left(d(u, v)^2 + (\lambda - \mu)^2 \right)^{1/2}$$

and $\text{epi}(f)$ with the induced metric. According to [9, 13], let us give

DEFINITION 2.2. For every $u \in \mathscr{D}(f)$, let

$$|df|(u) = \begin{cases} \dfrac{|d\mathscr{G}_f|(u, f(u))}{\sqrt{1 - |d\mathscr{G}_f|(u, f(u))^2}} & \text{if } |d\mathscr{G}_f|(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathscr{G}_f|(u, f(u)) = 1. \end{cases}$$

When f is real-valued and continuous, the above definition turns out to be consistent with Definition 2.1 (see [13]). Let us give a criterion for obtaining an estimate of $|d\mathscr{G}_f|(u, \lambda)$ and $|df|(u)$. It is the corrected version of [5, Theorem 1.5.4], which contains a mistake in the statement.

PROPOSITION 2.3. *Let $(u, \lambda) \in \text{epi}(f)$. Assume there exist $\delta, c, \sigma > 0$ and a continuous map*

$$\mathscr{H}: \{v \in B_\delta(u): f(v) < \lambda + \delta\} \times [0, \delta] \to X$$

such that for any $v \in B_\delta(u)$ *with* $f(v) < \lambda + \delta$ *and any* $t \in [0, \delta]$ *we have*

$$d(\mathcal{H}(v, t), v) \le ct, \qquad f(\mathcal{H}(v, t)) \le f(v) - \sigma t.$$

Then we have

$$|d\mathcal{G}_f|(u, \lambda) \ge \frac{\sigma}{\sqrt{c^2 + \sigma^2}}.$$

In particular, if $\lambda = f(u)$, *it is* $|df|(u) \ge \sigma/c$.

Proof. By the change of variable $t = \tau/c$, we can reduce the question to the case $c = 1$. Now the same argument of [5, Theorem 1.5.4] works and it is not necessary to assume the lower semicontinuity of f. ∎

DEFINITION 2.4.   We say that $u \in \mathcal{D}(f)$ is a (*lower*) *critical point of* f if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (*lower*) *critical value of* f if there exists a (lower) critical point $u \in \mathcal{D}(f)$ of f with $f(u) = c$.

DEFINITION 2.5.   Let $c \in \mathbb{R}$. A sequence (u_h) in $\mathcal{D}(f)$ is said to be a *Palais–Smale sequence at level* c (($PS)_c$-*sequence*, for short) for f, if $f(u_h) \to c$ and $|df|(u_h) \to 0$.

We say that f satisfies the *Palais–Smale condition at level* c (($PS)_c$ for short), if every $(PS)_c$-sequence (u_h) for f admits a convergent subsequence (u_{h_k}) in X.

For every $c \in \mathbb{R}$, let

$$K_c = \{u \in \mathcal{D}(f): |df|(u) = 0, f(u) = c\}.$$

LEMMA 2.6.   *Let* Y *be a metric space and let* A *be a subset of* Y. *Assume that for every neighborhood* U *of* A *there exists a deformation* $\eta: Y \times [0, 1] \to Y$ *such that* $\eta(Y \times \{1\}) \subseteq U$ *and* $\eta(A \times [0, 1]) \subseteq U$. *Then the inclusion map* $i: A \to Y$ *induces an isomorphism* $i^*: H^*(Y) \to H^*(A)$ *for Alexander–Spanier cohomology.*

Proof. Let $\omega \in H^*(Y)$ be such that $i^*(\omega) = 0$. Since A is tautly imbedded in Y (see [23]), there exists a neighborhood U of A such that $j^*(\omega) = 0$, where $j: U \to Y$ is the inclusion map. Let $\eta: Y \times [0, 1] \to Y$ be a deformation, according to the assumption. Since $j \circ \eta(\cdot, 1)$ is homotopic to the identity of Y, j^* is a monomorphism. It follows that $\omega = 0$, so that i^* is a monomorphism.

Now let $\omega \in H^*(A)$. Again from tautness we deduce that there exists a neighborhood U of A such that ω is in the range of $k^*: H^*(U) \to H^*(A)$, where $k: A \to U$ is the inclusion map. Let $\eta: Y \times [0, 1] \to Y$ be a deformation as in the assumption. Since $\eta(\cdot, 1) \circ i$ is homotopic to k, the range of k^* is contained in the range of i^*. In particular, ω belongs to the range of i^*, which is therefore an epimorphism. ∎

In the next result we prove a variant of the noncritical interval theorem (see [5, Theorem 1.1.14] and [8, Theorem 2.10]). Let us point out that we allow the level set $f^{-1}(a)$ to contain infinitely many critical points.

THEOREM 2.7.   *Let $f: X \to \mathbb{R}$ be continuous and let $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ $(a < b)$. Assume that f has no critical points u with $a < f(u) \le b$, that $(PS)_c$ holds, and that f^c is complete whenever $c \in [a, b[$. Then the inclusion map $i: f^a \to f^b$ induces an isomorphism $i^*: H^*(f^b) \to H^*(f^a)$ for Alexander–Spanier cohomology.*

Proof.   Let U be a neighborhood of f^a in f^b. Since K_a is compact, there exits $\rho > 0$ such that $B_{2\rho}(K_a) \subseteq U$. By the deformation theorem [9, Theorem 2.14], there exist $\varepsilon > 0$ with $a + \varepsilon < b$ and a deformation $\mathscr{H}: X \times [0, 1] \to X$ such that

$$\forall u \in X, \quad \forall t \in [0, 1]: \quad d(\mathscr{H}(u, t), u) \le \rho t, \quad f(\mathscr{H}(u, t)) \le f(u),$$
$$\mathscr{H}\big((f^{a+\varepsilon} \setminus B_\rho(K_a)) \times \{1\}\big) \subseteq f^{a-\varepsilon}.$$

It follows that

$$\mathscr{H}(f^{a+\varepsilon} \times \{1\}) \subseteq f^{a-\varepsilon} \cup B_{2\rho}(K_a) \subseteq U.$$

By the noncritical interval theorem [5, Theorem 1.1.14], there exists a strong deformation retraction $\mathscr{K}: f^b \times [0, 1] \to f^b$ of f^b into $f^{a+\varepsilon}$. Define a deformation $\eta: f^b \times [0, 1] \to f^b$ by

$$\eta(u, t) = \begin{cases} \mathscr{K}(u, 2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \mathscr{H}(\mathscr{K}(u, 1), 2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then we have $\eta(f^b \times \{1\}) \subseteq U$ and $\eta(f^a \times [0, 1]) \subseteq U$. From the previous lemma, the assertion follows. ∎

Finally, if Y is a Banach space and $f: Y \to \mathbb{R}$ is a locally Lipschitzian function, we denote by $\partial f(u)$ the Clarke subdifferential of f at u [7].

