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The Fredholm alternative for the one-dimensional p-Laplacian

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Abstract

In this paper we consider the solvability of the boundary value problem

$$\big(\varphi_p(u')\big)' + \lambda_1\varphi_p(u) = f(u) + h(t), \qquad u(0) = u(T) = 0,$$

where $p > 1$, $\varphi_p(u) = |u|^{p-2}u$. By using generalized polar coordinates transformation method, we improve and generalize some results obtained recently in [J. Differential Equations 151 (1999) 386].
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1. Introduction

In this paper, we are concerned with the solvability of the following Dirichlet boundary value problem

$$\big(\varphi_p(u')\big)' + \lambda_1\varphi_p(u) = f(u) + h(t) \quad \text{in } (0,T), \tag{1.1}$$

$$u(0) = u(T) = 0, \tag{1.2}$$

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where $p > 1$, $T > 0$, $\varphi_p(u) = |u|^{p-2}u$, $\lambda_1 = (p-1)(\pi_p/T)^p$, $\pi_p = 2\pi/(p\sin(\pi/p))$, $f \in C(R)$ is bounded and $\lim_{u\to+\infty} f(u) =: f(+\infty)$ exists, $h \in L^\infty(0,T)$.

For $p = 2$, $f(u) \equiv 0$, (1.1)–(1.2) reduces to the linear problem

$$u'' + (\pi/T)^2 u = h(t), \tag{1.3}$$

$$u(0) = u(T) = 0, \tag{1.4}$$

whose solvability is fully described by the classical linear Fredholm alternative, that is, (1.3)–(1.4) is solvable if and only if

$$\int_0^T h(t)\sin\left(\frac{\pi t}{T}\right) dt = 0 \quad \text{for } h \in L^\infty(0,T). \tag{1.5}$$

However, for $p \neq 2$, $f(u) \equiv 0$, (1.1) reduces to

$$\left(\varphi_p(u')\right)' + \lambda_1 \varphi_p(u) = h(t). \tag{1.6}$$

In this case, the situation is quite different. For example, in [1] a counter-example is given, which shows that the following condition

$$\int_0^T h(t)\sin_p\left(\frac{\pi_p t}{\Gamma}\right) dt = 0 \tag{1.7}$$

is not necessary for the solvability of problem (1.6)–(1.2), where $\sin_p t$ is the unique solution of the initial value problem

$$\left(\varphi_p(u')\right)' + (p-1)\varphi_p(u) = 0, \tag{1.8}$$

$$u(0) = 0, \qquad u'(0) = 1. \tag{1.9}$$

A more surprising result is obtained recently in [5], the authors obtained the following result.

Theorem A. *Let us assume that $h \in C^1[0,T]$, $h \not\equiv 0$, satisfies condition* (1.7). *Then problem* (1.6)–(1.2) *has at least one solution. Moreover, if $p \neq 2$, then the set of all possible solutions is bounded in $C^1[0,T]$.*

Since for $p = 2$, the solution set of (1.3)–(1.4) is an unbounded continuum constituted by a one-dimensional linear manifold. Theorem A and the example in [1] reveal a striking difference between the case $p \neq 2$ and $p = 2$. Moreover, in Theorem 1.1, h is assumed to be in $C^1[0,T]$, while in linear case, h need only to be in $L^\infty(0,T)$. The authors claimed in [5] that they do not know whether the C^1 assumption on h can be weakened to require $h \in L^\infty(0,T)$. For more recent results on this research area, we refer [2–4,6,7] and the references therein.

In this paper, by using a different approach, we give an affirmative answer to this question, moreover, we consider the more general problem (1.1)–(1.2).

The main result of this paper is the following theorem.

Theorem 1. *Assume $h \in L^\infty(0,T)$, f is continuous and bounded, moreover the limit $\lim_{u\to+\infty} f(u) =: f(+\infty)$ exists and $H(t) = f(+\infty) + h(t) \not\equiv 0$. Let*

$$I_H = \int_0^T H(t) \sin_p\left(\frac{\pi_p t}{T}\right) dt. \tag{1.10}$$

If $I_H = 0$ and $\lim_{u\to+\infty} |x|^{p-1}[f(x) - f(+\infty)] = 0$, then problem (1.1)–(1.2) has at least one solution. Moreover, if $p \neq 2$, the set of all possible solutions is bounded in $C^1[0,T]$.

Remark 1. After the submission of the original manuscript, the referees informed the author of the reference [7]. In [7], the authors obtained the similarly results, but the methods used in [7] and in this paper are different, neither contains the other. Moreover, Eq. (1.1) is more general than the equation in [5] and [7] and the method used in this paper can be easily applied to the Fučik like asymmetric nonlinearity which contains (1.1) as a special case, see the result obtained in Section 4.

