

A note on generalized Weyl's theorem

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Abstract

We prove that if either T or T^* has the single-valued extension property, then the spectral mapping theorem holds for B-Weyl spectrum. If, moreover T is isoloid, and generalized Weyl's theorem holds for T , then generalized Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$. An application is given for algebraically paranormal operators.

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1. Introduction

Let H be an infinite dimensional separable Hilbert space and let $\mathcal{B}(H)$ denote the space of all bounded linear operators acting on H . For $T \in \mathcal{B}(H)$, let T^* , $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $r(T)$ denote the adjoint, the null space, the range, the spectrum, the point spectrum, the approximate point spectrum and the spectral radius of T , respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim N(T)$, and $\beta(T) = \text{codim } R(T)$. A bounded linear operator T is called an *upper semi-Fredholm* (respectively *lower semi-Fredholm*) if $R(T)$ is closed and $\alpha(T)$ (respectively $\beta(T)$) is finite. If T is either upper or lower semi-Fredholm then T is called a semi-Fredholm operator.

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The *index* of a semi-Fredholm operator T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite then T is a *Fredholm* operator. An operator T is called *Weyl* if it is a Fredholm operator of index zero. We denote by $\sigma_W(T)$ the *Weyl spectrum* of T defined as the set of all λ in \mathbb{C} for which $T - \lambda I$ is not a Weyl operator.

For each nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_0 = T$). If for some n , $R(T^n)$ is closed and T_n is an upper (respectively lower) semi-Fredholm operator then T is called an *upper* (respectively *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or lower semi-B-Fredholm operators. If moreover, T_n is a Fredholm operator then T is called a *B-Fredholm* operator. From [6, Proposition 2.1] if T_n is a semi-Fredholm operators then T_m is also a semi-Fredholm operator for each $m \geq n$, and $\text{ind}(T_m) = \text{ind}(T_n)$. Then the *index* of a semi-B-Fredholm is defined as the index of the semi-Fredholm operator T_n (see [5,6]).

In [2, Theorem 2.7] it is proved that an operator T is a B-Fredholm operator if and only if $T = F \oplus N$, where F is a Fredholm operator and N is a nilpotent operator.

An operator $T \in \mathcal{B}(H)$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

Berkani [3, Theorem 4.5] has shown that every normal operator $T \in \mathcal{B}(H)$ satisfies

$$\sigma(T) \setminus E(T) = \sigma_{BW}(T), \quad (1.1)$$

where $E(T)$ is the set of all isolated eigenvalues of T . We say that *generalized Weyl's theorem* holds for T if equality (1.1) holds. This gives a generalization of the classical Weyl's theorem. Recall that the classical Weyl's theorem asserts that for every normal operator $T \in \mathcal{B}(H)$,

$$\sigma(T) \setminus E_0(T) = \sigma_W(T), \quad (1.2)$$

where $E_0(T)$ denotes the set of the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity [15].

Let $SBF_+(H)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(H)$ the class of all $T \in SBF_+(H)$ such that $\text{ind}(T) \leq 0$. Also let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } SBF_+^-(H)\}$$

be called the *semi-B-essential approximate point spectrum*. We say that T obeys *generalized a -Weyl's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T),$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_{ap}(T)$ [5, Definition 2.13]. This also gives a generalization of *a -Weyl's theorem*. We recall that T obeys *a -Weyl's theorem* if

$$\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = E_0^a(T),$$

where $E_0^a(T)$ is the set of all isolated points of $\sigma_{ap}(T)$ which are eigenvalues of finite multiplicity and $\sigma_{SF_+^-}(T)$ is set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not an upper semi-Fredholm operators with $\text{ind}(T - \lambda I) \leq 0$.

From [5,13] we have the following implications:

$$\begin{aligned} \text{generalized } a\text{-Weyl's theorem} &\Rightarrow \text{generalized Weyl's theorem} \\ &\Rightarrow \text{Weyl's theorem,} \\ \text{generalized } a\text{-Weyl's theorem} &\Rightarrow a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem.} \end{aligned}$$

Generalized Weyl's theorem has been studied in [5]. In particular, it is shown that generalized Weyl's theorem implies Weyl's theorem. It has been extended from normal operators to hyponormal operators [4], and to p -hyponormal and M -hyponormal operators by Cao et al. [7]. In this paper we prove that if T or T^* has the SVEP then the spectral mapping theorem holds for the B-Weyl spectrum $\sigma_{BW}(T)$ and for the semi-B-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$. We also obtain conditions sufficient for $f(T)$ to satisfy the generalized Weyl's theorem or generalized a -Weyl's theorem. An application is given for algebraically paranormal operators.

