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Existence and iteration of positive solutions for a generalized right-focal boundary value problem with p-Laplacian operator

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Abstract

In the paper, we obtain the existence of positive solutions and establish a corresponding iterative scheme for the following third-order generalized right-focal boundary value problem with p-Laplacian operator:

$$\big(\phi_p(u'')\big)'(t)=q(t)f\big(t,u(t)\big),\quad 0\leqslant t\leqslant 1,$$
$$u(0)=\sum_{i=1}^{m}\alpha_i u(\xi_i),\qquad u'(\eta)=0,\qquad u''(1)=\sum_{i=1}^{n}\beta_i u''(\theta_i).$$

The main tool is the monotone iterative technique.

Keywords: Iteration; Positive solution; Multi-point boundary value problem; p-Laplacian

1. Introduction

The purpose of this paper is to consider the existence of positive solutions and establish a corresponding iterative scheme for the following third-order generalized right-focal boundary

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value problem with p-Laplacian operator:

$$\begin{cases} \big(\phi_p(u'')\big)'(t) = q(t) f\big(t, u(t)\big), & 0 \leqslant t \leqslant 1, \\ u(0) = \displaystyle\sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, & u''(1) = \displaystyle\sum_{i=1}^{n} \beta_i u''(\theta_i), \end{cases} \tag{1}$$

where $\phi_p(s) = |s|^{p-2}s$, $1 < p \leqslant 2$, and the following conditions hold:

(H1) $1/2 \leqslant \eta \leqslant 1$; $0 < \xi_1 < \xi_2 < \cdots < \xi_m < \eta$ and $\theta_i \in (0, 1)$ $(i = 1, 2, \ldots, n)$;
(H2) $0 \leqslant \alpha_i < 1$ $(i = 1, 2, \ldots, m)$ satisfy $0 \leqslant \sum_{i=1}^{m} \alpha_i < 1$; $0 \leqslant \beta_i < 1$ $(i = 1, 2, \ldots, n)$ satisfy $0 \leqslant \sum_{i=1}^{n} \beta_i < 1$;
(H3) $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$; $q(t) \in L^1[0, 1]$ is nonnegative on $(0, 1)$ and $q(t)$ is not identically zero on any compact subinterval of $(0, 1)$. Furthermore, $q(t)$ satisfies

$$0 < \int_0^1 q(t)\, dt < +\infty.$$

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. Since then, there was much attention focused on the study of nonlinear multi-point boundary value problems (see [2–5], to name a few). As for the linear two-point right-focal boundary value problem, many authors have been published results, we refer the reader to Henderson [6,7], Agarwal and O'Regon [8,9], Wong and Agarwal [10], Chyan and Davis [11] and references therein. For more on focal problems and relative topics, see the book by Agarwal [15].

Very recently, in [12–14], Anderson and Davis got the existence of multiple positive solutions for the following third-order three-point, i.e., the generalized right-focal boundary value problems:

$$\begin{cases} u''' = f(t, u), & 0 \leqslant t \leqslant 1, \\ u(0) = 0, \qquad u'(t_1) = 0, & \gamma u(1) + \delta u''(1) = 0, \end{cases} \tag{2}$$

where $\gamma \geqslant 0$, $\delta > 0$ ($\gamma = 0$, $\delta = 1$ in [12,13]) and $t_1 > 1/2$. It is easy to see that (1) contains (2) as a special case when $\gamma = 0$. The point worth to mention in works [12–14] is that the Green's function for the third-order three-point generalized right-focal boundary value problem

$$\begin{cases} u''' = 0, & 0 \leqslant t \leqslant 1, \\ u(0) = 0, \qquad u'(t_1) = 0, & \gamma u(1) + \delta u''(1) = 0, \end{cases}$$

was obtained, and the properties of the Green's function were presented, which make it possible for [12–14] to use Krasnoselskii's theorem, Leggett–Williams theorem and five functional fixed-point theorem as tools.