2. Generalized polar coordinates

Let $u = \sin_p t$ be the solution of (1.8)–(1.9). Then from [5], for $t \in [0, \pi_p/2]$, it can be implicitly given by the formula

$$t = \int_0^{\sin_p t} \frac{ds}{(1 - s^p)^{1/p}}$$

and for $t \in [\pi_p/2, \pi_p]$, $\sin_p t = \sin_p(\pi_p - t)$; for $t \in [\pi_p, 2\pi_p]$, $\sin_p t = -\sin_p(2\pi_p - t)$. Moreover $\sin_p t$ is a $2\pi_p$-periodic C^2 function. Let $q = p/(p-1)$ be the conjugate exponent of p, then Eq. (1.1) can be written as first order differential system

$$u' = \varphi_q(v), \qquad v' = -\lambda_1 \varphi_p(u) + f(u) + h(t). \tag{2.1}$$

For simplicity, we can assume $T = \pi_p$ in (1.1)–(1.2) in the rest of this paper. In this case, $\lambda_1 = p - 1$ and (1.10) becomes

$$I_H = \int_0^{\pi_p} H(t) \sin_p t \, dt. \tag{2.2}$$

First, we introduce the generalized polar coordinates. Let $(S(t), C(t))$ be the solution of the auxiliary system

$$x' = \varphi_q(y), \qquad y' = -\lambda_1 \varphi_p(x) \tag{2.3}$$

satisfying the initial condition $(S(0), C(0)) = (0, 1)$.

Then it is easy to verify that $(S(t), C(t))$ is $2\pi_p$-periodic and

$$S(t) = \sin_p t, \qquad C(t) = \varphi_p\bigl(S'(t)\bigr), \tag{2.4}$$

$$\bigl|C(t)\bigr|^q + \bigl|S(t)\bigr|^p \equiv 1, \quad t \in [0, 2\pi_p]. \tag{2.5}$$

Under the generalized polar coordinates transformation $T:(r,\theta)\to(u,v)$:

$$T:\quad u=r^{q/2}S(\theta),\qquad v=r^{p/2}C(\theta),\quad r>0,\ \theta\in[0,2\pi_p] \tag{2.6}$$

and by using (2.5), system (2.1) is transformed into the following system:

$$\begin{aligned}\frac{dr}{dt}&=\frac{2}{p}r^{(2-p)/2}(p-1)S'(\theta)\big[f\big(r^{q/2}S(\theta)\big)+h(t)\big],\\ \frac{d\theta}{dt}&=1-r^{-p/2}S(\theta)\big[f\big(r^{q/2}S(\theta)\big)+h(t)\big],\quad t\in(0,T).\end{aligned} \tag{2.7}$$

Condition (1.2) can be seen as $\theta(0)=0$ or $\theta(0)=\pi_p$, $\theta(\pi_p)=k\pi_p$, $k\in Z$. It is easy to see $\theta(0)=0$ and $\theta(0)=\pi_p$ implies that $u'(0)>0$ and $u'(0)<0$, respectively. Without loss of generality, we may assume $\theta(0)=0$.

Let $\rho=r^{p/2}$, $F(\rho S(\theta))=f(\rho^{1/(p-1)}S(\theta))$, then (2.7) can be further simplified as

$$\begin{aligned}\frac{d\rho}{dt}&=(p-1)S'(\theta)\big[F\big(\rho S(\theta)\big)+h(t)\big],\\ \frac{d\theta}{dt}&=1-\rho^{-1}S(\theta)\big[F\big(\rho S(\theta)\big)+h(t)\big],\quad t\in(0,\pi_p).\end{aligned} \tag{2.8}$$

Let $(\rho(t),\theta(t))$ be the solution of (2.8) with initial value $(\rho(0),\theta(0))=(\rho_0,0)$. Then for $\rho_0\gg 1$, we have the following estimates.

Lemma 1. *Assume $f\in C(R)$ and f is bounded with finite limit $\lim_{u\to+\infty}f(u)=f(+\infty)$, $h\in L^\infty(0,\pi_p)$. Then*

$$\theta(\pi_p)=\pi_p-I_F/\rho_0+O\big(\rho_0^{-2}\big) \tag{2.9}$$

as $\rho_0\to+\infty$, where $O(\rho_0^{-2})$ is uniform with respect to all $f+h$ with $\|f+h\|_{L^\infty}\leqslant C_0$ for some $C_0>0$ fixed and

$$\begin{aligned}I_F&=\int_0^{\pi_p}H(t)\sin_p t\,dt\\ &=f(+\infty)\frac{2}{p}(p-1)^{2/p}B\left(\frac{2}{p},1-\frac{1}{p}\right)+\int_0^{\pi_p}h(t)\sin_p t\,dt,\end{aligned} \tag{2.10}$$

where $B(r,s)=\int_0^1 t^{r-1}(1-t)^{s-1}\,dt$ is the β function for $r>0$, $s>0$.