2. Preliminary results

Let $\mathcal{O}(U, H)$ be the Fréchet space of all H -valued analytic functions on an open subset U of \mathbb{C} . We say that $T \in \mathcal{B}(H)$ has the *single-valued extension property* at $\lambda \in \mathbb{C}$ (the SVEP for short) if for every open disk $D(\lambda, r)$, the map

$$T_U : \mathcal{O}(D(\lambda, r), H) \rightarrow \mathcal{O}(D(\lambda, r), H), \quad f \mapsto (z - T)f$$

is injective. When T satisfies the SVEP at every point of \mathbb{C} then we say that T satisfies the SVEP; (see [11]). Note that when the interior of $\sigma_p(T)$ is void then T satisfies the SVEP and when T^* satisfies the SVEP then $\sigma(T) = \sigma_{ap}(T)$ [12].

In the sequel we let $\mathcal{H}(\sigma(T))$ denote the space of all analytic functions in an open neighborhood of $\sigma(T)$. In [10, Corollary 2.6] it is shown that the spectral mapping theorem holds for the Weyl spectrum $\sigma_W(T)$ under the assumption that T or T^* has the SVEP. For the B-Weyl spectrum $\sigma_{BW}(T)$, it is known that the inclusion $\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T))$ holds for every $f \in \mathcal{H}(\sigma(T))$ with no other restriction on T (see [4]). The next theorem shows that the B-Weyl spectrum obeys the spectral mapping theorem whenever T or T^* has the SVEP.

Theorem 2.1. *Let $T \in \mathcal{B}(H)$. If either T or T^* has the SVEP, then*

$$f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)) \quad \text{for all } f \in \mathcal{H}(\sigma(T)).$$

Proof. Suppose $\lambda \notin \sigma_{BW}(f(T))$, then $f(T) - \lambda I$ is a B-Fredholm operator of index zero and

$$f(T) - \lambda I = (T - \lambda_1 I) \cdots (T - \lambda_p I)g(T),$$

where $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ and g is an analytic function nonvanishing on $\sigma(T)$. Hence $g(T)$ is invertible. Since $f(T) - \lambda I$ is a B-Fredholm operator then it follows from [2, Theorem 3.4]

that $T - \lambda_i I$ is a B-Fredholm operator for each i , $1 \leq i \leq p$. Since $\text{ind}(f(T) - \lambda I) = 0$, we deduce from [3, Theorem 3.2] that

$$\text{ind}(T - \lambda_1 I) + \cdots + \text{ind}(T - \lambda_p I) = 0. \quad (2.1)$$

For some n large enough, $T - (\lambda_i + 1/n)I$ is a Fredholm operator (see [3, Remark A]) and $\text{ind}(T - (\lambda_i + 1/n)I) = \text{ind}(T - \lambda_i I)$.

Case 1: Assume that T has the SVEP. From [1, Theorem 2.6], it follows that for each i , $1 \leq i \leq p$, $\text{ind}(T - (\lambda_i + 1/n)I) \leq 0$. Then $\text{ind}(T - \lambda_i I) \leq 0$. Hence from equality (2.1) we have $\text{ind}(T - \lambda_i I) = 0$ and so $\lambda_i \notin \sigma_{BW}(T)$.

Case 2: If T^* has the SVEP then it follows from [1, Theorem 2.6] that for each i , $1 \leq i \leq p$, $\text{ind}(T - (\lambda_i + 1/n)I) \geq 0$. Therefore, for each i , $1 \leq i \leq p$, $\text{ind}(T - \lambda_i I) \geq 0$ and so $\text{ind}(T - \lambda_i I) = 0$.

Now, if $\lambda \in f(\sigma_{BW}(T))$ then $\lambda = f(\mu)$ for some $\mu \in \sigma_{BW}(T)$. Hence

$$f(\mu) - \lambda I = (\mu - \lambda_1 I) \cdots (\mu - \lambda_p I)g(\mu) = 0.$$

This implies that $\mu = \lambda_{i_0}$ for some $1 \leq i_0 \leq p$. Therefore $\lambda_{i_0} \in \sigma_{BW}(T)$ which leads to a contradiction. Thus $\lambda \notin f(\sigma_{BW}(T))$. \square

A bounded linear operator T is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T . It is well known that every hyponormal operator is isoloid [14, Theorem 2]. T is said to be *transaloid* if $r(T - \lambda) = \|T - \lambda\|$ for all $\lambda \in \mathbb{C}$. In [10, Theorem 2.5], it was shown that $f(T)$ obeys Weyl's theorem for transaloid operator T with the SVEP. But in light of the proof of Theorem 2.5 in [10] more can be said: if T is isoloid, satisfies the SVEP and obeys Weyl's theorem then $f(T)$ obeys Weyl's theorem for all $f \in \mathcal{H}(\sigma(T))$. The following theorem gives the generalized Weyl's theorem analog of this result.