We recall that the methods used in [6–14] make full use of the fact that $u^{(n)}(t)$ is linear with respect to u and thus the corresponding Green's function exists. But, as for (1), when $p \neq 2$, $\phi_p(s)$ is not linear with respect to s, and thus, the corresponding Green's function does not exist. Therefore, the methods used in [6–14] are not available to (1).

On the other hand, one can see that all the results obtained in [1–14] are only the existence of solutions or positive solutions under some conditions. Seeing such a fact, we may ask "*How can we find the solutions since they exist definitely?*" Motivated by this question, in this paper, by improving the classical monotone iterative technique of Amann [16], we obtain not only the existence of positive solutions for (1), but also give an iterative scheme for approximating the solutions. It is worth stating that the first term of our iterative scheme is a constant function or a simple function. Therefore, the iterative scheme is significant and feasible. Meanwhile, we give a way to find the solution which will be useful from an application viewpoint. To our knowledge, this is the first paper to use the monotone iterative technique to deal with a multi-point boundary value problem with p-Laplacian operator.

We consider the Banach space $E = C[0,1]$ equipped with norm $\|w\| = \max_{0\leqslant t\leqslant 1}|w(t)|$. In this paper, a positive solution w^* of (1) means a solution w^* of (1) satisfying $w^*(t) > 0$, $0 < t < 1$. We recall that the function w is said to be concave on $[0,1]$, if

$$w\big(\lambda t_2 + (1-\lambda)t_1\big) \geqslant \lambda w(t_2) + (1-\lambda)w(t_1), \quad t_1, t_2, \lambda \in [0,1].$$

We denote

$$C^+[0,1] = \big\{w \in C[0,1]:\ w(t) \geqslant 0, t \in [0,1]\big\},$$
$$P = \left\{ w \in C[0,1] \,\middle|\, \begin{array}{l} w(t) \text{ is concave and nonnegative valued on } [0,1] \\ w(t) \text{ is nondecreasing on } [0,\eta] \\ w(t) \text{ is nonincreasing on } [\eta,1] \end{array} \right\}.$$

It is easy to see that P is a cone in $C[0,1]$. For $w \in P$, we have $\|w\| = w(\eta)$ and

$$\|w\| \min\left\{\frac{t}{\eta}, \frac{1-t}{1-\eta}\right\} \leqslant w(t) \leqslant \|w\|, \quad t \in [0,1]. \tag{3}$$

We know easily that when $p > 1$, $\phi_p(s)$ is strictly increasing on $(-\infty, +\infty)$. So ϕ_p^{-1} exists. Moreover, $\phi_p^{-1} = \phi_q$, where $1/p + 1/q = 1$. Furthermore, when $1 < p \leqslant 2$, $(\phi_p^{-1})'(s) = \phi_q'(s)$ is nonnegative and nonincreasing on $(-\infty, 0)$.

The paper is organized as follows. After this section, some lemmas will be established in Section 2. In Section 3, we give our main results Theorem 3.1. An example is also given to demonstrate our results.

2. Preliminary

In this section, we always suppose that (H1)–(H3) hold.

Lemma 2.1. *Suppose $g(r) \in C[0,1]$ is nonpositive and nondecreasing on $[0,1]$. Then, for any $t \in [0,1]$, we have $\int_0^t \int_s^\eta g(r)\,dr\,ds \leqslant 0$ ($\eta \in [1/2, 1]$ is defined in (1)).*

Proof. When $t \in [0,\eta]$, the conclusion is obvious.

When $t \in [\eta, 1]$, since $g(r)$ is nondecreasing on $[0,1]$, then

$$g(0) \leqslant g(r) \leqslant g(\eta) \leqslant 0, \quad r \in [0,\eta];$$
$$g(\eta) \leqslant g(r) \leqslant g(1) \leqslant 0, \quad r \in [\eta,1].$$