Proof. Since f and h are bounded, we obtain from the first equation of (2.8)

$$\rho(t)=\rho_0+(p-1)\int_0^t S'\big(\theta(\tau)\big)\big[F\big(\rho S\big(\theta(\tau)\big)\big)+h(\tau)\big]\,d\tau, \tag{2.11}$$

and

$$\rho^{-1}(t) = \rho_0^{-1} - \rho_0^{-2}(p-1)\int_0^t S'\big(\theta(\tau)\big)[F+h]\,d\tau + O\big(\rho_0^{-3}\big), \quad t \in (0, \pi_p). \tag{2.12}$$

Substituting (2.12) into the second equation of (2.8) yields

$$\begin{aligned}\frac{d\theta}{dt} &= 1 - \rho_0^{-1}S(\theta)[F+h] \\ &\quad + (p-1)\rho_0^{-2}S(\theta)[F+h]\int_0^t S'\big(\theta(\tau)\big)[F+h]\,d\tau + O\big(\rho_0^{-3}\big).\end{aligned} \tag{2.13}$$

From $\theta(0) = 0$, we obtain for $t \in [0, \pi_p]$,

$$\begin{aligned}\theta(t) &= t - \rho_0^{-1}\int_0^t S\big(\theta(\tau)\big)\big[F\big(\rho S\big(\theta(\tau)\big)\big) + h(\tau)\big]\,d\tau \\ &\quad + \rho_0^{-2}\int_0^\tau S'\big(\theta(\tau)\big)[F+h]\int_0^\tau S\big(\theta(x)\big)[F+h]\,dx\,d\tau \\ &\quad + (p-1)\rho_0^{-2}\int_0^t S\big(\theta(x)\big)[F+h]\int_0^x S'\big(\theta(\tau)\big)[F+h]\,d\tau\,dx + O\big(\rho_0^{-3}\big) \\ &= t - \rho_0^{-1}\int_0^t S(\tau)[F+h]\,d\tau + O\big(\rho_0^{-2}\big).\end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\theta(\pi_p) = \pi_p - \rho_0^{-1}\int_0^{\pi_p} H(t)S(t)\,dt + O\big(\rho_0^{-2}\big). \tag{2.15}$$

Since $S(t) = \sin_p t$, we obtain from (1.8)–(1.9) that $u = \sin_p t$ satisfies the equation

$$dt = \frac{du}{(1-u^p)^{1/p}}. \tag{2.16}$$

Hence

$$\begin{aligned}\int_0^{\pi_p} S(t)\,dt &= 2\int_0^{\pi_p/2} \sin_p t\,dt = 2\int_0^1 \frac{u\,du}{(1-u^p)^{1/p}} \\ &= \frac{2}{p}\int_0^1 v^{2/p-1}(1-v)^{-1/p}\,dv = \frac{2}{p}B\left(\frac{2}{p}, 1-\frac{1}{p}\right),\end{aligned}$$

which leads to (2.10), where we used the transformation: $v = u^p$. □

Lemma 2. *Let the conditions of Lemma* 1 *hold and* $I_H = 0$. *Moreover, we assume*

$$\lim_{x\to+\infty} |x|^{p-1}\big[f(x) - f(+\infty)\big] = 0. \tag{2.17}$$

Then

$$\theta(\pi_p) = \pi_p + \rho_0 J_H + o\big(\rho_0^{-2}\big), \tag{2.18}$$

where

$$\begin{aligned} J_H = \frac{2-p}{2}\Bigg[&\int_0^{\pi_p/2} \frac{(\int_t^{\pi_p/2} S'(\tau)H(\tau)\,d\tau)^2}{|S'(t)|^p}\,dt \\ &+ \int_0^{\pi_p/2} \frac{(\int_t^{\pi_p/2} S'(\pi_p-\tau)H(\tau)\,d\tau)^2}{|S'(\pi_p-\tau)|^p}\,dt\Bigg]. \end{aligned} \tag{2.19}$$

