

Some Simple Algorithms for the Evaluations and Representations of the Riemann Zeta Function at Positive Integer Arguments

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Many interesting solutions of the so-called Basler problem of evaluating the Riemann zeta function $\zeta(s)$ when $s = 2$, which was of vital importance to Euler and the Bernoulli brothers (Jakob and Johann Bernoulli), have appeared in the mathematical literature ever since Euler first solved this problem in the year 1736. The main object of the present paper is to investigate rather systematically several interesting evaluations and representations of $\zeta(s)$ when $s \in \mathbb{N} \setminus \{1\}$. In one of many computationally useful special cases considered here, it is observed that $\zeta(3)$ can be represented by means of a series which converges much more rapidly than that in Euler's celebrated formula as well as the series used recently by Apéry in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of this series are capable of producing an accuracy of seven decimal places. © 2000 Academic Press

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1. INTRODUCTION AND DEFINITIONS

The Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, a)$, which are defined usually by

$$\zeta(s) := \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1-2^{-s}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} & (\Re(s) > 1) \\ \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} & (\Re(s) > 0; s \neq 1) \end{cases} \quad (1.1)$$

and

$$\zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1; a \neq 0, -1, -2, \dots), \quad (1.2)$$

so that, obviously,

$$\zeta(s, 1) = \zeta(s) = (2^s - 1)^{-1} \zeta\left(s, \frac{1}{2}\right) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1, \quad (1.3)$$

can indeed be continued meromorphically to the whole complex s -plane with a simple pole at $s = 1$ (with residue 1) (see, for details, Titchmarsh [32]). More generally, in terms of Lerch's transcendent (or the Hurwitz-Lerch zeta function) $\Phi(z, s, a)$ defined by (cf., e.g., Erdélyi *et al.* [9, p. 27, Eq. 1.11(1)]; see also Whittaker and Watson [34, p. 280, Example 8])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \quad (1.4)$$

$$(a \neq 0, -1, -2, \dots; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1),$$

it is easily observed that

$$\Phi(1, s, a) = \zeta(s, a) \quad (1.5)$$

and

$$\Phi\left(1, s, \frac{1}{2}\right) = \zeta\left(s, \frac{1}{2}\right) = (2^s - 1) \zeta(s). \quad (1.6)$$

The solution of the so-called *Basler problem* (cf., e.g., Spiess [26, p. 66]),

$$\zeta(2) := \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad (1.7)$$

was first found in 1736 by Leonhard Euler (1707–1783), although Jakob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748) did their utmost to sum the series in (1.7). In fact, the former of these Bernoulli brothers did not live to see the solution of the problem, and the solution became

known to the latter soon after Euler found it (see, for details, Knopp [16, p. 238]).

The Basler problem (1.7) has been solved in the mathematical literature in many different ways. In addition to numerous papers containing elementary proofs of (1.7), many of which are referenced by Stark [31], there are a fairly large number of books on complex analysis and advanced calculus in which (1.7) is proven by using Cauchy's residue calculus, Weierstrass' product theorem, Parseval's theorem, Fourier series expansions, and so on. Some of these books were referred to by Choe [5] who seems to have rediscovered one of Euler's remarkably elementary proofs of (1.7) (cf., e.g., Ayoub [3, p. 1079]), which is based upon the integration of the Taylor series expansion of $\arcsin x$ near $x = 0$

$$\arcsin x = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{x^{2k+1}}{2k+1} \quad (-1 \leq x \leq 1) \quad (1.8)$$

with respect to t (where $x = \sin t$), by means of Wallis' integral formula in the form:

$$\int_0^{\pi/2} \sin^{2k+1} t \, dt = \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{1 \cdot 3 \cdot 5 \cdots (2k+1)} \quad (k \in \mathbb{N} := \{1, 2, 3, \dots\}). \quad (1.9)$$

Motivated largely by the aforementioned (Euler's) proof of (1.7) detailed by Choe [5], Choi and Rathie [6] gave an essentially analogous derivation of (1.7) by appealing to the Gauss summation theorem (cf., e.g., Slater [25, p. 243, Eq. (III.3)]),

$${}_2F_1(\alpha, \beta; \gamma; 1) := \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \quad (1.10)$$

$$(\Re(\gamma - \alpha - \beta) > 0; \quad \gamma \neq 0, -1, -2, \dots),$$

where $(\lambda)_k := \Gamma(\lambda + k)/\Gamma(\lambda)$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_k = k!$). Subsequently, Choi *et al.* [7] showed how the derivation of (1.7) by Choi and Rathie [6] can be accomplished *without* using the Gauss summation theorem (1.10). They also presented several other evaluations of $\zeta(2)$ and related sums including (for example) a fairly straightforward evaluation based upon the theory of hypergeometric series. Indeed, in terms of a generalized hypergeometric ${}_mF_n$ function with m numerator and n denominator parameters (cf. Slater [25, Chap. 2]; see also Srivastava and Karlsson [30, p. 19 *et seq.*]), it readily follows from the definition (1.4) that

$$\Phi(z, n, a) = a^{-n} {}_{n+1}F_n(1, a, \dots, a; a+1, \dots, a+1; z) \quad (1.11)$$

$$(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

which, by means of the relationships (1.5) and (1.6), immediately yields the hypergeometric representations

$$\zeta(n, a) = a^{-n} {}_{n+1}F_n(1, a, \dots, a; a+1, \dots, a+1; 1) \quad (n \in \mathbb{N} \setminus \{1\}) \quad (1.12)$$

and

$$\zeta(n) = \left(\frac{2^n}{2^n - 1} \right) {}_{n+1}F_n \left(1, \frac{1}{2}, \dots, \frac{1}{2}; \frac{3}{2}, \dots, \frac{3}{2}; 1 \right) \quad (n \in \mathbb{N} \setminus \{1\}). \quad (1.13)$$

