



Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps

M.O. Osilike¹

Department of Mathematics, University of Nigeria, Nsukka, Nigeria

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Abstract

Convergence theorems for approximation of common fixed points of strictly pseudocontractive mappings of Browder–Petryshyn type are proved in Banach spaces using an implicit iteration scheme recently introduced by Xu and Ori [Numer. Funct. Anal. Optim. 22 (2001) 767–773].

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1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^*: \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j .

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Browder and Petryshyn [2] if there exists $\lambda > 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|x - y - (Tx - Ty)\|^2, \quad (1)$$

E-mail address: osilike@yahoo.com.

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for all $x, y \in D(T)$ and for all $j(x - y) \in J(x - y)$. Without loss of generality we may assume $\lambda \in (0, 1)$. If I denotes the identity operator, then (1) can be written in the form

$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq \lambda \|(I - T)x - (I - T)y\|^2. \quad (2)$$

In Hilbert spaces H , (1) (and hence (2)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad k = (1 - 2\lambda) < 1, \quad (3)$$

and we can assume also that $k \geq 0$, so that $k \in [0, 1)$.

The class of strictly pseudocontractive mappings has been studied by several authors (see for example [2,5,6,8,9,12,13]). It is shown in [8] that a strictly pseudocontractive map is L -Lipschitzian (i.e., $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in D(T)$ and for some $L > 0$). It is clear that in Hilbert spaces the important class of *nonexpansive* mappings (mappings T for which $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in D(T)$) is a subclass of the class of strictly pseudocontractive maps.

Let K be a nonempty convex subset of E , and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of K . In [15], Xu and Ori introduced the following implicit iteration process. For $x_0 \in K$ and $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad n \geq 1, \quad (4)$$

where $T_n = T_{n \bmod N}$.

Using this iteration process, they proved the following convergence theorem for nonexpansive maps in Hilbert spaces.

Theorem XO [15, p. 770]. *Let H be a Hilbert space and let K be a nonempty closed convex subset of H . Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in K : T_i x = x\}$. Let $x_0 \in K$ and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ defined implicitly by (4) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.*

Observe that if K is a nonempty convex subset of E and $T : K \rightarrow K$ is a strictly pseudocontractive mapping, then for every $u \in K$, and $t \in (0, 1)$, the operator $S_t : K \rightarrow K$ defined by $S_t x = tu + (1 - t)Tx$ satisfies

$$\langle S_t x - S_t y, j(x - y) \rangle = (1 - t)\langle Tx - Ty, j(x - y) \rangle \leq (1 - t)\|x - y\|^2,$$

for all $x, y \in K$. Thus S_t is *strongly pseudocontractive* (see for example [1,11]). Since S_t is also Lipschitz, it follows from [1,4,11] that S_t has a unique fixed point $x_t \in K$. Thus there exists a unique $x_t \in K$ such that $x_t = tu + (1-t)Tx_t$. This implies that the implicit iteration scheme of Xu and Ori above can be employed for the approximation of common fixed points of a finite family of strictly pseudocontractive maps.

It is our purpose in this paper to first extend Theorem XO from the class of nonexpansive maps to the more general class of strictly pseudocontractive maps. We then obtain a necessary and sufficient condition for the strong convergence of the scheme to a common fixed point of a finite family of strictly pseudocontractive maps defined on a nonempty closed convex subset of an arbitrary Banach space.

In the sequel we shall need the following.

A Banach space E is said to satisfy *Opial's condition* if whenever $\{x_n\}$ is a sequence in E which converges weakly to x , then

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \quad \text{for all } y \in E, y \neq x.$$

It is well known that every Hilbert space satisfies the Opial condition (see for example [7]).

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at a point $p \in E$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p , then $Tx = p$.