Theorem 2.2. *Let $T \in \mathcal{B}(H)$ satisfy the following:*

- (i) *either T or T^* has the SVEP;*
- (ii) *T is isoloid;*
- (iii) *generalized Weyl's theorem holds for T .*

Then generalized Weyl's theorem holds for $f(T)$, for all $f \in \mathcal{H}(\sigma(T))$.

Proof. Generalized Weyl's theorem holds for T implies that $\sigma(T) \setminus E(T) = \sigma_{BW}(T)$. Since T is isoloid, then it follows from [4, Lemma 2.9] that $f(\sigma_{BW}(T)) = \sigma(f(T)) \setminus E(f(T))$. Hence from Theorem 2.1 it follows that

$$\sigma(f(T)) \setminus E(f(T)) = \sigma_{BW}(f(T)).$$

Thus generalized Weyl's theorem holds for $f(T)$. \square

The condition “ T is isoloid” is crucial in Theorem 2.2. To see this consider the following example.

Example 1. Let I_1 and I_2 be the identities on \mathbb{C} and l_2 , respectively. Let S_1 and S_2 defined on l_2 by

$$S_1(x_1, x_2, \dots) = \left(0, \frac{1}{3}x_1, \frac{1}{3}x_2, \dots\right), \quad S_2(x_1, x_2, \dots) = \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, \dots\right).$$

Let $T_1 = I_1 \oplus S_1$ and $T_2 = S_2 - I_2$. Since T_2 has no eigenvalue, then

$$\{-1\} = \sigma(T_2) = \sigma_{BW}(T_2)$$

(for second equality see, for instance, [4, Remark 3.5]). Clearly

$$\sigma(T_1) = \left\{\lambda \in \mathbb{C}: |\lambda| \leq \frac{1}{3}\right\} \cup \{1\}.$$

Now we claim that

$$\sigma_{BW}(T_1) = \left\{\lambda \in \mathbb{C}: |\lambda| \leq \frac{1}{3}\right\}.$$

Since $T_1 - (I_1 \oplus I_2) = 0 \oplus (S_1 - I_2)$ and $S_1 - I_2$ is invertible then it follows from [3, Lemma 4.1] that $T_1 - (I_1 \oplus I_2)$ is a B-Fredholm operator of index zero. Thus $1 \notin \sigma_{BW}(T_1)$. If there exists some $0 \leq |\lambda| \leq 1/3$ such that $T_1 - \lambda(I_1 \oplus I_2)$ is a B-Fredholm of index zero, then it follows from [4, Lemma 3.9] that $S_1 - \lambda I_2$ is a B-Fredholm operator of index zero, which is a contradiction since $S_1 - \lambda I_2$ has no eigenvalue. Therefore $\sigma_{BW}(T_1) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1/3\}$. Now let $T = T_1 \oplus T_2$. Then T has the SVEP and

$$\sigma(T) = \{-1\} \cup \left\{\lambda \in \mathbb{C}: |\lambda| \leq \frac{1}{3}\right\} \cup \{1\}.$$

Moreover -1 is not an eigenvalue of T . Hence T is not isoloid. Also by [4, Lemma 3.9] it is not difficult to see that

$$\sigma_{BW}(T) = \{-1\} \cup \left\{\lambda \in \mathbb{C}: |\lambda| \leq \frac{1}{3}\right\}.$$

Since $E(T) = \{1\}$ then generalized Weyl's theorem holds for T . Now from Theorem 2.1 we have

$$\sigma_{BW}(T^2) = \left\{\lambda \in \mathbb{C}: |\lambda| \leq \frac{1}{9}\right\} \cup \{1\},$$

which equals to $\sigma(T^2)$. Since $E(T^2) = \{1\}$ then generalized Weyl's theorem does not hold for T^2 .

In [10, Theorem 3.1] it is shown that the spectral mapping theorem holds for the essential approximate point spectrum $\sigma_{SF_+}(T)$ whenever T or T^* has the SVEP. In the following we give the analogous for the semi-B-essential approximate point spectrum $\sigma_{SBF_+}(T)$.