In particular, (1.13) with $n = 2$ yields

$$\zeta(2) = \frac{4}{3} {}_3F_2 \left(1, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; 1 \right), \quad (1.14)$$

which can be evaluated by applying Dixon's summation theorem [25, p. 243, Entry (III.8)] or Whipple's summation theorem [25, p. 245, Entry (III.23)].

Since

$$\zeta(n) = \int_0^1 \cdots \int_0^1 \frac{dt_1 \cdots dt_n}{1 - t_1 \cdots t_n} \quad (n \in \mathbb{N} \setminus \{1\}), \quad (1.15)$$

it is not difficult to provide yet another solution of the Basler problem (1.7) by using the changes of variables

$$t_1 = \frac{v+u}{\sqrt{2}} \quad \text{and} \quad t_2 = \frac{v-u}{\sqrt{2}}$$

in the double integral resulting from (1.15) when $n = 2$ (cf., e.g., Ojha and Singh [22]).

In view of the aforementioned developments stemming from the Basler problem (1.7), we aim at presenting here a systematic investigation of the problem of evaluation and representation of $\zeta(n)$ ($n \in \mathbb{N} \setminus \{1\}$). In one of many computationally useful special cases considered in this paper, we observe that $\zeta(3)$ can be represented by means of a series which converges much faster than that in Euler's celebrated formula (3.1) as well as the series (3.2) used recently by Apéry [1] in his proof of the irrationality of $\zeta(3)$. By symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux, we show (among other things) that only 50 terms of this series are capable of producing an accuracy of seven decimal places.

2. EVALUATION OF $\zeta(2n)$

The classical Bernoulli polynomials $B_n(x)$ of degree n in x are defined by the generating function:

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad (|z| < 2\pi). \quad (2.1)$$

In particular, for the Bernoulli numbers B_n defined by the generating function (2.1) with $x = 0$, it is known that (cf., e.g., Magnus *et al.* [18, pp. 26–27])

$$B_n = B_n(0) = (-1)^n B_n(1) \quad (n \in \mathbb{N}_0), \quad (2.2)$$

$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N}), \quad (2.3)$$

and

$$B_{2n} = 2(-1)^{n+1} \frac{(2n)!}{(2\pi)^{2n}} \zeta(2n) \quad (n \in \mathbb{N}_0). \quad (2.4)$$

We begin by recalling the following summation formula for the Bernoulli polynomials (cf. Hansen [13, p. 342, Entry (50.11.10)]),

$$\begin{aligned} & \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{(2k)!} B_{2k}(x) B_{n-2k}(y) \\ &= \frac{1}{2} \{ (x+y-1)nB_{n-1}(x+y) + (-1)^n(x-y)nB_{n-1}(x-y) \\ & \quad - (n-1)[B_n(x+y) + (-1)^n B_n(x-y)] \}, \end{aligned} \quad (2.5)$$

which, upon setting $x = 1$ and $y = 0$ (and replacing n by $2n$), yields

$$\begin{aligned} & \sum_{k=0}^n \binom{2n}{2k} B_{2k} B_{2n-2k} \\ &= -nB_{2n-1} - (2n-1)B_{2n} \quad (n \in \mathbb{N}_0), \end{aligned} \quad (2.6)$$

since

$$B_0 = 1 \quad \text{and} \quad B_2 = \frac{1}{6}.$$

Finally, by transposing to the right-hand side the terms for $k = 0$ and $k = n$ in (2.6), and making use of (2.3), we obtain

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1)B_{2n} \quad (n \in \mathbb{N} \setminus \{1\}) \quad (2.7)$$

or, equivalently,

$$\sum_{k=1}^{n-1} \zeta(2k) \zeta(2n-2k) = \left(n + \frac{1}{2}\right) \zeta(2n) \quad (n \in \mathbb{N} \setminus \{1\}), \quad (2.8)$$

in view of the relationship (2.4).

By setting $n = 2, 3, 4, \dots$ in (2.8), and applying the well-known result (1.7), we find that

$$\zeta(4) = \frac{2}{5} \{\zeta(2)\}^2 = \frac{\pi^4}{90}, \quad (2.9)$$

$$\zeta(6) = \frac{4}{7} \zeta(2) \zeta(4) = \frac{\pi^6}{945}, \quad (2.10)$$

$$\zeta(8) = \frac{4}{9} \zeta(2) \zeta(6) + \frac{2}{9} \{\zeta(4)\}^2 = \frac{\pi^8}{9450}, \quad (2.11)$$

and so on.