Theorem OU [8, p. 444]. *Let E be a real q -uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \rightarrow K$ a strictly pseudocontractive map. Then $(I - T)$ is demiclosed at zero.*

Lemma OAA [10, p. 80]. *Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^\infty$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

2. Main results

Theorem 1. *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H . Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in K$ and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n, \quad n \geq 1,$$

where $T_n = T_{n \bmod N}$, converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. We shall use the well known identity

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \quad (5)$$

which holds for all $x, y \in H$ and for all $t \in [0, 1]$. Let $p \in F$, then

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T_n x_n - p)\|^2 \\ &= \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|T_n x_n - p\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2.\end{aligned}\tag{6}$$

Since each T_i is strictly pseudocontractive, then there exists $k_i \in (0, 1)$ such that

$$\|T_i x - T_i y\|^2 \leq \|x - y\|^2 + k_i \|x - T_i x - (y - T_i y)\|^2, \quad i = 1, 2, \dots, N.$$

Let $k = \max_{1 \leq i \leq N} \{k_i\}$. Then

$$\|T_i x - T_i y\|^2 \leq \|x - y\|^2 + k \|x - T_i x - (y - T_i y)\|^2, \quad k \in (0, 1).$$

Thus we obtain from (6) that

$$\begin{aligned}\|x_n - p\|^2 &\leq \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) [\|x_n - p\|^2 + k \|x_n - T_n x_n\|^2] \\ &\quad - \alpha_n(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 \\ &= \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 + k \alpha_n^2 (1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 \\ &= \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) [1 - k \alpha_n] \|x_{n-1} - T_n x_n\|^2.\end{aligned}$$

Hence

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2 - (1 - \alpha_n) [1 - k \alpha_n] \|x_{n-1} - T_n x_n\|^2.$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, then there exists a positive integer N such $\alpha_n \leq (1 - k)$, $\forall n \geq N$. Thus

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2 - k [1 - k(1 - k)] \|x_{n-1} - T_n x_n\|^2.\tag{7}$$

It now follows from (7) that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. It also follows from (7) that

$$k [1 - k(1 - k)] \|x_{n-1} - T_n x_n\|^2 \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2, \quad \forall n \geq N.$$

Thus

$$k [1 - k(1 - k)] \sum_{j=N+1}^n \|x_{j-1} - T_j x_j\|^2 \leq \|x_N - p\|^2,$$

so that $\sum_{n=1}^{\infty} \|x_{n-1} - T_n x_n\|^2 < \infty$. Thus $\lim_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0$. Observe that

$$\|x_n - T_n x_n\| = \alpha_n \|x_{n-1} - T_n x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$\|x_n - x_{n-1}\| = (1 - \alpha_n) \|x_{n-1} - T_n x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus $\|x_n - x_{n+i}\| \rightarrow 0$ as $n \rightarrow \infty$, $\forall i = 1, 2, \dots, N$.

$$\|x_n - T_{n+i} x_n\| \leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| + \|T_{n+i} x_{n+i} - T_{n+i} x_n\|.$$

Since

$$\|T_i x - T_i y\| \leq L_i \|x - y\|, \quad \forall i = 1, 2, \dots, N,$$

if we choose $L = \max_{1 \leq i \leq N} \{L_i\}$, then $\|T_i x - T_i y\| \leq L \|x - y\|$, $\forall i = 1, 2, \dots, N$. Thus

$$\begin{aligned} \|x_n - T_{n+i} x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| + L \|x_{n+i} - x_n\| \\ &= (1 + L) \|x_{n+i} - x_n\| + \|x_{n+i} - T_{n+i} x_{n+i}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} \|x_n - T_{n+i} x_n\| = 0, \quad \forall i = 1, 2, \dots, N. \quad (8)$$

It follows from (8) (see also [14,15]) that $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$, $\forall l \in I = \{1, 2, \dots, N\}$. Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}_{j=1}^\infty$ which converges weakly to some $u \in K$, and hence we have $\lim_{j \rightarrow \infty} \|x_{n_j} - T_l x_{n_j}\| = 0$. Since every Hilbert space is uniformly convex and 2-uniformly smooth (see for example [8]), it follows from Theorem OU that $(I - T_l)$ is demiclosed at zero, so that $u \in F(T_l)$. Since $l \in I$ is arbitrary, then $u \in F$. Thus we have a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ which converges weakly to a common fixed point u of $\{T_i\}_{i=1}^\infty$. If $\{x_n\}$ has another subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges weakly to $z \neq u$, then we must have $z \in F$ and since $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists and H is an Opial space, it follows from a standard argument that $z = u$. Thus $\{x_n\}$ converges weakly to $u \in F$. \square

Lemma. Let E be a real Banach space and let K be a nonempty closed convex subset of E . Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence satisfying the conditions:

- (i) $0 < \alpha_n < 1$,
- (ii) $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$,
- (iii) $\sum_{n=1}^\infty (1 - \alpha_n)^2 < \infty$.