Thus,

$$\int_0^t \int_s^\eta g(r)\,dr\,ds = \int_0^\eta \int_s^\eta g(r)\,dr\,ds - \int_\eta^t \int_\eta^s g(r)\,dr\,ds$$

$$\leqslant \int_0^\eta \int_s^\eta g(\eta)\,dr\,ds - \int_\eta^t \int_\eta^s g(\eta)\,dr\,ds = g(\eta)t\left(\eta - \frac{t}{2}\right) \leqslant 0. \qquad \Box$$

For any fixed $x \in C^+[0,1]$, suppose u is a solution of the following BVP:

$$\begin{cases} \left(\phi_p(u'')\right)'(t) = q(t) f\left(t, x(t)\right), \quad 0 \leqslant t \leqslant 1, \\ u(0) = \sum_{i=1}^m \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^n \beta_i u''(\theta_i). \end{cases} \tag{4}$$

Then

$$u''(t) = \phi_p^{-1}\left[A_x - \int_t^1 q(s) f\left(s, x(s)\right) ds\right],$$

we use $\xi_i \in (0, \eta)$ $(i = 1, 2, \ldots, m)$ and the boundary conditions $u(0) = \sum_{i=1}^m \alpha_i u(\xi_i)$, $u'(\eta) = 0$ to obtain

$$u(t) = -\frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \left[\int_t^\eta \phi_p^{-1}\left(A_x - \int_s^1 q(\tau) f\left(\tau, x(\tau)\right) d\tau\right) ds\right] dt$$
$$- \int_0^t \left[\int_s^\eta \phi_p^{-1}\left(A_x - \int_r^1 q(\tau) f\left(\tau, x(\tau)\right) d\tau\right) dr\right] ds, \quad t \in [0,1],$$

where A_x satisfies the third boundary condition $u''(1) = \sum_{i=1}^n \beta_i u''(\theta_i)$, i.e.,

$$\phi_p^{-1}(A_x) = \sum_{i=1}^n \beta_i \phi_p^{-1}\left(A_x - \int_{\theta_i}^1 q(s) f\left(s, x(s)\right) ds\right). \tag{5}$$

Lemma 2.2. *For any fixed $x \in C^+[0,1]$, there exists a unique $A_x \in (-\infty, +\infty)$ satisfying Eq.* (5).

Proof. For any fixed $x \in C^+[0,1]$, define

$$H_x(c) = \phi^{-1}(c) - \sum_{i=1}^n \beta_i \phi_p^{-1}\left(c - \int_{\theta_i}^1 q(s) f\left(s, x(s)\right) ds\right),$$

then $H_x(c) \in C((-\infty, +\infty), R)$ and $H_x(0) \geqslant 0$. In what follows, we will consider two cases to prove that $H_x(c) = 0$ has a unique solution on $(-\infty, +\infty)$, which means that there exists a unique $A_x \in (-\infty, +\infty)$ satisfying Eq. (5).

Case 1. $H_x(0) = 0$. Then

$$\sum_{i=1}^n \beta_i \phi_p^{-1}\left(\int_{\theta_i}^1 q(s) f\left(s, x(s)\right) ds\right) = 0.$$

So,

$$\beta_i \phi_p^{-1}\left(\int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds\right) = 0, \quad i = 1, 2, \ldots, n.$$

Therefore,

$$\phi_p(\beta_i) \int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds = 0, \quad i = 1, 2, \ldots, n.$$

Then

$$\begin{aligned} H_x(c) &= \phi_p^{-1}(c) - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(c - \int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds\right) \\ &= \phi_p^{-1}(c) - \sum_{i=1}^{n} \phi_p^{-1}\left(\phi_p(\beta_i)\left[c - \int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds\right]\right) \\ &= \phi_p^{-1}(c) - \sum_{i=1}^{n} \phi_p^{-1}\left(\phi_p(\beta_i) c - \phi_p(\beta_i) \int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds\right) \\ &= \phi_p^{-1}(c) - \sum_{i=1}^{n} \beta_i \phi_p^{-1}(c) = \left(1 - \sum_{i=1}^{n} \beta_i\right) \phi^{-1}(c), \end{aligned}$$

Obviously, there exists a unique $c = 0$ satisfying $H_x(c) = 0$.