In order to give a *direct* proof of the summation formula (2.8), let us put (cf. Prudnikov [23])

$$\Omega(x) := \sum_{k=1}^{\infty} \exp(-\pi^2 k^2 x) \quad (x > 0), \quad (2.12)$$

so that

$$\int_0^x \Omega(x-t) \Omega(t) dt = x \Omega(x) + \frac{3}{2} \sum_{k=1}^{\infty} \frac{\exp(-\pi^2 k^2 x)}{\pi^2 k^2}, \quad (2.13)$$

which, by virtue of (1.7), readily yields

$$\lim_{x \rightarrow 0} \int_0^x \Omega(x-t) \Omega(t) dt = \frac{1}{4} \quad (2.14)$$

and

$$\int_0^x \Omega(x-t) \Omega(t) dt = O(xe^{-x}) \quad (x \rightarrow \infty). \quad (2.15)$$

Making use of (2.13) and (2.15), it is easily seen that

$$\begin{aligned} \mathcal{J}_s &:= \int_0^{\infty} \int_0^{\infty} (x+y)^{s-2} \Omega(x) \Omega(y) dx dy \\ &= \int_0^{\infty} x^{s-2} dx \int_0^x \Omega(x-t) \Omega(t) dt \\ &= (s + \tfrac{1}{2}) \frac{\Gamma(s-1)}{\pi^{2s}} \zeta(2s) \quad (\Re(s) > 1). \end{aligned} \quad (2.16)$$

On the other hand, since

$$\int_0^{\infty} t^{s-1} \Omega(t) dt = \frac{\Gamma(s)}{\pi^{2s}} \zeta(2s) \quad (\Re(s) > 0; s \neq \tfrac{1}{2}), \quad (2.17)$$

which is an immediate consequence of the definition (2.12) and (1.1), an appeal to the Binomial Theorem for $(x + y)^{n-2}$ in the form

$$(x + y)^{n-2} = \sum_{k=1}^{n-1} \binom{n-2}{k-1} x^{k-1} y^{n-k-1} \quad (n \in \mathbb{N} \setminus \{1\}) \quad (2.18)$$

would lead us also to

$$\mathcal{J}_n = \frac{(n-2)!}{\pi^{2n}} \sum_{k=1}^{n-1} \zeta(2k) \zeta(2n-2k) \quad (n \in \mathbb{N} \setminus \{1\}). \quad (2.19)$$

Now the summation formula (2.8) follows upon equating the two values of $\mathcal{J}_n$ given by (2.16) (*with* $s = n$) and (2.19).

Remark 1. The recursion formula (2.7) is a well-known (rather classical) result for Bernoulli numbers. It appears (for example) in Nielsen's book [21] and was also derived independently by Underwood [33]. The equivalent result (2.8) was proven, by elementary methods, by Williams [35, p. 20, Theorem I] (see also Apostol [2, p. 427]).

Remark 2. In terms of the $\mathcal{L}$ -function defined by [cf. Eq. (1.4)]

$$\mathcal{L}(s) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = 2^{-s} \Phi\left(-1, s, \frac{1}{2}\right) \quad (\Re(s) > 0), \quad (2.20)$$

Williams [35, p. 22, Theorem II] gave an interesting companion of the result (2.8) in the form

$$\sum_{k=1}^n \mathcal{L}(2k-1) \mathcal{L}(2n-2k+1) = \left(n - \frac{1}{2}\right) (1 - 2^{-2n}) \zeta(2n), \quad (2.21)$$

which appears erroneously in Hansen [13, p. 357, Entry (54.7.1)]. Since $\mathcal{L}(1)$ is the well-known Gregory series for $\frac{\pi}{4}$ (with $\mathcal{L}(2)$ being the familiar Catalan constant G), by setting $n = 1$ in (2.21) we immediately obtain

$$\zeta(2) = \frac{8}{3} \{\mathcal{L}(1)\}^2 = \frac{\pi^2}{6}. \quad (2.22)$$

Remark 3. In view of the constraint $\Re(s) > 0$ ($s \neq \frac{1}{2}$) associated with the Mellin transform in (2.17), an earlier attempt by Kalla and Villalobos [15] to extend the summation formula (2.8) by expressing $\mathcal{J}_s$ as a non-terminating (infinite) sum analogous to (2.19) cannot be justified. Thus their *main* result [15, p. 17, Eq. (17)] holds true only in the finite form given already by (2.8).

3. SERIES REPRESENTATIONS FOR $\zeta(n)$ ($n \in \mathbb{N} \setminus \{1\}$)

On the subject of series representations for $\zeta(3)$, the result

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}, \quad (3.1)$$

which is contained in a 1772 paper entitled “*Exercitationes Analyticae*” by Euler (see, e.g., Ayoub [3, pp. 1084–1085]), was rediscovered (among others) by Ramaswami [24] (see also Srivastava [27, p. 7, Eq. (2.23)]) and (more recently) by Ewell [10]. And, as pointed out by (for example) Chen and Srivastava [4, pp. 180–181], another interesting series representation,

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}, \quad (3.2)$$

which played a key rôle in Apéry’s proof [1] of the irrationality of $\zeta(3)$, was proven independently by (among others) Hjortnaes [14], Gosper [12], and Apéry [1]. Furthermore, in their recent work on the Ray–Singer torsion and topological field theories, Nash and O’Connor [19, 20] obtained a number of remarkable integral expressions for $\zeta(3)$, including (for example) the following result [20, p. 1489 *et seq.*]:

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 - \frac{8}{7} \int_0^{\pi/2} z^2 \cot z \, dz. \quad (3.3)$$