Theorem 2.3. *If either T or T^* has the SVEP then*

$$f(\sigma_{SBF_+}(T)) = \sigma_{SBF_+}(f(T)) \quad \text{for all } f \in \mathcal{H}(\sigma(T)).$$

Proof. Let $\lambda \notin \sigma_{SBF_+^-}(f(T))$, then $f(T) - \lambda I \in SBF_+^-(H)$ and

$$f(T) - \lambda I = (T - \lambda_1 I) \cdots (T - \lambda_p I)g(T),$$

where $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ and $g(T)$ is invertible. Since $f(T) - \lambda I$ is an upper semi-B-Fredholm operator then it follows from [6, Corollary 4.4] that $T - \lambda_i I$ is an upper semi-B-Fredholm operator for each i , $1 \leq i \leq p$. Hence

$$\text{ind}(f(T) - \lambda I) = \text{ind}(T - \lambda_1 I) + \cdots + \text{ind}(T - \lambda_p I) \leq 0. \quad (2.2)$$

Now from [6, Corollary 3.2] there exists some integer n such that for each i , $1 \leq i \leq p$, $T - (\lambda_i + 1/n)I$ is an upper semi-Fredholm operator and $\text{ind}(T - (\lambda_i + 1/n)I) = \text{ind}(T - \lambda_i I)$.

If T has the SVEP then it follows from [1, Theorem 2.6] that $\text{ind}(T - \lambda_i I) \leq 0$. Hence $\lambda \notin f(\sigma_{SBF_+^-}(T))$.

Now if T^* has the SVEP then we have from [1, Theorem 2.6] that $\text{ind}(T - \lambda_i I) \geq 0$. Hence $\text{ind}(T - \lambda_i I) = 0$ and so $T - \lambda_i I$ is a B-Fredholm operator of index zero. Thus $\lambda \notin f(\sigma_{SBF_+^-}(T))$.

For the converse inclusion $\sigma_{SBF_+^-}(f(T)) \subseteq f(\sigma_{SBF_+^-}(T))$. Let $\lambda \in \sigma_{SBF_+^-}(f(T)) \setminus f(\sigma_{SBF_+^-}(T))$. Suppose that

$$f(T) - \lambda I = (T - \lambda_1 I) \cdots (T - \lambda_p I)g(T),$$

where $\lambda_1, \dots, \lambda_p \notin \sigma_{SBF_+^-}(T)$ and $g(T)$ is invertible. Hence it follows from [6, Proposition 4.3] that $f(T) - \lambda I$ is an upper semi-B-Fredholm operator and $\text{ind}(f(T) - \lambda I) = \text{ind}(T - \lambda_1 I) + \cdots + \text{ind}(T - \lambda_p I) \leq 0$. So $\lambda \notin \sigma_{SBF_+^-}(f(T))$, contradiction. \square

A bounded linear operator T is called *a-isoloid* if every isolated point of $\sigma_{ap}(T)$ is an eigenvalue of T . Note that every *a-isoloid* operator is isoloid and the converse is not true in general.

Theorem 2.4. Let $T \in \mathcal{B}(H)$ satisfy the following:

- (i) either T or T^* has the SVEP;
- (ii) T is *a-isoloid*;
- (iii) generalized *a-Weyl's theorem* holds for T .

Then generalized *a-Weyl's theorem* holds for $f(T)$, for all $f \in \mathcal{H}(\sigma(T))$.

Proof. This follows at once from Theorem 2.3 and [7]. \square

Remark 2.1. The SVEP itself generally is not enough to guarantee that an operator satisfy either the generalized Weyl's theorem or the generalized *a-Weyl's theorem*. Let T defined on l_2 by

$$T(x_1, x_2, \dots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \dots \right).$$

Then T has the SVEP and $\sigma(T) = \sigma_{BW}(T) = E(T) = \{0\}$. Thus T does not obey generalized Weyl's theorem (and nor generalized *a-Weyl's theorem*).

3. Main results

An operator $T \in \mathcal{B}(H)$ is *algebraically paranormal* if there exists a nonconstant complex polynomial \mathcal{P} such that $\mathcal{P}(T)$ is paranormal. Recall that $T \in \mathcal{B}(H)$ is said to be *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\| \|x\| \quad \text{for all } x \in H.$$

For an hyponormal operator T , it is shown in [4] that generalized Weyl's theorem holds for $f(T)$, for every $f \in \mathcal{H}(\sigma(T))$. Also in [9] it is shown that for algebraically paranormal operator, Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$. In the following we can give more.