Case 2. $H_x(0) \neq 0$. Then $H_x(0) > 0$.

(i) When $c \in (0, +\infty)$,

$$\begin{aligned} H_x(c) &= \phi_p^{-1}(c) - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(c - \int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds\right) \geqslant \phi_p^{-1}(c) - \sum_{i=1}^{n} \beta_i \phi_p^{-1}(c) \\ &= \left(1 - \sum_{i=1}^{n} \beta_i\right) \phi^{-1}(c) > 0. \end{aligned}$$

So when $c \in (0, +\infty)$, $H_x(c) \neq 0$.

(ii) When $c \in (-\infty, 0)$,

$$\begin{aligned} H_x(c) &= \phi_p^{-1}(c) - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(c - \int_{\theta_i}^{1} q(s) f\big(s, x(s)\big)\, ds\right) \\ &= \phi_p^{-1}(c)\left[1 - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 - \frac{\int_{\theta_i}^{1} q(s) f(s, x(s))\, ds}{c}\right)\right] = \phi_p^{-1}(c) \bar{H}(c), \end{aligned}$$

where

$$\bar{H}_x(c) = 1 - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 - \frac{\int_{\theta_i}^1 q(s) f(s, x(s))\, ds}{c}\right).$$

Since $H_x(0) > 0$, that is $\sum_{i=1}^n \beta_i \phi_p^{-1}(\int_{\theta_i}^1 q(s) f(s, x(s))\, ds) > 0$. As a result, there must exist $i_0 \in \{1, 2, \dots, n\}$ such that $\beta_{i_0} \phi_p^{-1}(\int_{\theta_{i_0}}^1 q(s) f(s, x(s))\, ds) > 0$. Thus, we get that $\bar{H}_x(c)$ is strictly decreasing on $(-\infty, 0)$; $\int_0^1 q(s) f(s, x(s))\, ds > 0$ and $\sum_{i=1}^n \beta_i > 0$. Let

$$\bar{c} = -\frac{\phi_p(\sum_{i=1}^n \beta_i)}{1 - \phi_p(\sum_{i=1}^n \beta_i)} \int_0^1 q(s) f\big(s, x(s)\big)\, ds,$$

then $\bar{c} < 0$ and we have

$$\bar{H}_x(\bar{c}) = 1 - \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 + \frac{\big[1 - \phi_p\big(\sum_{i=1}^n \beta_i\big)\big] \int_{\theta_i}^1 q(s) f(s, x(s))\, ds}{\phi_p\big(\sum_{i=1}^n \beta_i\big) \int_0^1 q(s) f(s, x(s))\, ds}\right) \geqslant 0.$$

So, $H_x(\bar{c}) = \phi_p^{-1}(\bar{c}) \bar{H}(\bar{c}) \leqslant 0$. Remembering $H_x(0) > 0$, the intermediate value theorem guarantees that there exists $c_0 \in [\bar{c}, 0) \subset (-\infty, 0)$ such that $H_x(c_0) = 0$. If there exist two constants $c_i \in (-\infty, 0)$ $(i = 1, 2)$ satisfying $H_x(c_1) = H_x(c_2) = 0$, then $\bar{H}_x(c_1) = \bar{H}_x(c_2) = 0$. So $c_1 = c_2$ since $\bar{H}_x(c)$ is strictly decreasing on $(-\infty, 0)$. Therefore, $H_x(c) = 0$ has a unique solution on $(-\infty, 0)$.

Combing (i), (ii) and $H_x(0) \neq 0$, we obtain that $H_x(c) = 0$ has a unique solution on $(-\infty, +\infty)$.

The proof of Lemma 2.2 is completed. □

Remark 2.1. From the proof of Lemma 2.2, we know that for any fixed $x \in C^+[0, 1]$,

$$A_x \in \left[-\frac{\phi_p\big(\sum_{i=1}^n \beta_i\big)}{1 - \phi_p\big(\sum_{i=1}^n \beta_i\big)} \int_0^1 q(s) f\big(s, x(s)\big)\, ds, 0\right].$$

Moreover, if $H_x(0) = 0$, then $A_x = 0$; if $H_x(0) \neq 0$, then $A_x \neq 0$.