3. GEODESICS ON NONSMOOTH SETS

Let M be a subset of $\mathbb{R}^n$. In the sequel, each $\gamma \in W^{1,2}(a, b; \mathbb{R}^n)$ will be identified with its continuous representative $\tilde{\gamma}: [a, b] \to \mathbb{R}^n$. Moreover, we will denote by $\|\cdot\|_{1,2}$ and $\|\cdot\|_p$ the usual norms in $W^{1,2}(a, b; \mathbb{R}^n)$ and $L^p(a, b; \mathbb{R}^n)$, $1 \le p \le \infty$. We set

$$W^{1,2}(a, b; M) := \big\{\gamma \in W^{1,2}(a, b; \mathbb{R}^n): \gamma(s) \in M \text{ for each } s \in [a, b]\big\}$$

and we define a functional $\mathscr{E}_{a,b}: W^{1,2}(a,b;M) \to \mathbb{R}$ by

$$\mathscr{E}_{a,b}(\gamma) := \frac{1}{2}\int_a^b |\gamma'(s)|^2 \, ds.$$

The next definition is suggested by Proposition 2.3 (see also the proof of Theorem 3.8 below).

DEFINITION 3.1. Let $a, b \in \mathbb{R}$ with $a < b$. A curve $\gamma \in W^{1,2}(a,b;M)$ is said to be *energy-stationary* if it is not possible to find $\delta, c, \sigma > 0$ and a map

$$\mathscr{H}: \{\eta \in W^{1,2}(a,b;M): \|\eta - \gamma\|_{1,2} < \delta\} \times [0,\delta] \to W^{1,2}(a,b;M)$$

with the following properties:

(a) $\mathscr{H}$ is continuous from the topology of $L^2(a,b;\mathbb{R}^n) \times \mathbb{R}$ to that of $L^2(a,b;\mathbb{R}^n)$;

(b) for every $\eta \in W^{1,2}(a,b;M)$ with $\|\eta - \gamma\|_{1,2} < \delta$ and $t \in [0,\delta]$, we have

$$(\mathscr{H}(\eta, t) - \eta) \in W_0^{1,2}(a,b;\mathbb{R}^n),$$

$$\|\mathscr{H}(\eta, t) - \eta\|_2 \le ct, \qquad \mathscr{E}_{a,b}(\mathscr{H}(\eta,t)) \le \mathscr{E}_{a,b}(\eta) - \sigma t.$$

PROPOSITION 3.2. *Let* $\gamma \in W^{1,2}(a,b;M)$ *be energy-stationary. Then for every* $[\alpha, \beta] \subseteq [a,b]$, *the restriction* $\gamma_{\restriction [\alpha,\beta]}$ *is energy-stationary.*

Proof. Set $\hat{\gamma} = \gamma_{\restriction[\alpha,\beta]}$. By contradiction, assume there exist $\delta, c, \sigma > 0$ and

$$\mathscr{H}: \{\eta \in W^{1,2}(\alpha,\beta;M): \|\eta - \hat{\gamma}\|_{1,2} < \delta\} \times [0,\delta] \to W^{1,2}(\alpha,\beta;M)$$

according to Definition 3.1. For any $\eta \in W^{1,2}(a,b;M)$, set $\hat{\eta} = \eta_{\restriction[\alpha,\beta]}$ and define

$$\mathscr{K}: \{\eta \in W^{1,2}(a,b;M): \|\eta - \gamma\|_{1,2} < \delta\} \times [0,\delta] \to W^{1,2}(a,b;M)$$

by

$$\mathscr{K}(\eta,t)(s) = \begin{cases} \mathscr{H}(\hat{\eta}, t)(s) & \text{if } s \in [\alpha, \beta], \\ \eta(s) & \text{if } s \notin [\alpha, \beta]. \end{cases}$$

It is readily seen that $\mathscr{K}$ has all the properties required in Definition 3.1. It follows that γ is not energy-stationary, a contradiction. $\blacksquare$

DEFINITION 3.3.   Let I be an interval in $\mathbb{R}$ with $\mathrm{int}(I) \neq \emptyset$. A continuous map $\gamma: I \to M$ is said to be a *geodesic on M* if every $s \in \mathrm{int}(I)$ admits a neighborhood $[a, b]$ in I such that $\gamma_{[a, b]}$ belongs to $W^{1,2}(a, b; M)$ and is energy-stationary.

DEFINITION 3.4.   A *closed geodesic on M* is a geodesic $\gamma: \mathbb{R} \to M$ which is periodic of period 1.

THEOREM 3.5.   *Let I be an interval in $\mathbb{R}$ with $\mathrm{int}(I) \neq \emptyset$ and let $\gamma: I \to M$ be a geodesic on M. Then γ is Lipschitzian and $|\gamma'|$ is almost everywhere equal to a constant.*

Proof.   First, we have $\gamma \in W^{1,2}_{\mathrm{loc}}(\mathrm{int}(I); \mathbb{R}^n)$. For each $s \in \mathrm{int}(I)$, let $[a_s, b_s]$ be a neighborhood of s as in Definition 3.3. It is sufficient to show that

$$\forall s \in \mathrm{int}(I), \quad \forall \varphi \in C_c^\infty(]a_s, b_s[): \qquad \int_{a_s}^{b_s} \varphi'(\lambda)|\gamma'(\lambda)|^2 \, d\lambda = 0.$$

By contradiction, let $s \in \mathrm{int}(I)$ and $\varphi \in C_c^\infty(]a_s, b_s[)$ be such that

$$\sigma := \frac{1}{3} \int_{a_s}^{b_s} \varphi'(\lambda)|\gamma'(\lambda)|^2 \, d\lambda > 0.$$