Theorem 3.1. *Let $T \in \mathcal{B}(H)$ be an algebraically paranormal operator. Then generalized Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Since T is isoloid by [9, Lemma 2.3] and has the SVEP [8, Corollary 2.10], then we see from Theorem 2.2 that it suffices to prove that generalized Weyl's theorem holds for T .

Let $\lambda \in E(T)$. Then λ is isolated in $\sigma(T)$. Thus we can represent T as the direct sum

$$T = T_1 \oplus T_2, \quad \text{where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T is algebraically paranormal operator then T_1 is also algebraically paranormal operator. If $\lambda = 0$, then T_1 is quasiniptotent. Hence it follows from [9, Lemma 2.2] that T_1 is nilpotent. Since T_2 is invertible, then we deduce from [3, Lemma 4.1] that T is B-Fredholm of index zero. Now if $\lambda \neq 0$, let \mathcal{P} be a nonconstant polynomial such that $\mathcal{P}(T)$ is paranormal. Since $\sigma(\mathcal{P}(T_1)) = \mathcal{P}(\sigma(T_1)) = \mathcal{P}(\lambda)$ then it follows from [9, Lemma 2.1] that $\mathcal{P}(T) = \mathcal{P}(\lambda)$. Put $\mathcal{Q}(z) = \mathcal{P}(z) - \mathcal{P}(\lambda)$. Then we can write

$$\mathcal{Q}(z) = a_0(z - \lambda)^n(z - \lambda_1) \cdots (z - \lambda_p),$$

with $n \neq 0$ and $\lambda_i \neq \lambda$ for each i , $1 \leq i \leq p$. Hence

$$0 = \mathcal{Q}(T_1) = a_0(T_1 - \lambda I_1)^n(T_1 - \lambda_1 I_1) \cdots (T_1 - \lambda_p I_1).$$

Since $T_1 - \lambda_i I_1$ is invertible for every $1 \leq i \leq p$, then $(T_1 - \lambda I_1)^n = 0$ and so $T_1 - \lambda I_1$ is nilpotent. Since $T_2 - \lambda I_2$ is invertible it follows from [3, Lemma 4.1] that $T - \lambda I$ is a B-Fredholm of index zero. Thus $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$.

Conversely let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is a B-Fredholm operator of index zero. Hence it follows from [3, Lemma 4.1] that there exists H_1, H_2 closed subspaces of H such that $H = H_1 \oplus H_2$, $(T - \lambda I) | H_1$ is a Fredholm operator of index zero and $(T - \lambda I) | H_2$ is a nilpotent operator.

Let $T_i = T | H_i$ and $I_i = I | H_i$ for $i = 1, 2$. Since T is algebraically paranormal then T_1 is also algebraically paranormal. Hence it follows from [9, Theorem 2.4] that

$$\sigma(T_1) \setminus \sigma_W(T_1) = E_0(T_1).$$

Case 1: $\lambda \in \sigma(T_1)$. Since $T_1 - \lambda I_1$ is a Fredholm operator of index zero then $\lambda \in E_0(T_1)$ and so λ is isolated in $\sigma(T_1)$. Since $T - \lambda I = (T_1 - \lambda I_1) \oplus (T_2 - \lambda I_2)$ and $T_2 - \lambda I_2$ is

nilpotent then $\sigma(T - \lambda I) \setminus \{0\} = \sigma(T_1 - \lambda I_1) \setminus \{0\}$. Therefore 0 is isolated in $\sigma(T - \lambda I)$, i.e., λ is isolated in $\sigma(T)$. But since $\lambda \in \sigma_p(T_1)$ then $\lambda \in E(T)$.

Case 2: If $\lambda \notin \sigma(T_1)$. Then we also deduce from $T - \lambda I = (T_1 - \lambda I_1) \oplus (T_2 - \lambda I_2)$ that λ is isolated in $\sigma(T)$. Since $T - \lambda I$ is not invertible then $\lambda \in E(T)$. \square

The following corollary is an immediate consequence of Theorem 3.1 and [4, Proposition 3.6].