For any $x \in C^+[0, 1]$, let A_x be the unique constant satisfying Eq. (5) corresponding to x, then we have:

Lemma 2.3. $A_x : C^+[0, 1] \to R$ *has the following properties*:

(a) A_x *is continuous with respect to* x.
(b) *Assume* $f(t, x)$ *is nondecreasing with respect to* x *on* $[0, 1] \times [0, +\infty)$, *then* A_x *is nonincreasing with respect to* x *on* $C^+[0, 1]$.

Proof. (a) Suppose $\{x_n\} \in C^+[0, 1]$ with $x_n \to x_0 \in C^+[0, 1]$ in $C^+[0, 1]$. Let $\{A_n\}$ $(n = 0, 1, 2, \dots)$ be constants decided by Eq. (5) corresponding to x_n $(n = 0, 1, 2, \dots)$. Since $x_n \to x_0$

uniformly on $[0,1]$ and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, we get that for $\varepsilon=1$, there exists $N>0$ such that, when $n>N$, for any $r\in[0,1]$,

$$0\leqslant q(r)f\big(r,x_n(r)\big)\leqslant q(r)\big[1+f\big(r,x_0(r)\big)\big]\leqslant q(r)\Big[1+\max_{r\in[0,1]}f\big(r,x_0(r)\big)\Big]. \tag{6}$$

So,

$$\begin{aligned}A_n&\in\left[-\frac{\phi_p\big(\sum_{i=1}^n\beta_i\big)}{1-\phi_p\big(\sum_{i=1}^n\beta_i\big)}\int_0^1 q(s)f\big(s,x_n(s)\big)\,ds,0\right]\\&\subseteq\left[-\frac{\phi_p\big(\sum_{i=1}^n\beta_i\big)\big[1+\max_{r\in[0,1]}f(r,x_0(r))\big]}{1-\phi_p\big(\sum_{i=1}^n\beta_i\big)}\int_0^1 q(s)\,ds,0\right],\end{aligned}$$

which means that $\{A_n\}$ is bounded.

Suppose A_n does not converge to A_0. Then there exist two subsequences $\{A_{n_k}^{(1)}\}$ and $\{A_{n_k}^{(2)}\}$ of $\{A_n\}$ with $A_{n_k}^{(1)}\to c_1$ and $A_{n_k}^{(2)}\to c_2$ since $\{A_n\}$ is bounded, but $c_1\neq c_2$.

By construction of $\{A_n\}$ $(n=0,1,2,\ldots)$, we have

$$\phi_p^{-1}\big(A_{n_k}^{(1)}\big)=\sum_{i=1}^n\beta_i\phi_p^{-1}\left(A_{n_k}^{(1)}-\int_{\theta_i}^1 q(s)f\big(s,x_{n_k}^{(1)}(s)\big)\,ds\right). \tag{7}$$

Combing (6) and using Lebesgue's dominated convergence theorem in (7), we get

$$\begin{aligned}\phi_p^{-1}(c_1)&=\lim_{n_k\to\infty}\sum_{i=1}^n\beta_i\phi_p^{-1}\left(A_{n_k}^{(1)}-\int_{\theta_i}^1 q(s)f\big(s,x_{n_k}^{(1)}(s)\big)\,ds\right)\\&=\sum_{i=1}^n\beta_i\phi_p^{-1}\left(\lim_{n_k\to\infty}A_{n_k}^{(1)}-\lim_{n_k\to\infty}\int_{\theta_i}^1 q(s)f\big(s,x_{n_k}^{(1)}(s)\big)\,ds\right)\\&=\sum_{i=1}^n\beta_i\phi_p^{-1}\left(c_1-\int_{\theta_i}^1 q(s)f\big(s,x_0(s)\big)\,ds\right).\end{aligned}$$

Since $\{A_n\}$ $(n=0,1,2,3,\ldots)$ is unique, we get $c_1=A_0$.