Let $\delta > 0$ be such that $\delta \|\varphi'\|_\infty < 1$ and let $\psi: [a_s, b_s] \times [0, \delta] \to [a_s, b_s]$ be the smooth function such that

$$\forall \lambda \in [a_s, b_s], \quad \forall t \in [0, \delta]: \qquad \lambda = \psi(\lambda, t) - t\varphi(\psi(\lambda, t)).$$

Define $\mathscr{H}: W^{1,2}(a_s, b_s; M) \times [0, \delta] \to W^{1,2}(a_s, b_s; M)$ by

$$\mathscr{H}(\eta, t)(\mu) = \eta(\mu - t\varphi(\mu)).$$

It is easy to see that $\mathscr{H}$ is continuous from the topology of $L^2 \times \mathbb{R}$ to that of L^2 and that

$$(\mathscr{H}(\eta, t) - \eta) \in W_0^{1,2}(a_s, b_s; \mathbb{R}^n),$$

$$\|\mathscr{H}(\eta, t) - \eta\|_2 \leq \|\eta'\|_2 \|\varphi\|_\infty t,$$

and

$$\begin{aligned}
\mathscr{E}_{a_s, b_s}(\mathscr{H}(\eta, t)) &= \frac{1}{2} \int_{a_s}^{b_s} (1 - t\varphi'(\mu))^2 |\eta'(\mu - t\varphi(\mu))|^2 \, d\mu \\
&= \frac{1}{2} \int_{a_s}^{b_s} (1 - t\varphi'(\psi(\lambda, t)))|\eta'(\lambda)|^2 \, d\lambda \\
&= \mathscr{E}_{a_s, b_s}(\eta) - \frac{t}{2} \int_{a_s}^{b_s} \varphi'(\psi(\lambda, t))|\eta'(\lambda)|^2 \, d\lambda.
\end{aligned}$$

By decreasing δ, we may assume that

$$\|\eta - \gamma\|_{1,2} < \delta, 0 \le t \le \delta \Rightarrow \int_{a_s}^{b_s} \varphi'(\psi(\lambda, t))|\eta'(\lambda)|^2 \, d\lambda \ge 2\sigma.$$

Therefore, if we restrict $\mathscr{H}$ to

$$\{\eta \in W^{1,2}(a_s, b_s; M): \|\eta - \gamma\|_{1,2} < \delta\} \times [0, \delta],$$

we have

$$\|\mathscr{H}(\eta, t) - \eta\|_2 \le (\|\gamma'\|_2 + \delta)\|\varphi\|_\infty t,$$
$$\mathscr{E}_{a_s, b_s}(\mathscr{H}(\eta, t)) \le \mathscr{E}_{a_s, b_s}(\eta) - \sigma t.$$

It follows that $\gamma_{[[a_s, b_s]]}$ is not energy-stationary, a contradiction. ∎

In order to apply the techniques of the previous section to the study of closed geodesics, we have to introduce a variational structure suitable for such a problem. As in [3, 21] we set

$$X = \{\gamma \in W^{1,2}(0, 1; M): \gamma(0) = \gamma(1)\} \tag{3.6}$$

and we define a functional $f: L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\gamma'(s)|^2 \, ds & \text{if } \gamma \in X, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

THEOREM 3.8.   *If $\gamma \in X$ is a critical point of f, then γ is the restriction to $[0, 1]$ of a closed geodesic on M.*

Proof. Let us show that, if γ is a critical point of f, then γ is energy-stationary on $[0, 1]$. By contradiction, let $\delta, c, \sigma > 0$ and let $\mathscr{H}$ be as in Definition 3.1. There exists $\delta' \in]0, \delta]$ such that

$$\forall \eta \in X: \qquad \|\eta - \gamma\|_2 < \delta', \quad f(\eta) < f(\gamma) + \delta' \Rightarrow \|\eta - \gamma\|_{1,2} < \delta.$$

Then it is easy to deduce, by Proposition 2.3, that γ is not a critical point of f. The contradiction shows that γ is energy-stationary on $[0, 1]$.

Now define $\hat{\gamma} \in X$ by

$$\hat{\gamma}(s) = \begin{cases} \gamma(s + \tfrac{1}{2}) & \text{if } 0 \le s \le \tfrac{1}{2}, \\ \gamma(s - \tfrac{1}{2}) & \text{if } \tfrac{1}{2} \le s \le 1. \end{cases}$$

It is easy to see that also $\hat{\gamma}$ is energy-stationary on $[0, 1]$, whence the assertion. ∎

4. A COMPARISON WITH A PREVIOUS NOTION

In this section, we compare our definition of a geodesic with that of [2]. In the following, H will denote a real Hilbert space endowed with the scalar product $(\cdot \mid \cdot)$.

DEFINITION 4.1.　A subset M of H is said to be *locally closed* if for every $u \in M$ there exists $r > 0$ such that $M \cap \overline{B_r(u)}$ is closed in H.

DEFINITION 4.2.　A locally closed subset M of H is said to be *p-convex* if for every $u \in M$ there exist $r_u > 0$ and $p_u \geq 0$ such that

$$\forall v, w \in M \cap \overline{B_{r_u}(u)}, \quad \exists z \in M \cap \overline{B_{r_u}(u)}:$$

$$\left\| z - \frac{v + w}{2} \right\| \leq p_u \|v - w\|^2.$$

The above notion was introduced in [12, Definition 1.34], in the particular case where p_u is independent of u and was developed in [2–4], following an approach which is equivalent to the more general Definition 4.2. See in particular [4, Proposition 1.12].

If M is a p-convex, locally closed subset of H and $u \in M$, we denote by $T_u M$ and $N_u M$ the tangent and the normal cone to M at u, as defined in [7] (here we identify H with its dual space, so that $N_u M \subseteq H$). Let us mention that, in this case, the tangent and normal cones may be introduced in other equivalent ways, according to [2, Proposition 2.14]. Finally, we denote by $P_u \colon H \to T_u M$ the orthogonal projection on $T_u M$.

In the next result we recall the main properties we need for our purposes.