Proof. Since $I_H = 0$, it follows from (2.14) that

$$\begin{aligned} \theta(\pi_p) = \pi_p - \rho_0^{-1} &\int_0^{\pi_p} S(t)\big[F\big(\rho(t)S\big(\theta(t)\big)\big) - f(+\infty)\big]\,dt \\ &+ \rho_0^{-2}\Bigg[(p-1)\int_0^{\pi_p} S(t)\Bigg(\int_0^t S'(\tau)[F+h]\,d\tau\Bigg)(F+h)\,dt \\ &+ \int_0^{\pi_p} S'(t)[F+h]\Bigg(\int_0^t S(\tau)[F+h]\,d\tau\Bigg)\,dt\Bigg] + O\big(\rho_0^{-3}\big). \end{aligned} \tag{2.20}$$

From $\lim_{x\to+\infty} |x|^{p-1}[f(x) - f(+\infty)] = 0$, we have

$$\begin{aligned} &\lim_{\rho_0\to+\infty} \Bigg|\int_0^{\pi_p} S(t)\big[F\big(\rho(t)S\big(\theta(t)\big)\big) - f(+\infty)\big]\,dt\Bigg|\rho_0^{-1} \\ &\quad\leqslant \lim_{\rho_0\to+\infty} \int_0^{\pi_p} \big|S(t)\big|\rho_0^{-1}\big|\big[F\big(\rho_0 S(t)\big) - f(+\infty)\big]\big|\,dt = 0. \end{aligned} \tag{2.21}$$

It follows from (2.20) and (2.21) that

$$\begin{aligned} J_H = &\int_0^{\pi_p} S'(t)H(t)\Bigg(\int_0^t S(\tau)H(\tau)\,d\tau\Bigg)\,dt \\ &+ (p-1)\int_0^{\pi_p} S(t)H(t)\Bigg(\int_0^t S'(\tau)H(\tau)\,d\tau\Bigg)\,dt. \end{aligned} \tag{2.22}$$

Set $u(t)=\int_0^t S(\tau)H(\tau)\,d\tau$, $v(t)=\int_0^t S'(\tau)H(\tau)\,d\tau$, then by $u(\pi_p)=I_H=0$, we obtain

$$J_H=\int_0^{\pi_p}u(t)v'(t)\,dt+(p-1)\int_0^{\pi_p}u'(t)v(t)\,dt$$
$$=u(t)v(t)\Big|_0^{\pi_p}+(p-2)\int_0^{\pi_p}u'(t)v(t)\,dt=(p-2)\int_0^{\pi_p}u'(t)v(t)\,dt.$$

Set $a=\pi_p/2$ and

$$L=\int_0^{\pi_p}u'(t)v(t)\,dt=\int_0^{a}u'(t)v(t)\,dt+\int_a^{\pi_p}u'(t)v(t)\,dt=:L_1+L_2.$$

Then $J_H=(p-2)L$ and

$$L_1=\int_0^a S(t)H(t)\left(\int_0^t S'(t)H(\tau)\,d\tau\right)dt=\int_0^a\left(\int_t^a S(\tau)H(\tau)\,d\tau\right)S'(t)H(t)\,dt.$$

Set $U(t)=\int_t^a S'(\tau)H(\tau)\,d\tau$, $V(t)\int_t^a=S(\tau)H(\tau)\,d\tau$, then

$$L_1=-\int_0^a U'(t)V(t)\,dt=U(0)V(0)+\int_0^a U(t)V'(t)\,dt$$
$$=U(0)V(0)+\int_0^a\frac{U(t)U'(t)S(t)}{S'(t)}\,dt$$
$$=U(0)V(0)+\frac{1}{2}\frac{U^2(t)S(t)}{S'(t)}\bigg|_0^a-\frac{1}{2}\int_0^a U^2(t)\left(1-\frac{S(t)S''(t)}{(S'(t))^2}\right)dt. \tag{2.23}$$

Claim 1. $\lim_{t\to a}U^2(t)S(t)/S'(t)=0$.

In fact, by the definition of U, S and L'Hospital's rule, we obtain

$$\lim_{t\to a}\frac{U^2(t)S(t)}{S'(t)}=\lim_{t\to a}\frac{U^2(t)}{S'(t)}\lim_{t\to a}S(t)=\lim_{t\to a}\frac{U^2(t)}{S'(t)}=\lim_{t\to a}\frac{2U(t)U'(t)}{S''(t)}$$
$$=\lim_{t\to a}\frac{-2U(t)|S'(t)|^{p-2}S'(t)}{|S(t)|^{p-2}S(t)}=0.$$

Claim 2. $1-S(t)S''(t)/(S'(t))^2=1/|S'(t)|^p$.