Similarly, $c_2=A_0$. So $c_1=c_2$, which is a contradiction. Therefore, for any $x_n\to x_0$, $A_n\to A_0$, which means that $A_x:C^+[0,1]\to R$ is continuous.

(b) For any $x_i\in C^+[0,1]$ $(i=1,2)$, let A_i $(i=1,2)$ be two constants decided by Eq. (5) corresponding to x_i $(i=1,2)$. Suppose $x_1\geqslant x_2$; in the following, we will prove $A_1\leqslant A_2$.

(i) If $A_2=0$, then by Remark 1, we know that $A_1\leqslant 0=A_2$.
(ii) If $A_1=0$, then by Remark 1, we know that $H_{x_1}(0)=0$, i.e.,

$$\sum_{i=1}^n\beta_i\phi_p^{-1}\left(\int_{\theta_i}^1 q(s)f\big(s,x_1(s)\big)\,ds\right)=0.$$

Thus,

$$0 \leqslant H_{x_2}(0) = \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(\int_{\theta_i}^{1} q(s) f\big(s, x_2(s)\big)\, ds\right)$$
$$\leqslant \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(\int_{\theta_i}^{1} q(s) f\big(s, x_1(s)\big)\, ds\right) = 0.$$

So, $H_{x_2}(0) = 0$, which means $A_2 = 0$.

(iii) $A_i \neq 0$ $(i = 1, 2)$. Then $A_i < 0$ and $H_{x_i}(0) \neq 0$ $(i = 1, 2)$. By the definition of A_i $(i = 1, 2)$,

$$\phi_p^{-1}(A_j) = \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(A_j - \int_{\theta_i}^{1} q(s) f\big(s, x_j(s)\big)\, ds\right), \quad j = 1, 2.$$

So,

$$\sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 - \frac{\int_{\theta_i}^{1} q(s) f(s, x_1(s))\, ds}{A_1}\right) = \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 - \frac{\int_{\theta_i}^{1} q(s) f(s, x_2(s))\, ds}{A_2}\right). \tag{8}$$

Suppose that $A_1 > A_2$, then $0 > 1/A_2 > 1/A_1$. Since $x_1 \geqslant x_2$ and $f(t, x)$ is nondecreasing with respect to x, we have

$$\int_{\theta_i}^{1} q(s) f\big(s, x_1(s)\big)\, ds \geqslant \int_{\theta_i}^{1} q(s) f\big(s, x_2(s)\big)\, ds, \quad i = 1, 2, \ldots, n.$$

On the other hand, since $H_{x_2}(0) = \sum_{i=1}^{n} \beta_i \phi_p^{-1}(\int_{\theta_i}^{1} q(s) f(s, x_2(s))\, ds) \neq 0$, there must exist $i_0 \in \{1, 2, \ldots, n\}$ such that $\beta_{i_0} \phi_p^{-1}(\int_{\theta_{i_0}}^{1} q(s) f(s, x_2(s))\, ds) \neq 0$. Thus, when $i \neq i_0$,

$$\frac{1}{A_1}\int_{\theta_i}^{1} q(s) f\big(s, x_1(s)\big)\, ds \leqslant \frac{1}{A_1}\int_{\theta_i}^{1} q(s) f\big(s, x_2(s)\big)\, ds \leqslant \frac{1}{A_2}\int_{\theta_i}^{1} q(s) f\big(s, x_2(s)\big)\, ds;$$

when $i = i_0$,

$$\frac{1}{A_1}\int_{\theta_{i_0}}^{1} q(s) f\big(s, x_1(s)\big)\, ds \leqslant \frac{1}{A_1}\int_{\theta_{i_0}}^{1} q(s) f\big(s, x_2(s)\big)\, ds < \frac{1}{A_2}\int_{\theta_{i_0}}^{1} q(s) f\big(s, x_2(s)\big)\, ds.$$

Therefore,

$$\sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 - \frac{1}{A_1}\int_{\theta_i} q(s) f\big(s, x_1(s)\big)\, ds\right) > \sum_{i=1}^{n} \beta_i \phi_p^{-1}\left(1 - \frac{1}{A_2}\int_{\theta_i}^{1} q(s) f\big(s, x_2(s)\big)\, ds\right),$$

which is a contradiction to (8). As a result, $A_1 \leqslant A_2$.