Since [9, p. 51, Eq. 1.20(3)]

$$z \cot z = -2 \sum_{k=0}^{\infty} \zeta(2k) \left(\frac{z}{\pi}\right)^{2k} \quad (|z| < \pi), \quad (3.4)$$

the result (3.3) is obviously equivalent to the series representation (cf. Dąbrowski [8, p. 202]; see also Chen and Srivastava [4, p. 191, Eq. (3.19)]):

$$\zeta(3) = \frac{2\pi^2}{7} \left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} \right). \quad (3.5)$$

Moreover, by integrating by parts, it is easily seen that

$$\int_0^{\pi/2} z^2 \cot z \, dz = -2 \int_0^{\pi/2} z \log \sin z \, dz, \quad (3.6)$$

so that the result (3.3) is equivalent also to the integral representation

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 + \frac{16}{7} \int_0^{\pi/2} z \log \sin z \, dz, \quad (3.7)$$

which was proved in the aforementioned 1772 paper by Euler (cf., e.g., Ayoub [3, p. 1084]).

Next, since

$$i \cot iz = \coth z = \frac{2}{e^{2z} - 1} + 1 \quad (i := \sqrt{-1}), \quad (3.8)$$

by replacing z in the known expansion (3.4) by $\frac{1}{2}i\pi z$, it is easily seen that (cf., e.g., Koblitz [17, p. 25]; see also Erdélyi *et al.* [9, p. 51, Eq. 1.20(1)])

$$\frac{\pi z}{e^{\pi z} - 1} + \frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \zeta(2k)}{2^{2k-1}} z^{2k} \quad (|z| < 2). \quad (3.9)$$

By setting $z = it$ in (3.9), multiplying both sides by t^{m-1} ($m \in \mathbb{N}$), and then integrating the resulting equation from $t = 0$ to $t = \tau$ ($0 < \tau < 2$), if we apply the (readily derivable) integral formula

$$\int_0^\tau t^m e^{\lambda t} dt = \frac{(-1)^m m! e^{\lambda \tau}}{\lambda^{m+1}} \left(\sum_{j=0}^m \frac{(-\lambda \tau)^j}{j!} - e^{-\lambda \tau} \right) \quad (\lambda \neq 0), \quad (3.10)$$

we shall obtain

$$\begin{aligned} m! \left(\frac{i}{\pi \tau} \right)^m & \left\{ \sum_{k=1}^{\infty} \frac{\cos(k\pi\tau) + i \sin(k\pi\tau)}{k^{m+1}} - \zeta(m+1) \right\} \\ & + \sum_{k=1}^{\infty} \frac{\cos(k\pi\tau)}{k} + i \left(\frac{\pi\tau}{2(m+1)} + \sum_{k=1}^{\infty} \frac{\sin(k\pi\tau)}{k} \right) \\ & + \sum_{j=1}^{m-1} \frac{m!}{(m-j)!} \left(\frac{i}{\pi\tau} \right)^j \sum_{k=1}^{\infty} \frac{\cos(k\pi\tau) + i \sin(k\pi\tau)}{k^{j+1}} \\ & = \sum_{k=0}^{\infty} \frac{\zeta(2k)}{k + \frac{1}{2}m} \left(\frac{\tau}{2} \right)^{2k} \quad (m \in \mathbb{N}; 0 < \tau < 2), \end{aligned} \quad (3.11)$$

which, in the special case when $\tau = 1$, would immediately simplify to the form

$$\begin{aligned} & \frac{i\pi}{2(m+1)} - \left(2^{m+1} - 1 \right) \left(\frac{i}{2\pi} \right)^m m! \zeta(m+1) \\ & = \log 2 + \sum_{j=1}^{m-1} \frac{m!}{(m-j)!} (2^j - 1) \left(\frac{i}{2\pi} \right)^j \zeta(j+1) \\ & \quad + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k + \frac{1}{2}m) 2^{2k}} \quad (m \in \mathbb{N}), \end{aligned} \quad (3.12)$$

where we have also applied the last part of the definition (1.1). Furthermore, in (3.11), (3.12), and elsewhere in this paper, an empty sum is interpreted (as usual) to be nil.

Now, for a given sequence $\{\Omega_n\}_{n=1}^\infty$, it is easily verified that

$$\sum_{j=1}^{m-1} \Omega_j = \sum_{j=1}^{[(m-1)/2]} \Omega_{2j} + \sum_{j=1}^{[m/2]} \Omega_{2j-1}, \quad (3.13)$$

where, just as in (2.5), $[\kappa]$ denotes the greatest integer in κ . Thus we find from (3.12) that

$$\begin{aligned} \frac{i\pi}{2(m+1)} - \left(2^{m+1} - 1\right) \left(\frac{i}{2\pi}\right)^m m! \zeta(m+1) \\ = \log 2 + \sum_{j=1}^{[(m-1)/2]} (-1)^j \binom{m}{2j} \frac{(2j)! (2^{2j} - 1)}{(2\pi)^{2j}} \zeta(2j+1) \\ + i \sum_{j=1}^{[m/2]} (-1)^{j-1} \binom{m}{2j-1} \frac{(2j-1)! (2^{2j-1} - 1)}{(2\pi)^{2j-1}} \zeta(2j) \\ + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k + \frac{1}{2}m) 2^{2k}} \quad (m \in \mathbb{N}). \end{aligned} \quad (3.14)$$

Setting $m = 2n$ ($n \in \mathbb{N}$) in (3.14), and then equating the real and imaginary parts in the resulting equation, we obtain