THEOREM 4.3.　*Let M be a p-convex, locally closed subset of H. Then the following facts hold*:

(a)　*there exists an open neighborhood A of M such that any $u \in A$ admits one and only one point in M with minimal distance from u*;

(b)　*the induced projection $\pi \colon A \to M$ is Lipschitzian of constant* 2;

(c)　*for every $u \in A$, we have*

$$u - \pi(u) \in N_{\pi(u)} M;$$

(d)　*if (u_h) is a sequence converging to u in M and (ν_h) is a sequence weakly converging to ν in H with $\nu_h \in N_{u_h} M$, we have that $\nu \in N_u M$*;

(e)　*there exists a continuous function $\tilde{p} \colon M \to [0, +\infty[$ such that*

$$\forall u, w \in M, \quad \forall \nu \in N_u M: \quad (\nu \mid w - u) \leq \tilde{p}(u)\|\nu\| \|w - u\|^2.$$

Proof. Property (e) follows from [4, Proposition 1.12]. The remaining statements follow from [2, Propositions 2.2 and 2.9 and Remark 2.10]. ∎

THEOREM 4.4. *Let M be a p-convex, locally closed subset of H. Then for every $u \in M$ and $v \in H$, we have*

$$\limsup_{\substack{w \to u, \, w \in M \\ t \to 0^+}} \frac{\|w + tv - \pi(w + tv)\|}{t} \leq \|v - P_u v\|.$$

Proof. Let $w_h \in M$, $w_h \to u$, and $t_h \to 0$, $t_h > 0$. Without loss of generality, we may suppose that $w_h + t_h v - \pi(w_h + t_h v) \neq 0$ and that

$$\frac{w_h + t_h v - \pi(w_h + t_h v)}{\|w_h + t_h v - \pi(w_h + t_h v)\|} \to \nu, \qquad \|\nu\| \leq 1.$$

Observe that we have

$$\|w_h - \pi(w_h + t_h v)\| = \|\pi(w_h) - \pi(w_h + t_h v)\| \leq 2 t_h \|v\|, \quad (4.5)$$

$$\frac{w_h + t_h v - \pi(w_h + t_h v)}{\|w_h + t_h v - \pi(w_h + t_h v)\|} \in N_{\pi(w_h + t_h v)} M,$$

so that $\nu \in N_u M$. On the other hand, we have

$$(w_h + t_h v - \pi(w_h + t_h v) | w_h - \pi(w_h + t_h v))$$
$$\leq \tilde{p}(\pi(w_h + t_h v)) \|w_h + t_h v - \pi(w_h + t_h v)\|$$
$$\cdot \|w_h - \pi(w_h + t_h v)\|^2,$$

namely,

$$\|w_h + t_h v - \pi(w_h + t_h v)\|^2$$
$$\leq (w_h + t_h v - \pi(w_h + t_h v) | t_h v)$$
$$+ \tilde{p}(\pi(w_h + t_h v)) \|w_h + t_h v - \pi(w_h + t_h v)\|$$
$$\cdot \|w_h - \pi(w_h + t_h v)\|^2.$$

Taking into account (4.5), we deduce that

$$\frac{\left\| w_h + t_h v - \pi(w_h + t_h v) \right\|}{t_h}$$

$$\leq \left(\frac{w_h + t_h v - \pi(w_h + t_h v)}{\left\| w_h + t_h v - \pi(w_h + t_h v) \right\|} \middle| v \right) + \tilde{p}(\pi(w_h + t_h v)) 4 t_h \|v\|^2.$$

It follows that

$$\limsup_h \frac{\left\| w_h + t_h v - \pi(w_h + t_h v) \right\|}{t_h} \leq (\nu | v - P_u v) + (\nu | P_u v)$$

$$\leq \|\nu\| \|v - P_u v\| \leq \|v - P_u v\|,$$

whence the assertion. ∎

Now assume that M is a p-convex, locally closed subset of $\mathbb{R}^n$.

LEMMA 4.6.   *Let* $a, b \in \mathbb{R}$ *with* $a < b$, $\gamma \in W^{1,2}(a, b; M)$, *and* $\xi \in W_0^{1,2}(a, b; \mathbb{R}^n)$. *Then*

$$\liminf_{\substack{\eta \to \gamma, \, \eta \in W^{1,2}(a, b; M) \\ t \to 0^+}} \frac{\frac{1}{2} \int_a^b |\eta' + t\xi'|^2 \, ds - \frac{1}{2} \int_a^b |\pi(\eta + t\xi)'|^2 \, ds}{t}$$

$$\geq -2 \int_a^b p(\gamma) |\xi - P_\gamma \xi| |\gamma'|^2 \, ds.$$

Proof.   Let (η_h) be a sequence in $W^{1,2}(a, b; M)$ strongly convergent to γ in the $W^{1,2}$-topology and let (t_h) be a sequence in $]0, +\infty[$ convergent to 0. We have

$$\frac{\left| \eta_h(s) + t_h \xi(s) - \pi(\eta_h(s) + t_h \xi(s)) \right|}{t_h} \leq 3 |\xi(s)| \qquad \forall s \in [a, b]$$

and

$$\limsup_h \frac{\left| \eta_h(s) + t_h \xi(s) - \pi(\eta_h(s) + t_h \xi(s)) \right|}{t_h}$$

$$\leq \left| \xi(s) - P_{\gamma(s)} \xi(s) \right| \qquad \forall s \in [a, b]$$

by Theorem 4.4. Then the same argument of [2, Lemma 3.3] works, with Lebesgue's Theorem replaced by Fatou's Lemma. ∎

Finally, let us prove the main result of this section.