In fact, from (2.5) and $S(t)$ satisfies $|u'|^{p-2}u'' + |u|^{p-2}u = 0$, we obtain

$$1 - \frac{S(t)S''(t)}{(S'(t))^2} = 1 + \left|S(t)\right|^p / \left|S'(t)\right|^p = \frac{1}{|S'(t)|^p}.$$

It follows from (2.23), Claims 1 and 2 that

$$L_1 = U(0)V(0) - \frac{1}{2}\int_0^a \frac{(\int_t^a S(\tau)H(\tau)\,d\tau)^2}{|S'(t)|^p}\,dt. \tag{2.24}$$

Now we calculate L_2: let $A(t) = \int_0^t S'(\tau)H(\tau)\,d\tau$, $B(t) = \int_0^t S(\tau)H(\tau)\,d\tau$, then from $I_H = 0$, we have $B(\pi_p) = 0$ and

$$L_2 = \int_a^{\pi_p} A(t)B'(t)\,dt = -A(a)B(a) - \int_a^{\pi_p} A'(t)B(t)\,dt.$$

Let $t = \pi_p - x$, $\tau = \pi_p - y$, then it is easy to verify that

$$\int_a^{\pi_p} A'(t)B(t)\,dt = \int_0^a S'(\pi_p - x)H(\pi_p - x)\left(\int_0^x S(\pi_p - y)H(\pi_p - y)\,dy\right)dx.$$

Similar to the calculation of L_1, we can verify that

$$L_2 = -A(a)B(a) - \frac{1}{2}\int_0^a \frac{(\int_t^a S'(\pi_p - \tau)H(\pi_p - \tau)\,d\tau)^2}{|S'(\pi_p - t)|^p}\,dt. \tag{2.25}$$

It is evident that $A(a) = U(0)$, $B(a) = V(0)$, hence we obtain from (2.23)–(2.25)

$$L = L_1 + L_2 = -\frac{1}{2}\left[\int_0^a \frac{(\int_t^a S'(\tau)H(\tau)\,d\tau)^2}{|S'(t)|^p}\,dt + \int_0^a \frac{(\int_t^a S'(\pi_p - \tau)H(\pi_p - \tau)\,d\tau)^2}{|S'(\pi_p - t)|^p}\,dt\right]. \qquad \square$$

Remark 2. Let $f \equiv 0$, $S(t) = \sin_p t$, $S'(t) = \cos_p t$. Then J_H reduces to

$$J_H = \frac{2-p}{2}\left[\int_0^a \frac{(\int_t^a h(\tau)\cos_p \tau\,d\tau)^2 + (\int_t^a h(\pi_p - \tau)\cos_p \tau\,d\tau)^2}{\cos_p^p t}\,dt\right],$$

which differs from J_h in [5] only by a constant.

Moreover, the expression of J_h in [5] contains a typing error: π_p should be $\pi_p/2 = a$ in the upper limit of the second integral.

3. Proof of Theorem 1

By using a similar method used in [5], we prove Theorem 1.

Let $X = C_0^1[0,\pi_p] = \{u \in C^1[0,\pi_p]\colon\ u(0) = u(\pi_p) = 0\}$ and $R^+ = [0,+\infty)$. For $u \in R$, $\bar F(u,\cdot) \in L^\infty(0,\pi_p)$ and $\lambda \in R^+$, define an operator $G_{\lambda,\bar F} : X \to X$ by $G_{\lambda,\bar F}(v) = u$ if and only if

$$\big(\varphi_p(u')\big)' = \lambda\big[\bar F\big(v,\lambda^{1/p}t\big) - \varphi_p(v)\big], \tag{3.1_λ}$$

$$u(0) = u(\pi_p) = 0. \tag{3.2}$$

Standard arguments based on the Arzela–Ascoli theorem implies that $G_{\lambda.\bar F}$ is a well-defined operator which is compact from X in X^*. Moreover, $G_{\lambda,\bar F}$ depends continuously on the perturbations of $\bar F$ and λ.

The following lemma is a direct consequence of the results of [5] and [6].

Lemma 3. *Let* $\deg[I - G_{\lambda,F}; B_R(0), 0]$ *be the Leray–Schauder degree of* $I - G_{\lambda.\bar F}$ *with respect to* $B_R(0)$ *and* 0, *where* $R > 0$ *and* $B_R(0) = \{u \in X\colon\ \|u\| < R\}$, I *is the identity operator. Then for* $0 < \varepsilon \ll 1$ *and any* $R > 0$,

$$\deg\big[I - G_{1-\varepsilon,0}; B_R(0), 0\big] = 1, \tag{3.3}$$

$$\deg\big[I - G_{1+\varepsilon,0}; B_R(0), 0\big] = -1. \tag{3.4}$$

Proof of Theorem 1. We distinguish between the two cases $1 < p < 2$ and $p > 2$.