Corollary 3.1. *Let $T \in \mathcal{B}(H)$ be an algebraically paranormal operator and F be a finite rank operator commuting with T . Then generalized Weyl's theorem holds for $f(T) + F$ for every $f \in \mathcal{H}(\sigma(T))$.*

With the same argument as in [7, Corollary 3.5] we have

Corollary 3.2. *If $T \in \mathcal{B}(H)$ is algebraically paranormal and if $\sigma_{BW}(T) = \{0\}$ then T is normal.*

Theorem 3.5 of [10] affirms that if T^* has the SVEP and if T is transaloid and a -isoloid then a -Weyl's theorem holds for $f(T)$, for all $f \in \mathcal{H}(\sigma(T))$. If T^* is algebraically paranormal, then we have:

Theorem 3.2. *Let T^* be algebraically paranormal operator. Then generalized a -Weyl's theorem holds for T .*

Proof. Let $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+}^-(T)$. Then $T - \lambda I$ is an upper semi-B-Fredholm operator and $\text{ind}(T - \lambda I) \leq 0$. Hence for n large enough, $T - (\lambda + \frac{1}{n})I$ is an upper semi-Fredholm operator and $\text{ind}(T - (\lambda + \frac{1}{n})I) = \text{ind}(T - \lambda I)$ [6, Corollary 3.2]. Hence $\text{ind}(T - (\lambda + 1/n)I) \leq 0$. But since T^* has the SVEP then it follows from [1, Theorem 2.6] that $\text{ind}(T - (\lambda + 1/n)I) \geq 0$. Thus $\text{ind}(T - (\lambda + 1/n)I) = 0$. Therefore $\text{ind}(T - \lambda I) = 0$ and so $T - \lambda I$ is a B-Fredholm operator with index zero. Since T^* has the SVEP, then $\sigma(T) = \sigma_{ap}(T)$ and then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Hence it follows from Theorem 3.1 that $\lambda \in E(T)$. Thus $\lambda \in E^a(T)$.

For the converse, let $\lambda \in E^a(T)$. Then λ is an isolated point of $\sigma_{ap}(T)$ ($= \sigma(T)$ since T^* has the SVEP). Hence $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$. Now we present T^* as the direct sum

$$T^* = T_1 \oplus T_2, \quad \text{where } \sigma(T_1) = \{\bar{\lambda}\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\bar{\lambda}\}.$$

Then we argue as in the proof of Theorem 3.1 to show that $T^* - \bar{\lambda}I$ is a B-Fredholm operator of index zero. \square

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References

- [1] P. Aiena, O. Monsalve, Operators which do not have the single valued extension property, *J. Math. Anal. Appl.* 250 (2000) 435–448.
- [2] M. Berkani, On a class of quasi-Fredholm operators, *Integral Equations Operator Theory* 34 (1999) 244–249.
- [3] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl theorem, *Proc. Amer. Math. Soc.* 130 (2002) 1717–1723.
- [4] M. Berkani, A. Arroud, Generalized Weyl's theorem and hyponormal operators, *J. Austral. Math. Soc.* 76 (2004) 291–302.
- [5] M. Berkani, J.J. Koliha, Weyl type theorems for bounded linear operators, *Acta Sci. Math. (Szeged)* 69 (2003) 359–376.
- [6] M. Berkani, M. Sarih, On semi B-Fredholm operators, *Glasgow Math. J.* 43 (2001) 457–465.
- [7] X. Cao, M. Guo, B. Meng, Weyl type theorem for p -hyponormal and M -hyponormal operators, *Studia Math.* 163 (2004) 177–188.
- [8] N.N. Chourasia, P.B. Ramanujan, Paranormal operators on Banach spaces, *Bull. Austral. Math. Soc.* 21 (1980) 161–168.
- [9] R.E. Curto, Y.M. Han, Weyl's theorem for algebraically paranormal operators, *Integral Equations Operator Theory* 47 (2003) 307–314.
- [10] R.E. Curto, Y.M. Han, Weyl's theorem, a -Weyl's theorem and local spectral theory, *J. London Math. Soc.* (2) 67 (2003) 499–509.
- [11] J.K. Finch, The single valued extension property on a Banach space, *Pacific J. Math.* 58 (1975) 61–69.
- [12] K.B. Laursen, M.M. Neumann, *An Introduction to Local Spectral Theory*, Clarendon, Oxford, 2000.
- [13] V. Rakočević, Operators obeying a -Weyl's theorem, *Rev. Roumaine Math. Pures Appl.* 34 (1989) 915–919.
- [14] J. Stampfli, Hyponormal operators, *Pacific J. Math.* 12 (1962) 1453–1458.
- [15] H. Weyl, Über beschränkte quadratische formen, deren Differenz vollsteig ist, *Rend. Circ. Mat. Palermo* 27 (1909) 373–392.