THEOREM 4.7.   *Let $M \subseteq \mathbb{R}^n$ be locally closed and p-convex. Moreover, let I be an interval in $\mathbb{R}$ with $\text{int}(I) \neq \varnothing$ and let $\gamma: I \to M$ be a continuous map. Then the following facts are equivalent*:

(a)   *γ is a geodesic on M*;
(b)   *$\gamma \in W^{2,1}_{\text{loc}}(\text{int}(I); \mathbb{R}^n)$ and*

$$\gamma''(s) \in N_{\gamma(s)}M \quad a.e. \text{ in } \text{int}(I).$$

Proof.   (a) $\Rightarrow$ (b)   Let $s \in \text{int}(I)$ and let $[a_s, b_s]$ be a neighborhood of s as in Definition 3.3. It is sufficient to show that $\gamma \in W^{2,1}(a_s, b_s; \mathbb{R}^n)$ and $\gamma''(s) \in N_{\gamma(s)}M$ a.e. in $]a_s, b_s[$. Let us argue by contradiction. From [2, Lemma 3.5], it follows that we can find $\sigma > 0$ and $\xi \in W^{1,2}_0(a_s, b_s; \mathbb{R}^n)$ such that

$$\int_{a_s}^{b_s} \gamma' \cdot \xi' \, d\lambda = -2\int_{a_s}^{b_s} \tilde{p}(\gamma)|\xi - P_\gamma \xi||\gamma'|^2 \, d\lambda - 3\sigma.$$

Take $\delta > 0$ sufficiently small and define

$$\mathcal{H}: \left\{\eta \in W^{1,2}(a_s, b_s; M): \|\eta - \gamma\|_{1,2} < \delta\right\} \times [0, \delta] \to W^{1,2}(a_s, b_s; M)$$

by

$$\mathcal{H}(\eta, t)(s) = \pi(\eta(s) + t\xi(s)).$$

Then $\mathcal{H}$ is continuous from the topology of $L^2 \times \mathbb{R}$ to that of L^2 and we have

$$(\mathcal{H}(\eta, t) - \eta) \in W^{1,2}_0(a_s, b_s; \mathbb{R}^n),$$
$$\|\mathcal{H}(\eta, t) - \eta\|_2 \leq 2\|\xi\|_2 t.$$

Moreover, by decreasing δ we may assume that

$$\int_{a_s}^{b_s} (\eta' - \gamma') \cdot \xi' \, d\lambda + \frac{\delta}{2}\int_{a_s}^{b_s} |\xi'|^2 \, d\lambda \leq \sigma$$

and, by Lemma 4.6, that

$$\frac{1}{2}\int_{a_s}^{b_s} |\eta' + t\xi'|^2 \, d\lambda \geq \frac{1}{2}\int_{a_s}^{b_s} |\pi(\eta + t\xi)'|^2 \, d\lambda$$
$$- t\left[2\int_{a_s}^{b_s} \tilde{p}(\gamma)|\xi - P_\gamma \xi||\gamma'|^2 \, d\lambda + \sigma\right].$$

It follows that

$$\mathscr{E}_{a_s, b_s}(\mathscr{H}(\eta, t))$$

$$\leq \mathscr{E}_{a_s, b_s}(\eta) + t\left[\int_{a_s}^{b_s} \eta' \cdot \xi' \, d\lambda + \frac{\delta}{2}\int_{a_s}^{b_s}|\xi'|^2 \, d\lambda\right.$$

$$\left.+ 2\int_{a_s}^{b_s} \tilde{p}(\gamma)|\xi - P_\gamma \xi| \, |\gamma'|^2 \, d\lambda + \sigma\right]$$

$$= \mathscr{E}_{a_s, b_s}(\eta) + t\left[\int_{a_s}^{b_s}(\eta' - \gamma') \cdot \xi' \, d\lambda - 2\sigma + \frac{\delta}{2}\int_{a_s}^{b_s}|\xi'|^2 \, d\lambda\right]$$

$$\leq \mathscr{E}_{a_s, b_s}(\eta) - \sigma t.$$

We deduce that γ is not energy-stationary on $[a_s, b_s]$, which is a contradiction.

(b) $\Rightarrow$ (a)   Let $s \in \text{int}(I)$ and let $[a_s, b_s]$ be a neighborhood of s such that $\gamma \in W^{2,1}(a_s, b_s; \mathbb{R}^n)$. From [2, Theorem 3.8] it follows that $\gamma \in W^{2,\infty}(a_s, b_s; \mathbb{R}^n)$. It is sufficient to show that γ is energy-stationary on $[a_s, b_s]$. By contradiction, let $\mathscr{H}$ be as in Definition 3.1. In particular, we have

$$\mathscr{E}_{a_s, b_s}(\gamma) \geq \mathscr{E}_{a_s, b_s}(\mathscr{H}(\gamma, t)) + \frac{\sigma}{c}\|\mathscr{H}(\gamma, t) - \gamma\|_2,$$

$$0 < t \leq \delta \Rightarrow \mathscr{H}(\gamma, t) \neq \gamma.$$

On the other hand, from [2, Theorems 3.7 and 3.9] we deduce that

$$\mathscr{E}_{a_s, b_s}(\mathscr{H}(\gamma, t)) \geq \mathscr{E}_{a_s, b_s}(\gamma) - \varphi_0\|\mathscr{H}(\gamma, t) - \gamma\|_2^2$$

for some constant $\varphi_0 > 0$. It follows that

$$\frac{\sigma}{2}\|\mathscr{H}(\gamma, t) - \gamma\|_2 \leq \varphi_0\|\mathscr{H}(\gamma, t) - \gamma\|_2^2;$$

hence $\sigma = 0$, which is absurd.   ∎

5. LIPSCHITZ NEIGHBORHOOD RETRACTS

In this section we establish the topological properties we need to get our main result. Let us recall a notion from [14].

DEFINITION 5.1.   A subset M of $\mathbb{R}^n$ is said to be a LNR if there exist an open neighborhood U of M in $\mathbb{R}^n$ and a locally Lipschitzian retraction r: $U \to M$.

If Y is a metric space, we shall consider

$$\Lambda(Y) := \{ \gamma \in C([0,1]; Y): \gamma(0) = \gamma(1) \}$$

endowed with the sup-metric ($\Lambda(Y)$ is called the *free loop space* of Y).

Now let M be a subset of $\mathbb{R}^n$ and let X and $f: L^2(0,1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ be defined as in (3.6), (3.7).

In the next lemma, we state without proof a simple variant of a well-known result (see, e.g., [19, Theorem 17.1]).