Case $1 < p < 2$. By (2.19), $J_H > 0$ and for $t > \pi_p$, we extend H to $[0, 2\pi_p]$ as a L^∞ function.

We claim that there exists a constant $R > 0$ such that for any $\lambda \in [1, 1+\varepsilon]$ the boundary value problem

$$\big(\varphi_p(u')\big)' + \lambda(p-1)\varphi_p(u) = \lambda\big[f(u) + h(\lambda^{1/p}t)\big], \tag{3.5}$$

$$u(0) = u(\pi_p) = 0 \tag{3.6}$$

has no solution with $\|u\|_{C^1[0,\pi_p]} \geqslant R$.

Suppose on the contrary that there exist sequences $\{u_n\}_{n=1}^\infty \subset C_0^1[0,\pi_p]$, $\{\lambda_n\}_{n=1}^\infty \subset [1, 1+\varepsilon]$, such that $\lambda_n \to \bar\lambda \in [1, 1+\varepsilon]$ and $\|u_n\|_{C^1[0,\pi_p]} \to \infty$ and u_n, λ_n satisfy (3.5)–(3.6). From (2.6) and $\rho = r^{p/2}$, we know that $\rho_{0,n} \to +\infty$. In this case, $v_n(t) = u_n(\lambda^{-1/p}t)$ solves the equation

$$\big(\varphi_p(v_n')\big)' + (p-1)\varphi_p(v_n) = f(v_n) + h(t),$$

$$v_n(0) = 0,$$

with $\rho_{0,n} \to +\infty$ and $u_n(\pi_p) = v_n(\lambda_n^{1/p}\pi_p) = 0$. But Lemma 2 and $I_H = 0$ imply $\theta_n(\pi_p) > \pi_p$ for $n \gg 1$. This contradicts the fact $u_n(\pi_p) = v_n(\lambda_n^{1/p}\pi_p) = 0$ because $1 \leqslant \lambda_n \leqslant 1+\varepsilon$ for all $n \in N$. Thus the claim is verified.

By this claim we see that for $0 < \varepsilon \ll 1$, the homotopy $\bar H : [1, 1+\varepsilon] \times X \to X$ defined by $\bar H(u,\lambda) = u - G_{\lambda,\bar F_\lambda}(u)$, where $\bar F_\lambda = f(u) + h(\lambda^{1/p}t)$, satisfies $\bar H(u,\lambda) \neq 0$ for all

$\lambda \in [1, 1+\varepsilon]$ and $\|u\|_{C^1[0,\pi_p]} \geqslant R$. Thus, from the homotopy invariance property of the Leray–Schauder degree, we obtain by (3.4),

$$\deg\bigl[I - G_{1,\bar F}; B_R(0), 0\bigr] = \deg\bigl[I - G_{1+\varepsilon,\bar F_{1+\varepsilon}}; B_R(0), 0\bigr] = -1.$$

This proves that for given $\bar F = f + h$ satisfying $I_H = 0$, the problem (1.1)–(1.2) has at least one solution. Moreover, it follows from above discussions that all possible solutions of (1.1)–(1.2) are priori bounded in the $C^1[0, \pi_p]$ norm.

The case $p > 2$ can be proved similarly. Thus Theorem 1 is proved. □

Remark 3. With only minor modifications, we can consider the following more general problem:

$$\bigl(\varphi_p(u')\bigr)' + (p-1)\bigl[\alpha\varphi_p\bigl(u^+\bigr) - \beta\varphi_p\bigl(u^-\bigr)\bigr] = f(u,t), \tag{3.7}$$

$$u(0) = u(T_0) = 0, \tag{3.8}$$

where $p > 1$, $T_0 = \alpha^{-1/p}\pi_p$, $u^\pm = \max\{\pm u, 0\}$, f is bounded and $\lim_{u\to+\infty} f(u,t) = H(t) \in L^\infty(0, T_0)$, $\alpha > 0$, $\beta > 0$ and

$$\alpha^{-1/p} + \beta^{-1/p} = 2. \tag{3.9}$$

In this case (2.9) becomes

$$\theta(T_0) = T_0 - \rho_0^{-1} I_H + O\bigl(\rho_0^{-2}\bigr), \quad \rho_0 \gg 1, \tag{3.10}$$

and

$$I_H = \int_0^{T_0} H(t)S(t)\,dt,$$

where $(S(t), C(t))$ is the solution of the initial value problem

$$u' = \varphi_q(v), \qquad v' = -\alpha\varphi_p\bigl(u^+\bigr) + \beta\varphi_p\bigl(u^-\bigr) \tag{3.11}$$