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)! (2^{2n+1} - 1)} \\ \cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n}{2j} \frac{(2j)! (2^{2j} - 1)}{(2\pi)^{2j}} \zeta(2j+1) \right. \\ \left. + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n) 2^{2k}} \right] \quad (n \in \mathbb{N}) \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n-1}}{(2n)! (2^{2n-1} - 1)} \\ \cdot \left[\frac{\pi}{2(2n+1)} + \sum_{j=1}^{n-1} (-1)^j \binom{2n}{2j-1} \right. \\ \left. \cdot \frac{(2j-1)! (2^{2j-1} - 1)}{(2\pi)^{2j-1}} \zeta(2j) \right] \quad (n \in \mathbb{N}). \end{aligned} \quad (3.16)$$

Similarly, if we set $m = 2n + 1$ ($n \in \mathbb{N}_0$) in (3.14), we shall obtain the following results which can provide series representations for $\zeta(2n + 1)$ and $\zeta(2n)$ when $n \in \mathbb{N}$:

$$\begin{aligned} \sum_{j=1}^n (-1)^{j-1} \binom{2n+1}{2j} \frac{(2j)! (2^{2j} - 1)}{(2\pi)^{2j}} \zeta(2j + 1) \\ = \log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k + n + \frac{1}{2}) 2^{2k}} \quad (n \in \mathbb{N}_0) \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n-1}}{(2n-1)! (2^{2n} - 1)} \\ \cdot \left[\frac{\pi}{4n} + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} \cdot \frac{(2j-1)! (2^{2j-1} - 1)}{(2\pi)^{2j-1}} \zeta(2j) \right] \quad (n \in \mathbb{N}). \end{aligned} \quad (3.18)$$

Indeed, in the particular cases when $n = 0$ and $n \in \mathbb{N}$, the summation formula (3.17) immediately yields the known sum:

$$\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1) 2^{2k}} = -\frac{1}{2} \log 2 \quad (3.19)$$

and

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)! (2^{2n} - 1)} \\ \cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n+1}{2j} \frac{(2j)! (2^{2j} - 1)}{(2\pi)^{2j}} \zeta(2j + 1) \right. \\ \left. + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k + n + \frac{1}{2}) 2^{2k}} \right] \quad (n \in \mathbb{N}), \end{aligned} \quad (3.20)$$

respectively.

Each of the recursion formulas (3.16) and (3.18) can be used to evaluate $\zeta(2n)$ ($n \in \mathbb{N}$). Formula (3.16) was proved, in a markedly different way, by Stark (cf., e.g., [31, p. 199, Eq. (5)]). Formulas (3.16) and (3.18), together, yield the series identity

$$\sum_{j=1}^{n-1} (-1)^{j-1} \binom{2n-1}{2j-1} \frac{(2j-1)! (2^{2j-1} - 1)}{(2\pi)^{2j-1}} \zeta(2j)$$

$$\begin{aligned}
&= \frac{\pi \{(2n-1)2^{2n-1} - 2n\}}{4n(2n+1)(2^{2n-1} - 1)} \\
&\quad + \frac{2^{2n} - 1}{2n(2^{2n-1} - 1)} \sum_{j=1}^{n-1} (-1)^{j-1} \binom{2n}{2j-1} \\
&\quad \cdot \frac{(2j-1)!(2^{2j-1} - 1)}{(2\pi)^{2j-1}} \zeta(2j) \quad (3.21) \\
&\quad (n \in \mathbb{N})
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\sum_{j=1}^{n-1} (-1)^{j-1} \binom{2n-1}{2j-2} \{2(n-j)(2^{2n-1} - 1) - 2^{2n-1}\} \\
&\quad \cdot \frac{(2j-2)!(2^{2j-1} - 1)}{(2\pi)^{2j-1}} \zeta(2j) \\
&= \frac{\pi \{(2n-1)2^{2n-1} - 2n\}}{4n(2n+1)} \quad (n \in \mathbb{N}), \quad (3.22)
\end{aligned}$$

which can also be used as a recurrence relation for evaluating $\zeta(2n)$ ($n \in \mathbb{N}$).

An integral representation for $\zeta(2n+1)$, which is equivalent to the series representation (3.15), was given earlier by Dąbrowski [8, p. 203, Eq. (16)], who [8, p. 206] also mentioned the existence of (but did not fully state) the series representation (3.20).

For $n = 1$, (3.15) immediately yields the series representation (3.5) which, in conjunction with the known sum (3.19), would lead us readily to Euler's formula (3.1). On the other hand, by setting $n = 1$ in the series representation (3.20), we obtain

$$\zeta(3) = \frac{4\pi^2}{9} \left(\frac{1}{2} \log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)2^{2k}} \right), \quad (3.23)$$

which was derived *independently* by (for example) Glasser [11, p. 446, Eq. (12)], Zhang and Williams [26, p. 1585, Eq. (2.13)], and Dąbrowski [8, p. 206] (see also Chen and Srivastava [4, p. 183, Eq. (2.15)]).