Let $x_0 \in K$ and let $\{x_n\}_{n=1}^\infty$ be defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1,$$

where $T_n = T_{n \bmod N}$. Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$,
- (ii) $\lim_{n \rightarrow \infty} d(x_n, F)$ exists, where $d(x_n, F) = \inf_{p \in F} \|x_n - p\|$,
- (iii) $\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.

Proof. It is now well known (see for example [3]) that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad (9)$$

for all $x, y \in E$ and for all $j(x - y) \in J(x - y)$. Let $p \in F$, then using (9) we obtain

$$\begin{aligned} \|x_n - p\|^2 &= \|\alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T_n x_n - p)\|^2 \\ &\leq \alpha_n^2 \|x_{n-1} - p\|^2 + 2(1 - \alpha_n) \langle T_n x_n - p, j(x_n - p) \rangle. \end{aligned} \quad (10)$$

Since $T_i : K \rightarrow K$, $i \in I$ is strictly pseudocontractive, we have

$$\langle T_i x - T_i y, j(x - y) \rangle \leq \|x - y\|^2 - \lambda_i \|x - T_i x - (y - T_i y)\|^2, \quad i \in I, \lambda_i \in (0, 1).$$

Let $\lambda = \min_{1 \leq i \leq N} \{\lambda_i\}$. Then

$$\langle T_i x - T_i y, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|x - T_i x - (y - T_i y)\|^2, \quad \lambda \in (0, 1).$$

Thus, it follows from (10) that

$$\begin{aligned} \|x_n - p\|^2 &\leq \alpha_n^2 \|x_{n-1} - p\|^2 + 2(1 - \alpha_n) [\|x_n - p\|^2 - \lambda \|x_n - T_n x_n\|^2] \\ &= \alpha_n^2 \|x_{n-1} - p\|^2 + 2(1 - \alpha_n) \|x_n - p\|^2 \\ &\quad - 2\lambda(1 - \alpha_n) \|x_n - T_n x_n\|^2. \end{aligned} \quad (11)$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 1$, then there exists a positive integer N such that $\alpha_n \geq 1 - (1 - \lambda)/2$, $\forall n \geq N$. Thus $1 - 2(1 - \alpha_n) \geq \lambda > 0$, $\forall n \geq N$. Hence it follows from (11) that for all $n \geq N$ we have

$$\begin{aligned} \|x_n - p\|^2 &\leq \left[\frac{\alpha_n^2}{[1 - 2(1 - \alpha_n)]} \right] \|x_{n-1} - p\|^2 - \left[\frac{2\lambda(1 - \alpha_n)\alpha_n^2}{[1 - 2(1 - \alpha_n)]} \right] \|x_{n-1} - T_n x_n\|^2 \\ &= \left[1 + \frac{(1 - \alpha_n)^2}{[1 - 2(1 - \alpha_n)]} \right] \|x_{n-1} - p\|^2 \\ &\quad - \left[\frac{2\lambda(1 - \alpha_n)\alpha_n^2}{[1 - 2(1 - \alpha_n)]} \right] \|x_{n-1} - T_n x_n\|^2 \\ &\leq \left[1 + \frac{1}{\lambda}(1 - \alpha_n)^2 \right] \|x_{n-1} - p\|^2 - 2\lambda(1 - \alpha_n)\alpha_n^2 \|x_{n-1} - T_n x_n\|^2 \\ &\leq \left[1 + \frac{1}{\lambda}(1 - \alpha_n)^2 \right] \|x_{n-1} - p\|^2 - \frac{\lambda}{2}(1 + \lambda)^2(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 \\ &= [1 + \sigma_n] \|x_{n-1} - p\|^2 - \frac{\lambda}{2}(1 + \lambda)^2(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2, \end{aligned} \quad (12)$$

where $\sigma_n = \frac{1}{\lambda}(1 - \alpha_n)^2$.