$$u(0) = 0, \qquad v(0) = 1, \tag{3.12}$$

which can be given by

$$S(t) = \begin{cases} \alpha^{-1/p}\sin_p(\alpha^{1/p}t), & t \in [0, T_0], \\ -\beta^{-1/p}\sin_p(\beta^{1/p}(t - T_0)), & t \in [T_0, 2\pi_p], \end{cases}$$

$$C(t) = \varphi_p\bigl(S'(t)\bigr), \quad t \in [0, 2\pi_p]. \tag{3.13}$$

If $I_H = 0$ and

$$\lim_{x\to+\infty} |x|^{p-1}\bigl[f(x,t) - H(t)\bigr] = 0,$$

then

$$\theta(T_0) = T_0 + \rho_0^{-2} J_H + o\bigl(\rho_0^{-2}\bigr),$$

where

$$J_H = \frac{2-p}{2}\left[\int\limits_0^{T_0/2} \frac{(\int_t^{T_0/2} S'(\tau)H(\tau)\,d\tau)^2}{|S'(t)|^p}\,dt + \int\limits_0^{T_0/2} \frac{(\int_t^{T_0/2} S'(T_0-\tau)H(\tau)\,d\tau)^2}{|S'(T_0-t)|^p}\,dt\right],$$

which implies that if $I_H = 0$ and $H(t) \not\equiv 0$, then the conclusions of Theorem 1 still hold.

4. Neumann boundary value problem

In this section, we state without proof the solvability of the following Neumann boundary value problem

$$\big(\varphi_p(u')\big)' + (p-1)\big[\alpha\varphi_p\big(u^+\big) - \beta\varphi_p\big(u^-\big)\big] = f(u,t) \quad \text{in } (0, T_0), \tag{4.1}$$

$$u'(0) = u'(T_0) = 0, \tag{4.2}$$

where $T_0 = \alpha^{-1/p}\pi_p$, $\alpha, \beta > 0$ and

$$\alpha^{-1/p} + \beta^{-1/p} = 2. \tag{4.3}$$

Let $(\bar{S}(t), \bar{C}(t))$ be the solution of the following system:

$$x' = \varphi_{p'}(y), \qquad y' = -(p-1)\big[\alpha\varphi_p\big(x^+\big) - \beta\varphi_p\big(x^-\big)\big]$$
$$x(0) = 1, \qquad y(0) = 0.$$

Then we have the following theorem.

Theorem 2. *Suppose f is continuous and bounded, the limit*

$$\lim_{u\to\pm\infty} f(u,t) = H(t)$$

exists and $H(t) \in L^\infty(0, T_0)$. Then

$$\theta(T_0) = T_0 - \frac{1}{\alpha\rho_0}\bar{I}_H + O\big(\rho_0^{-2}\big), \quad \rho_0 \gg 1,$$

where

$$\bar{I}_H = \int\limits_0^{T_0} \bar{S}(t)H(t)\,dt.$$

If $\alpha > 0$, $\beta > 0$ satisfy (4.3) and $\bar{I}_H = 0$, $H(t) \not\equiv 0$. $\bar{L} < +\infty$ where $\bar{L} > 0$ is given by

$$\bar{L} = \int\limits_0^{T_0/2}\left[\frac{(\int_0^t \bar{S}'(\tau)H(\tau)\,d\tau)^2}{|\bar{S}'(t)|^p}\,dt + \frac{(\int_0^t \bar{S}'(T_0-\tau)H(T_0-\tau)\,d\tau)^2}{|\bar{S}'(T_0-t)|^p}\,dt\right],$$

then

$$\theta(T_0) = T_0 - \rho_0^{-2}\bar{I}_H + o(\rho_0^{-2})$$

with

$$\bar{I}_H = \left(\frac{p-2}{2}\right)\bar{L}.$$

In this case, the problem (4.1)–(4.2) has at least one solution. Moreover, if $p \neq 2$, then the set of all possible solutions is bounded in $C^1[0, T_0]$.

5. Nonhomogeneous problems

In this section, we deal with the solvability of the following nonhomogeneous problem:

$$\left(\varphi_p(u')\right)' + \frac{(p-1)q}{p}\varphi_q(u) = f(u,t), \quad t \in (0, \pi_{pq}), \tag{5.1}$$

$$u(0) = u(\pi_{pq}) = 0, \tag{5.2}$$

where $q \geqslant p > 1$,

$$\pi_{pq} = \int_0^1 \frac{ds}{(1-s^q)^{1/p}} = \frac{2}{q}B\left(\frac{1}{q}, \frac{p}{p-1}\right),$$

f is bounded and continuous.