By suitably combining the series occurring in (3.5), (3.19), and (3.23), it is not difficult to derive several other series representations for $\zeta(3)$, which are analogous to Euler's formula (3.1). More generally, since

$$\begin{aligned}
&\frac{\lambda k^2 + \mu k + \nu}{(2k+2n-1)(2k+2n)(2k+2n+1)} \\
&= \frac{\mathcal{A}}{2k+2n-1} + \frac{\mathcal{B}}{2k+2n} + \frac{\mathcal{C}}{2k+2n+1}, \quad (3.24)
\end{aligned}$$

where, for convenience,

$$\mathcal{A} = \mathcal{A}_n(\lambda, \mu, \nu) := \frac{1}{2} \left[\lambda n^2 - (\lambda + \mu)n + \frac{1}{4}(\lambda + 2\mu + 4\nu) \right], \quad (3.25)$$

$$\mathcal{B} = \mathcal{B}_n(\lambda, \mu, \nu) := -(\lambda n^2 - \mu n + \nu), \quad (3.26)$$

and

$$\mathcal{C} = \mathcal{C}_n(\lambda, \mu, \nu) := \frac{1}{2} \left[\lambda n^2 + (\lambda - \mu)n + \frac{1}{4}(\lambda - 2\mu + 4\nu) \right], \quad (3.27)$$

by appealing to (3.15), (3.17) with n replaced by $n - 1$, and (3.20), we can derive the following unification of a large number of known (or new) series representations including (for example) Euler's result (3.1)

$$\begin{aligned} \zeta(2n+1) = & \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)! \{ (2^{2n+1} - 1)\mathcal{B} + (2n+1)(2^{2n} - 1)\mathcal{C} \}} \\ & \cdot \left[\frac{1}{4} \lambda \log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \right. \\ & \cdot \left\{ 2j(2j-1)\mathcal{A} + [\lambda(4n-1) - 2\mu]nj + \lambda n \left(n + \frac{1}{2} \right) \right\} \\ & \cdot \frac{(2j-2)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & \left. + \sum_{k=0}^{\infty} \frac{(\lambda k^2 + \mu k + \nu)\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}} \right] \\ & (n \in \mathbb{N}; \lambda, \mu, \nu \in \mathbb{C}), \quad (3.28) \end{aligned}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by (3.25), (3.26), and (3.27), respectively.

For $\lambda = 0$, the series representation (3.28) simplifies to the form:

$$\begin{aligned} \zeta(2n+1) = & \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)! \{ (2^{2n+1} - 1)(\mu n - \nu) - (2^{2n} - 1)(n + \frac{1}{2})[\mu(n + \frac{1}{2}) - \nu] \}} \\ & \cdot \left[\sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \left\{ j(2j-1) \left[\nu - \mu \left(n - \frac{1}{2} \right) \right] - 2\mu nj \right\} \right. \\ & \cdot \frac{(2j-2)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & \left. + \sum_{k=0}^{\infty} \frac{(\mu k + \nu)\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}} \right] \\ & (n \in \mathbb{N}; \mu, \nu \in \mathbb{C}). \quad (3.29) \end{aligned}$$

Furthermore, by setting

$$\lambda = \mu = 0 \quad \text{and} \quad \nu = 1$$

in (3.28) or (alternatively) by setting

$$\mu = 0 \quad \text{and} \quad \nu = 1$$

in (3.29), we immediately obtain the series representation

$$\begin{aligned} \zeta(2n+1) = & \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)! \{2^{2n}(2n-3) - 2n+1\}} \\ & \cdot \left[\sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \frac{(2j)! (2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \right. \\ & \left. + 2 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}} \right] \\ & (n \in \mathbb{N}), \end{aligned} \quad (3.30)$$

which, in the special case when $n = 1$, was given by Chen and Srivastava [4, p. 189, Eq. (2.45)].

Of the three representations (3.28), (3.29), and (3.30) for $\zeta(2n+1)$ ($n \in \mathbb{N}$), the infinite series in (3.30) converges most rapidly. Many other families of rapidly convergent series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) can be found in the recent works [28] and [29] (and indeed also in the numerous references already cited in each of these earlier works).

For various suitable special values of the parameters λ , μ , and ν , we can easily deduce from (3.28) and (3.29) several known (or new) series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$). For example, if we set

$$\mu = 2 \quad \text{and} \quad \nu = 2n+1$$

in our series representation (3.29), we shall obtain

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!(2^{2n+1}-1)} \left[\sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-1} \right. \\ & \cdot \frac{(2j)! (2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & \left. - \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n-1)(k+n)2^{2k}} \right] \quad (n \in \mathbb{N}), \end{aligned} \quad (3.31)$$

which, in the special case when $n = 1$, immediately yields Euler's formula (3.1).

The following additional series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$), which are analogous to (3.31), can also be deduced similarly from (3.29):

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)! \{(2n-1)2^{2n} - 2n\}} \left[2n \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \right. \\ & \cdot \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & \left. - \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n)(2k+2n+1)2^{2k}} \right] \quad (n \in \mathbb{N}) \quad (3.32) \end{aligned}$$

and

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)} \left[\sum_{j=1}^{n-1} (-1)^j \binom{4nj-2j+1}{2j-1} \binom{2n-1}{2j-2} \right. \\ & \cdot \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & \left. - 4 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n-1)(2k+2n+1)2^{2k}} \right] \quad (n \in \mathbb{N}). \quad (3.33) \end{aligned}$$