Since $\sum_{n=1}^{\infty} \sigma_n < \infty$, it follows from Lemma OAA that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Also it follows from (12) that

$$\|x_n - p\| \leq [1 + \sigma_n]^{\frac{1}{2}} \|x_{n-1} - p\| \leq [1 + \sigma_n] \|x_{n-1} - p\|.$$

Thus

$$d(x_n, F) \leq [1 + \sigma_n] d(x_{n-1}, F),$$

and it again follows from Lemma OAA that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. From (12) we have $\|x_n - p\|^2 \leq M$, $\forall n \geq 1$ and for some $M > 0$, so that

$$\frac{\lambda}{2}(1 + \lambda)^2 \sum_{j=N+1}^n (1 - \alpha_j) \|x_{n-1} - T_j x_j\|^2 \leq \|x_N - p\|^2 + M \sum_{j=N+1}^n \sigma_j,$$

and hence

$$\sum_{n=1}^{\infty} (1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 < \infty.$$

Since $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, then we must have

$$\liminf_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0.$$

Since

$$\|x_n - T_n x_n\| = \alpha_n \|x_{n-1} - T_n x_n\|,$$

we have $\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. \square

Theorem 2. Let E be a real Banach space and let K be a nonempty closed convex subset of E . Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:

- (i) $0 < \alpha_n < 1$,
- (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$,
- (iii) $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty$.

Let $x_0 \in K$ and let $\{x_n\}_{n=1}^{\infty}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1,$$

where $T_n = T_{n \bmod N}$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Proof. If $\{x_n\}$ converges strongly to a common fixed point p of the family $\{T_i\}_{i=1}^N$, then $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$. Since

$$0 \leq d(x_n, F) \leq \|x_n - p\|,$$

we have $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Conversely suppose $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, then our lemma implies that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Thus for arbitrary $\varepsilon > 0$, there exists a positive integer N_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4}, \quad \forall n \geq N_0.$$

Furthermore, $\sum_{n=1}^{\infty} \sigma_n < \infty$ implies that there exists a positive integer N_1 such that $\sum_{j=n}^{\infty} \sigma_j < \varepsilon/(4M)$, $\forall n \geq N_1$. Choose $N = \max\{N_0, N_1\}$.

Then $d(x_N, F) \leq \varepsilon/4$ and $\sum_{j=N}^{\infty} \sigma_j < \varepsilon/(4M)$. For all $n, m \geq N$ and for all $p \in F$ we have

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_N - p\| + M \sum_{j=N+1}^n \sigma_j + \|x_N - p\| + M \sum_{j=N+1}^m \sigma_j \end{aligned}$$

$$\leq 2\|x_N - p\| + 2M \sum_{j=N}^{\infty} \sigma_j.$$

Taking infimum over all $p \in F$, we obtain

$$\|x_n - x_m\| \leq 2d(x_N, F) + 2M \sum_{j=N}^{\infty} \sigma_j \leq \frac{2\varepsilon}{4} + \frac{2M\varepsilon}{4M} = \varepsilon.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose $\lim_{n \rightarrow \infty} x_n = u$. Observe that if $T : K \rightarrow K$ is strictly pseudocontractive and $\{p_n\}_{n=1}^{\infty}$ is a sequence in $F(T)$ which converges strongly to some p , then

$$\begin{aligned} \|p - Tp\| &\leq \|p - p_n\| + \|p_n - Tp\| = \|p - p_n\| + \|Tp_n - Tp\| \\ &\leq (1 + L)\|p - p_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus $p \in F(T)$, so that $F(T)$ is closed. It follows that $F(T_i)$ is closed for all $i \in I$, so that F is closed. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, we must have that $u \in F$. \square

Remark. Prototype of the sequence $\{\alpha_n\}$ in Theorem 1 is $\alpha_n = 1/(n+1)$, $n \geq 1$. For our Lemma and Theorem 2, a prototype for $\{\alpha_n\}$ is $\alpha_n = 1 - 1/(n+1)$, $n \geq 1$.

It is clear from the proof of Theorem 2 that under the hypothesis of Theorem 1, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^{\infty}$ if and only $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

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