If $q = p$, then (5.1)–(5.2) reduces to (1.1)–(1.2) with $f(u,t) = f(u) + h(t)$. Therefore we discuss the case $q > p$ only.

Similar to the results of [4], one can define (with minor modification) the following $2\pi_{pq}$-periodic function $u = \sin_{pq} t$ as the solution of the following initial value problem:

$$\left(\varphi_p(u')\right)' + \frac{(p-1)q}{p}\varphi_q(u) = 0, \tag{5.3}$$

$$u(0) = 0, \qquad u'(0) = 1, \tag{5.4}$$

which for $t \in [0, \pi_{pq}/2]$ can be given implicitly by the formula

$$t = \int_0^{\sin_{pq} t} \frac{ds}{(1-s^q)^{1/p}} \tag{5.5}$$

and $\sin_{pq} t = \sin_{pq}(\pi_{pq} - t)$ for $t \in [\pi_{pq}/2, \pi_{pq}]$, $\sin_{pq} t = -\sin_{pq}(2\pi_{pq} - t)$ for $t \in [\pi_{pq}, 2\pi_{pq}]$. Define $\cos_{pq} t = \frac{d}{dt}(\sin_{pq} t)$. Then

$$|\sin_{pq} t|^q + |\cos_{pq} t|^p \equiv 1, \quad t \in R. \tag{5.6}$$

Let $(\tilde{S}(t), \tilde{C}(t))$ be the solution of the following first order system:

$$x' = \varphi_{p^*}(y), \qquad y' = -\frac{q}{p}\varphi_q(x) \tag{5.7}$$

satisfying $(\tilde{S}(0), \tilde{C}(0)) = (0, 1)$, where $p^* = p/(p-1)$. Then we can verify that

$$\tilde{S}(t) = \sin_{pq} t, \qquad \tilde{C}(t) = \varphi\big(\tilde{S}'(t)\big) = \varphi_p(\cos_{pq} t). \tag{5.8}$$

Moreover, they satisfy

$$\big|\tilde{S}(t)\big|^q + \big|\tilde{C}(t)\big|^p \equiv 1, \quad t \in R. \tag{5.9}$$

Rewrite (5.1) as

$$u' = \varphi_{p^*}(v), \qquad v' = -\frac{q}{p^*}\varphi_q(u) + f(u,t) \tag{5.10}$$

and define the generalized polar coordinates transformation

$$u = r^{p^*/2}\tilde{S}(\theta), \qquad v = r^{q/2}\tilde{C}(\theta), \quad r > 0,\ \theta \in R. \tag{5.11}$$

Then one can verify that system (5.10) is changed into the form

$$\frac{dp}{dt} = (p-1)\tilde{S}'(\theta) f(u,t), \tag{5.12}$$

$$\frac{d\theta}{dt} = \rho^\sigma - \frac{p}{q}\rho^{-1}\tilde{S}(\theta) f(u,t), \tag{5.13}$$

where $\rho = r^{q/2}$, $u = \rho^{p^*/q}\tilde{S}(\theta)$, $\sigma = (q-p)/(q(p-1)) > 0$.

Theorem 3. *Let $q > p$ and $f : R \times [0, \pi_{pq}] \to R$ be continuous and bounded. Then the problem* (5.1)–(5.2) *has infinitely many solutions $u_n(t)$ and the number of zeros of u_n in $(0, \pi_{pq})$ increases to ∞ as $n \to \infty$, moreover, $\|u_n\| \to \infty$ as $n \to \infty$.*

Sketch of the proof. Now suppose $\theta(0) = 0$, then (5.2) is equivalent to $\theta(\pi_{pq}) = k\pi_{pq}$, $k \in Z$. The assumption $q > p$ implies $\sigma > 0$. Let $\rho(0) = \rho_0 \gg 1$, then it follows form (5.12)

$$\rho(t) = \rho_0 + O(1)$$

and

$$\rho^{-1}(t) = \rho_0^{-1} + o\big(\rho_0^{-1}\big), \quad t \in [0, \pi_p q]. \tag{5.14}$$

Substituting (5.14) into (5.13), we get

$$\theta(\pi_{pq}) = \rho_0^\sigma \pi_{pq} + o\big(\rho_0^\sigma\big) = \rho_0^\sigma \pi_{pq}\big(1 + o(1)\big). \tag{5.15}$$

From (5.14) and the fact that $\theta(\pi_{pq})$ depends continuously on ρ_0, we know that there exist infinitely many $n \in N$ such that

$$\theta(\pi_{pq}) = n\pi_{pq}$$

and $\rho_{0,n} \to \infty$ as $n \to \infty$. $\square$

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