The special case of each of the last two series representations (3.32) and (3.33) when $n = 1$ was given by Zhang and Williams [36, p. 1586]. For $n = 2$, our results (3.30) to (3.33) would readily yield the following (presumably new) series representations for $\zeta(5)$:

$$\zeta(5) = \frac{\pi^2}{13} \zeta(3) - \frac{8\pi^4}{13} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)(2k+4)(2k+5)2^{2k}}, \quad (3.34)$$

$$\zeta(5) = \frac{3\pi^2}{31} \zeta(3) + \frac{4\pi^4}{93} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)(2k+4)2^{2k}}, \quad (3.35)$$

$$\zeta(5) = \frac{\pi^2}{11} \zeta(3) + \frac{\pi^4}{33} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+4)(2k+5)2^{2k}}, \quad (3.36)$$

and

$$\zeta(5) = \frac{7\pi^2}{75} \zeta(3) + \frac{8\pi^4}{225} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)(2k+5)2^{2k}}. \quad (3.37)$$

Next, with a view to *further* improving the rate of convergence in the most rapidly convergent series representations (3.30) and (3.34) considered in this work, we observe that

$$\begin{aligned} & \frac{1}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)} \\ &= \frac{1}{6} \left(\frac{1}{2k+2n-1} - \frac{1}{2k+2n+2} \right) \\ & \quad - \frac{1}{2} \frac{1}{(2k+2n)(2k+2n+1)}. \end{aligned} \quad (3.38)$$

Thus, by applying our series representations (3.17) *with* n replaced by $n-1$, (3.15) *with* n replaced by $n+1$, and (3.32), we obtain

$$\begin{aligned} \zeta(2n+3) &= \frac{2\pi^2 \{2^{2n+2} + n(2n-3)(2^{2n}-1) - 1\}}{(n+1)(2n+1)(2^{2n+3}-1)} \zeta(2n+1) \\ &+ (-1)^{n-1} \frac{(2\pi)^{2n+2}}{(2n+2)!(2^{2n+3}-1)} \\ &\cdot \left[\sum_{j=1}^{n-1} (-1)^j \left\{ \binom{2n-1}{2j} - \binom{2n+2}{2j} \right\} \right. \\ &+ 6n \binom{2n-1}{2j-2} \left. \right\} \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ &+ 12 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)2^{2k}} \Big] \\ &\quad (n \in \mathbb{N}), \end{aligned} \quad (3.39)$$

where the series converges faster than that in (3.30).

In its special case when $n=1$, (3.39) readily yields the following improved version of the series representation (3.34) above (cf. [36, p. 1590, Eq. (3.14)]),

$$\zeta(5) = \frac{4\pi^2}{31} \zeta(3) + \frac{8\pi^4}{31} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)2^{2k}}, \quad (3.40)$$

in which $\zeta(3)$ can be replaced by its known value $-4\pi^2 \zeta'(-2)$ given by (cf., e.g., [29, p. 387, Eq. (1.15)])

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \zeta'(-2n) \quad (n \in \mathbb{N}) \quad (3.41)$$

for $n=1$.

Yet another rapidly convergent series representation for $\zeta(2n+3)$ ($n \in \mathbb{N}$), analogous to (3.39), can be derived by means of the identity

$$\begin{aligned} & \frac{1}{(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)} \\ &= \frac{1}{6} \left(\frac{1}{2k+2n} - \frac{1}{2k+2n+3} \right) \\ & \quad - \frac{1}{2} \frac{1}{(2k+2n+1)(2k+2n+2)}, \end{aligned} \quad (3.42)$$

together with our series representations (3.15), (3.20) *with* n replaced by $n+1$, and (3.31) *with* n replaced by $n+1$. We thus obtain the series representation

$$\begin{aligned} \zeta(2n+3) = & \frac{2\pi^2 \left\{ \frac{1}{3}(2n+1)(2n^2-4n+3)(2^{2n}-1) - 2^{2n+1} + 1 \right\}}{(n+1)(2n+1)\{(2n-3)2^{2n+2} - 2n\}} \zeta(2n+1) \\ & + (-1)^{n-1} \frac{(2\pi)^{2n+2}}{(2n+2)!\{(2n-3)2^{2n+2} - 2n\}} \\ & \cdot \left[\sum_{j=1}^{n-1} (-1)^j \left\{ \binom{2n}{2j} - \binom{2n+3}{2j} \right\} \right. \\ & + 3 \binom{2n+1}{2j-1} \left. \right\} \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & + 12 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)2^{2k}} \Big] \\ & (n \in \mathbb{N}), \end{aligned} \quad (3.43)$$

which, in the special case when $n=1$, yields

$$\zeta(5) = \frac{2\pi^2}{27} \zeta(3) - \frac{4\pi^4}{9} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2)(2k+3)(2k+4)(2k+5)2^{2k}}, \quad (3.44)$$

where the series obviously converges faster than that in (3.34).