LEMMA 5.2.   *Let U be an open subset of $\mathbb{R}^n$ and let*

$$\Lambda^1(U) := \{ \gamma \in W^{1,2}(0,1; U): \gamma(0) = \gamma(1) \}$$

be endowed with the $W^{1,2}$-metric. Then there exists a continuous map

$$\mathscr{K}: \Lambda(U) \times [0,1] \to \Lambda(U)$$

such that

$$\forall \gamma \in \Lambda(U): \quad \mathscr{K}(\gamma, 0) = \gamma, \quad \mathscr{K}(\gamma, 1) \in \Lambda^1(U);$$

$$\mathscr{K}(\cdot, 1): \Lambda(U) \to \Lambda^1(U) \quad \text{is continuous};$$

$$\mathscr{K}(\Lambda^1(U) \times [0,1]) \subseteq \Lambda^1(U);$$

$$\forall \gamma \in \Lambda^1(U), \quad \forall t \in [0,1]: \quad \left\| [\mathscr{K}(\gamma, t)]' \right\|_2 \leq \| \gamma' \|_2.$$

As we have already mentioned in the Introduction, we want to apply a variational argument to the continuous function $\mathscr{G}_f$: epi$(f) \to \mathbb{R}$. Therefore the next result gives us crucial information.

THEOREM 5.3.   *Assume that M is a LNR. Then the map*

$$\pi: \text{epi}(f) \to \Lambda(M)$$

$$(\gamma, \lambda) \mapsto \gamma$$

is a homotopy equivalence (epi(f) is endowed with the topology of $L^2 \times \mathbb{R}$).

Proof.   Let $r: U \to M$ be as in Definition 5.1 and $\mathscr{K}: \Lambda(U) \times [0,1] \to \Lambda(U)$ as in Lemma 5.2. The function $\{\gamma \mapsto f(r \circ \mathscr{K}(\gamma, 1))\}$ is locally bounded on $\Lambda(M)$. Let $\mu: \Lambda(M) \to \mathbb{R}$ be a continuous function such that

$$\forall \gamma \in \Lambda(M): \quad f(r \circ \mathscr{K}(\gamma, 1)) \leq \mu(\gamma).$$

Then the map

$$\varphi \colon \Lambda(M) \to \mathrm{epi}(f)$$
$$\gamma \mapsto (r \circ \mathscr{K}(\gamma, 1), \mu(\gamma))$$

is well defined and continuous. Since

$$(\pi \circ \varphi)(\gamma) = r \circ \mathscr{K}(\gamma, 1),$$

it is evident that $(\pi \circ \varphi)$ is homotopic to the identity map of $\Lambda(M)$.

Now observe that there exists a continuous function $L \colon \Lambda(U) \to [1, +\infty[$ such that

$$\forall \gamma \in \Lambda(U), \quad \forall s_1, s_2 \in [0, 1]:$$
$$\left| r(\gamma(s_2)) - r(\gamma(s_1)) \right| \leq L(\gamma) \left| \gamma(s_2) - \gamma(s_1) \right|.$$

It follows that

$$\forall(\gamma, \lambda) \in \mathrm{epi}(f), \quad \forall t \in [0, 1]:$$
$$f(r \circ \mathscr{K}(\gamma, t)) \leq L^2(\mathscr{K}(\gamma, t)) f(\gamma) \leq L^2(\mathscr{K}(\gamma, t)) \lambda.$$

Since

$$(\varphi \circ \pi)(\gamma, \lambda) = (r \circ \mathscr{K}(\gamma, 1), \mu(\gamma)),$$

we have that $(\varphi \circ \pi)$ is homotopic to the map

$$\left\{ (\gamma, \lambda) \mapsto (r \circ \mathscr{K}(\gamma, 1), L^2(\mathscr{K}(\gamma, 1))) \right\}.$$

This in turn is homotopic to the map

$$\left\{ (\gamma, \lambda) \mapsto (\gamma, L^2(\gamma)\lambda) \right\},$$

which is clearly homotopic to the identity map of $\mathrm{epi}(f)$. Therefore $(\varphi \circ \pi)$ is homotopic to the identity map of $\mathrm{epi}(f)$ and the proof is complete. ∎

The next result is a simple consequence of a well-known theorem of [24].

THEOREM 5.4. *Assume that M is an absolute neighborhood retract. Suppose also that M is compact, simply connected, and noncontractible in itself. Then there exists $q \in \mathbb{N}$ such that $H^q(M) = \{0\}$ and $H^q(\Lambda(M)) \neq \{0\}$, where H^* denotes Alexander–Spanier cohomology with coefficients in $\mathbb{R}$.*

Proof. Since M is a compact ANR, it follows from [25] that M is homotopically equivalent to a compact polyhedron. Therefore M has cohomology of finite type. On the other hand, from [24, Main Theorem and Addendum] we deduce that $H^q(\Lambda(M)) \neq \{0\}$ for infinitely many q's. ∎

COROLLARY 5.5. *Assume that M is a LNR. Suppose also that M is compact, simply connected, and noncontractible in itself. Then there exists $q \in \mathbb{N}$ such that $H^q(M) = \{0\}$ and $H^q(\text{epi}(f)) \neq \{0\}$.*

Proof. Being a LNR, M is clearly an ANR. Then the assertion follows from Theorems 5.3 and 5.4. ∎

PROPOSITION 5.6. *Assume that M is a LNR. Suppose also that M is connected, but not simply connected. Then there exists $\gamma \in X$ such that γ is not contractible to a point in M.*

Proof. Let $r: U \to M$ be as in Definition 5.1 and let $\eta \in \Lambda(M)$ be noncontractible in M. Let (η_h) be a sequence of smooth closed curves in $\mathbb{R}^n$ with (η_h) uniformly convergent to η. Since the range of η_h is contained in U eventually as $h \to \infty$, we have $r \circ \eta_h \in X$ for large h. On the other hand, it is easy to show that $r \circ \eta_h$ is homotopic to η again for large h. Then the assertion follows. ∎

6. LIPSCHITZ OBSTACLES

Let $g: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitzian function such that

$$\forall x \in \mathbb{R}^n: g(x) = 0 \Rightarrow 0 \notin \partial g(x)$$

and let

$$M = \{x \in \mathbb{R}^n: g(x) \leq 0\}.$$

According to [6], define a lower semicontinuous function $\lambda: \mathbb{R}^n \to [0, +\infty[$ by

$$\lambda(x) = \min\{|\alpha|: \alpha \in \partial g(x)\}.$$

The same argument of [6, Lemma 3.3] shows that there exists a locally Lipschitzian map

$$v: \{x \in \mathbb{R}^n: 0 \notin \partial g(x)\} \to \mathbb{R}^n$$

such that

$$0 \notin \partial g(x) \Rightarrow |v(x)| \leq 2\lambda(x),$$

$$0 \notin \partial g(x), \quad \alpha \in \partial g(x) \Rightarrow \langle \alpha, v(x) \rangle \geq \lambda(x)^2.$$

Let $\mathscr{O}$ be an open neighborhood of $\{x \in \mathbb{R}^n \colon 0 \in \partial g(x)\}$ and let $\vartheta\colon$ $\mathbb{R}^n \to [0, 1]$ be a locally Lipschitzian function such that

$$x \in \mathscr{O} \Rightarrow \vartheta(x) = 0,$$
$$g(x) = 0 \Rightarrow \vartheta(x) = 1.$$

Finally, define $\nu\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\nu(x) = \begin{cases} \vartheta(x) \dfrac{v(x)}{|v(x)|} & \text{if } 0 \notin \partial g(x), \\ 0 & \text{if } x \in \mathscr{O}. \end{cases}$$

Then ν is well defined and locally Lipschitzian. Moreover, we have

$$g(x) = 0 \Rightarrow |\nu(x)| = 1,$$
$$g(x) = 0, \quad \alpha \in \partial g(x) \Rightarrow \langle \alpha, \nu(x) \rangle \geq \tfrac{1}{2}\lambda(x).$$

THEOREM 6.1. *The set M is a LNR.*

Proof. Let $\eta\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the flow generated by the Cauchy problem

$$\frac{\partial \eta}{\partial t}(x, t) = -\nu(\eta(x, t))$$
$$\eta(x, 0) = x.$$

Since $\eta(\cdot, -1)$ is a homeomorphism, the set

$$U = \{\eta(x, -1)\colon g(x) < 0\}$$

is open in $\mathbb{R}^n$. The same arguments as in [6, Lemma 3.4] show that

$$\forall x \in M\colon g(\eta(x, 1)) < 0,$$

so that $M \subseteq U$. Finally, for every $x \in U$ with $g(x) \geq 0$ there exists one and only one $\tau(x) \in [0, 1[$ such that $g(\eta(x, \tau(x))) = 0$ and the function τ is locally Lipschitzian with $\tau(x) = 0$ on $g^{-1}(0)$. Then

$$r(x) = \begin{cases} \eta(x, \tau(x)) & \text{if } x \in U \text{ and } g(x) \geq 0 \\ x & \text{if } g(x) \leq 0 \end{cases}$$

is a locally Lipschitzian retraction of U onto M. ∎

LEMMA 6.2. *For every compact subset $K \subseteq M$ there exist $c, r > 0$ such that*

$$(1 - t)x + t(y - \rho\nu(y)) \in M$$

whenever $x \in M$, $y \in K$, $c|x - y| \le \rho \le r$, and $t \in [0, 1]$.

Proof. By contradiction, let $c = h$, let $r = \frac{1}{h}$, and let $x_h \in M$, $y_h \in K$, $h|x_h - y_h| \le \rho_h \le \frac{1}{h}$, and $t_h \in [0, 1]$ be such that

$$g\big((1 - t_h)x_h + t_h(y_h - \rho_h\nu(y_h))\big) > 0.$$

Of course, we have $\rho_h > 0$ and $t_h > 0$. Also, up to a subsequence, we have $y_h \to y \in K$. It follows that $x_h \to y$ and $g(y) = 0$. By Lebourg's theorem [7], we may find z_h between x_h and $(y_h - \rho_h\nu(y_h))$ and $\alpha_h \in \partial g(z_h)$ such that

$$t_h\langle \alpha_h, (y_h - \rho_h\nu(y_h) - x_h\rangle > 0,$$

namely,

$$\left\langle \alpha_h, \frac{y_h - x_h}{\rho_h} \right\rangle > \langle \alpha_h, \nu(y_h)\rangle.$$

Up to a subsequence, we have $\alpha_h \to \alpha \in \partial g(y)$. It follows that $\langle \alpha, \nu(y)\rangle \le 0$, which is absurd. ∎

Now let X and $f\colon L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ be defined as in (3.6), (3.7).

THEOREM 6.3. *For every $\gamma \in X$, $\lambda > f(\gamma)$, and $\varepsilon > 0$, there exist $\delta > 0$ and a map*

$$\mathscr{H}\colon \{\eta \in X\colon \|\eta - \gamma\|_2 < \delta, f(\eta) < \lambda + \delta\} \times [0, 1] \to X$$

such that $\mathscr{H}$ is continuous from the topology of $L^2(0, 1; \mathbb{R}^n) \times \mathbb{R}$ to that of $L^2(0, 1; \mathbb{R}^n)$ and

$$\|\mathscr{H}(\eta, t) - \eta\|_\infty \le \varepsilon t,$$
$$\|\mathscr{H}(\eta, t) - \eta\|_{1,2} \le (\varepsilon + \sqrt{8\lambda})t,$$
$$f(\mathscr{H}(\eta, t)) \le f(\eta) + t(f(\gamma) - f(\eta) + \varepsilon).$$

Proof. Let $K = \gamma([0, 1])$ and let $c, r > 0$ be given by the previous lemma. Let

$$\hat{\gamma}(s) = \gamma(s) - \rho\nu(\gamma(s)),$$

where $\rho \in \,]0, r]$ is such that

$$\|\hat{\gamma} - \gamma\|_{\infty} \leq \frac{\varepsilon}{2}, \qquad \|\hat{\gamma} - \gamma\|_{1,2} \leq \frac{\varepsilon}{2}, \qquad f(\hat{\gamma}) \leq f(\gamma) + \varepsilon.$$

Let $\delta > 0$ be such that

$$\sqrt{8\lambda + \delta^2 + 4\delta} \leq \frac{\varepsilon}{2} + \sqrt{8\lambda},$$

$$\forall \eta \in X: \qquad \|\eta - \gamma\|_2 < \delta, \quad f(\eta) < \lambda + \delta \Rightarrow \|\eta - \gamma\|_{\infty} \leq \min\left\{\frac{\rho}{c}, \frac{\varepsilon}{2}\right\}.$$