Finally, by applying the identity

$$\begin{aligned} & \frac{1}{2k(2k+2n-1)(2k+2n)(2k+2n+1)} \\ &= \frac{1}{2n(2n-1)(2n+1)} \frac{1}{2k} - \frac{1}{2(2n-1)} \frac{1}{2k+2n-1} \\ & \quad + \frac{1}{2n} \frac{1}{2k+2n} - \frac{1}{2(2n+1)} \frac{1}{2k+2n+1} \end{aligned} \quad (3.45)$$

in conjunction with our series representations (3.17) with n replaced by $n - 1$, (3.15), (3.20), and the known result (cf., e.g., [13, p. 356, Entry (54.5.3)]),

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} t^{2k} = \log [\pi t \csc(\pi t)] \quad (3.46)$$

with $t = \frac{1}{2}$, we arrive at the following series representation for $\zeta(2n + 1)$ ($n \in \mathbb{N}$)

$$\begin{aligned} \zeta(2n + 1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{2 \cdot (2n - 1)! \{n - 1 - (n - 2)2^{2n}\}} \\ & \cdot \left[\frac{12n^2 - 1}{2n^2(4n^2 - 1)^2} - \frac{\log \pi}{n(4n^2 - 1)} \right. \\ & - \sum_{j=2}^{n-1} (-1)^j \binom{2n-2}{2j-3} \frac{(2j-1)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\ & \left. + \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+n)(2k+2n-1)(2k+2n+1)2^{2k}} \right] \\ & (n \in \mathbb{N}), \end{aligned} \quad (3.47)$$

where we have also applied the fact that $\zeta(0) = -\frac{1}{2}$.

For $n = 1$, (3.47) reduces immediately to Wilton's formula (cf. Hansen [13, p. 357, Entry (54.5.9)]; see also Chen and Srivastava [4, p. 181, Equation (2.1)]):

$$\zeta(3) = \frac{\pi^2}{2} \left(\frac{11}{18} - \frac{1}{3} \log \pi + \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)(2k+1)(2k+3)2^{2k}} \right). \quad (3.48)$$

Furthermore, in its special case when $n = 2$, (3.47) would yield the following new companion of the series representations (3.40) and (3.44):

$$\zeta(5) = \frac{2\pi^4}{45} \left(\log \pi - \frac{47}{60} - 30 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+2)(2k+3)(2k+5)2^{2k}} \right), \quad (3.49)$$

which does *not* contain a term involving $\zeta(3)$ on the right-hand side.

By eliminating $\zeta(2n + 3)$ between our results (3.39) and (3.43), we can obtain a series representation for $\zeta(2n + 1)$ ($n \in \mathbb{N}$), which would converge as rapidly as the series in (3.47). We thus find that

$$\begin{aligned} & \zeta(2n + 1) \\ &= (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)! \Delta_n} \left[\sum_{j=1}^{n-1} (-1)^j \left\{ (2n-3)2^{2n+2} - 2n \right\} \right. \end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ \binom{2n-1}{2j} - \binom{2n+2}{2j} + 6n \binom{2n-1}{2j-2} \right\} \\
& - (2^{2n+3} - 1) \left\{ \binom{2n}{2j} - \binom{2n+3}{2j} + 3 \binom{2n+1}{2j-1} \right\} \left(\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \right. \\
& \left. + 12 \sum_{k=0}^{\infty} \frac{(\xi_n k + \eta_n) \zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)2^{2k}} \right] \\
& (n \in \mathbb{N}), \tag{3.50}
\end{aligned}$$

where, for convenience,

$$\begin{aligned}
\Delta_n := & (2^{2n+3} - 1) \left\{ \frac{1}{3} (2n+1)(2n^2 - 4n + 3)(2^{2n} - 1) - 2^{2n+1} + 1 \right\} \\
& - \{(2n-3)2^{2n+2} - 2n\} \{2^{2n+2} + n(2n-3)(2^{2n} - 1) - 1\}, \tag{3.51}
\end{aligned}$$

$$\xi_n := 2\{(2n-5)2^{2n+2} - 2n + 1\}, \tag{3.52}$$

and

$$\eta_n := (4n^2 - 4n - 7)2^{2n+2} - (2n+1)^2. \tag{3.53}$$

In its special case when $n = 1$, (3.50) yields the (presumably new) series representation

$$\zeta(3) = -\frac{6\pi^2}{23} \sum_{k=0}^{\infty} \frac{(98k+121)\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)2^{2k}}, \tag{3.54}$$

where the series obviously converges much more rapidly than that in *each* of the celebrated results (3.1) and (3.2).

We conclude this work by summarizing below the results of our symbolic and numerical computations with the series in (3.54) using *Mathematica* (Version 4.0) for Linux.

$$\begin{aligned}
\text{In}[1] := & (98k+121)\text{Zeta}[2k]/((2k+1)(2k+2) \\
& \cdot (2k+3)(2k+4)(2k+5)2^{2k})
\end{aligned}$$

$$\text{Out}[1] = \frac{(121+98k)\text{Zeta}[2k]}{2^{2k}(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)}$$

$$\text{In}[2] := \text{Sum}[\%, \{k, 1, \text{Infinity}\}]/\text{Simplify}$$

$$\text{Out}[2] = \frac{121}{240} - \frac{23 \text{Zeta}[3]}{6 \text{Pi}^2}$$

$$\text{In}[3] := \text{N}[\%]$$

$$\text{Out}[3] = 0.0372903$$

$$\text{In}[4] := \text{Sum}[\text{N}[\%1]/\text{Evaluate}, \{k, 1, 50\}]$$

$$\text{Out}[4] = 0.0372903$$

$$\text{In}[5] := \text{N Sum}[\%1/\text{Evaluate}, \{k, 1, \text{Infinity}\}]$$

$$\text{Out}[5] = 0.0372903$$

Since $\zeta(0) = -\frac{1}{2}$, Out [2] evidently validates our series representation (3.54) symbolically. Furthermore, our numerical computations in Out [3], Out [4], and Out [5], together, exhibit the fact that only 50 terms ($k = 1$ to $k = 50$) of the series in (3.54) can produce an accuracy of seven decimal places.

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