Then, from Lemma 6.2 we deduce that

$$\mathscr{H}(\eta, t) := (1 - t)\eta + t\hat{\gamma} \in X$$

whenever $\eta \in X$, $\|\eta - \gamma\|_2 < \delta$, $f(\eta) < \lambda + \delta$, and $t \in [0, 1]$. Of course, $\mathscr{H}$ is continuous from the topology of $L^2 \times \mathbb{R}$ to that of L^2. Moreover, we have

$$\|\mathscr{H}(\eta, t) - \eta\|_{\infty} = t\|\hat{\gamma} - \eta\|_{\infty} \leq (\|\hat{\gamma} - \gamma\|_{\infty} + \|\gamma - \eta\|_{\infty})t \leq \varepsilon t,$$

$$\|\mathscr{H}(\eta, t) - \eta\|_{1,2} = t\|\hat{\gamma} - \eta\|_{1,2} \leq (\|\hat{\gamma} - \gamma\|_{1,2} + \|\gamma - \eta\|_{1,2})t$$

$$\leq \left(\frac{\varepsilon}{2} + \left(\delta^2 + 2\|\gamma'\|_2^2 + 2\|\eta'\|_2^2\right)^{1/2}\right)t$$

$$= \left(\frac{\varepsilon}{2} + \left(\delta^2 + 4f(\gamma) + 4f(\eta)\right)^{1/2}\right)t$$

$$\leq \left(\frac{\varepsilon}{2} + \left(\delta^2 + 8\lambda + 4\delta\right)^{1/2}\right)t$$

$$\leq \left(\varepsilon + \sqrt{8\lambda}\right)t.$$

Finally, we have

$$f(\mathscr{H}(\eta, t)) = f(\eta + t(\hat{\gamma} - \eta)) \leq f(\eta) + t(f(\hat{\gamma}) - f(\eta))$$

$$\leq f(\eta) + t(f(\gamma) - f(\eta) + \varepsilon),$$

whence the assertion.   ∎

Now we can prove the main result of this section.

THEOREM 6.4.   *For every $(\gamma, \lambda) \in \mathrm{epi}(f)$ with $\lambda > f(\gamma)$, we have $|d\mathscr{G}_f|(\gamma, \lambda) = 1$.*

Proof.   Let $\sigma = \frac{1}{5}(\lambda - f(\gamma))$, let $\varepsilon \in \,]0, \sigma]$, and let

$$\mathscr{H}: \{\eta \in X: \|\eta - \gamma\|_2 < \delta, f(\eta) < \lambda + \delta\} \times [0, 1] \to X$$

be as in the previous theorem. In particular, it is $\|\mathscr{H}(\eta, t) - \eta\|_2 \leq \varepsilon t$.

By reducing δ, we can assume that

$$\delta \leq 1, \qquad \delta \leq \sigma, \qquad \delta(f(\gamma) + \sigma) \leq \sigma.$$

Define

$$\mathscr{K}: (B_\delta(\gamma, \lambda) \cap \mathrm{epi}(f)) \times [0, \delta] \to \mathrm{epi}(f)$$

by $\mathscr{K}((\eta, \mu), t) = (\mathscr{H}(\eta, t), \mu - \sigma t)$.

If $f(\eta) \geq \lambda - 3\sigma$, we have

$$f(\mathscr{H}(\eta, t)) \leq f(\eta) - (f(\eta) - f(\gamma) - \sigma)t \leq f(\eta) - \sigma t \leq \mu - \sigma t.$$

On the other hand, if $f(\eta) \leq \lambda - 3\sigma$, we have

$$\begin{aligned} f(\mathscr{H}(\eta, t)) &\leq (1 - t)f(\eta) + t(f(\gamma) + \varepsilon) \\ &\leq (1 - t)(\lambda - 3\sigma) + t(f(\gamma) + \sigma) \\ &\leq \lambda - 2\sigma \leq \lambda - \delta - \sigma \leq \mu - \sigma t. \end{aligned}$$

Therefore $\mathscr{K}$ takes actually its values in $\mathrm{epi}(f)$. Since

$$d(\mathscr{K}((\eta, \mu), t), (\eta, \mu)) \leq \sqrt{\varepsilon^2 + \sigma^2}\, t,$$

we have

$$|d\mathscr{G}_f|(\gamma, \lambda) \geq \frac{\sigma}{\sqrt{\varepsilon^2 + \sigma^2}}$$

and the assertion follows from the arbitrariness of ε. ∎

7. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1. Let X and $f\colon L^2(0, 1; \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ be defined as in (3.6), (3.7). Since M is closed in $\mathbb{R}^n$, f is lower semicontinuous.

Assume first that M is simply connected. Consider also the continuous function $\mathscr{G}_f\colon \mathrm{epi}(f) \to \mathbb{R}$. It is easy to see that $(\mathscr{G}_f)^0$ is homeomorphic to M. From Theorem 6.1 and Corollary 5.5 we deduce that $H^q((\mathscr{G}_f)^0) = \{0\}$ and $H^q(\mathrm{epi}(f)) \neq \{0\}$ for some $q \in \mathbb{N}$. Of course, $\mathrm{epi}(f)$, endowed with the metric of $L^2 \times \mathbb{R}$, is complete. Since M is compact, it is readily seen that $(\mathscr{G}_f)^b$ is compact for any $b \in \mathbb{R}$. In particular, $\mathscr{G}_f$ satisfies $(PS)_c$ for any $c \in \mathbb{R}$. By Theorem 2.7 there exists $(\gamma, \lambda) \in \mathrm{epi}(f)$ with $\lambda > 0$ and $|d\mathscr{G}_f|(\gamma, \lambda) = 0$. From Theorem 6.4 it follows that $\lambda = f(\gamma)$. Therefore $|df|(\gamma) = 0$ and $\gamma \in X$ is not a constant curve. From Theorem 3.8 the assertion follows.

Now assume that M is not simply connected. The set

$$X_1 = \{\gamma \in X: \gamma \text{ is not contractible in } M\}$$

is sequentially weakly closed in $W^{1,2}(0,1;\mathbb{R}^n)$ and nonempty by Theorem 6.1 and Proposition 5.6. Since M is compact, it is easy to show that f restricted to X_1 admits a minimum point γ. Of course, $|df|(\gamma) = 0$ and γ is not constant. As in the previous case, we get the existence of a nonconstant closed geodesic on M. ∎

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