The proof of Lemma 2.3 is completed. $\square$

For any $x \in C^+[0,1]$, define

$$(Tx)(t) = -\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i}\left[\int_t^\eta \phi_p^{-1}\left(A_x - \int_s^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\right)ds\right]dt$$
$$- \int_0^t\left[\int_s^\eta \phi_p^{-1}\left(A_x - \int_r^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\right)dr\right]ds, \quad t\in[0,1],$$

where A_x is the unique constant decided in Eq. (5) corresponding to x. By Lemma 2.2, we know that Tx is well defined. It is easy to verify that a fixed point of T in P must be a solution of (1) in P. Now, we have the following result:

Lemma 2.4. *$T: P \to P$ is completely continuous, i.e., T is continuous and compact. Moreover, if $f(t,x)$ is nondecreasing with respect to x on $[0,1]\times[0,+\infty)$, then Tx is also nondecreasing with respect to x on P.*

Proof. Firstly, we show $TP \subseteq P$. For any $x \in P$, from the definition of Tx, we know that $(Tx) \in C[0,1]$ and

$$(Tx)''(t) = \phi_p^{-1}\left(A_x - \int_t^1 q(s)f\big(s,x(s)\big)\,ds\right), \quad t\in[0,1], \tag{9}$$

$$(Tx)'(t) = \begin{cases} -\int_t^\eta \phi_p^{-1}\big(A_x - \int_s^1 q(\tau)f(\tau,x(\tau))\,d\tau\big)\,ds, & t\in[0,\eta],\\ \int_\eta^t \phi_p^{-1}\big(A_x - \int_s^1 q(\tau)f(\tau,x(\tau))\,d\tau\big)\,ds, & t\in[\eta,1],\end{cases} \tag{10}$$

$$(Tx)(0) = \sum_{i=1}^n \alpha_i (Tx)(\xi_i). \tag{11}$$

By Remark 1, $A_x \leqslant 0$; by (H3), $f(s,x(s)) \geqslant 0$. Thus, from (9), we have $(Tx)''(t) \leqslant 0$, which means that $(Tx)(t)$ is concave on $[0,1]$. From (10), we have $(Tx)'(t) \geqslant 0$, $t\in[0,\eta]$, and $(Tx)'(t) \leqslant 0$, $t\in[\eta,1]$, which means that $(Tx)(t)$ is nondecreasing on $[0,\eta]$ and $(Tx)(t)$ is nonincreasing on $[\eta,1]$. From (11) and $\xi_i \in (0,\eta)$ $(i=1,2,\dots,m)$, we have $(Tx)(0) \geqslant \sum_{i=1}^m \alpha_i (Tx)(0)$, which means $(Tx)(0) \geqslant 0$ since $\sum_{i=1}^m \alpha_i < 1$. Now, we concentrate on proving $(Tx)(1) \geqslant 0$. In fact, by the definition of Tx,

$$(Tx)(1) = -\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_0^{\xi_i}\left[\int_t^\eta \phi_p^{-1}\left(A_x - \int_s^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\right)ds\right]dt$$
$$- \int_0^1\left[\int_s^\eta \phi_p^{-1}\left(A_x - \int_r^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\right)dr\right]ds$$
$$= (Tx)(0) - \int_0^1\left[\int_s^\eta \phi_p^{-1}\left(A_x - \int_r^1 q(\tau)f\big(\tau,x(\tau)\big)\,d\tau\right)dr\right]ds. \tag{12}$$

Let $g(r) = A_x - \int_r^1 q(\tau) f(\tau, x(\tau))\, d\tau$. Then, $g(r) \leqslant 0$ and $g'(r) = q(r) f(r, x(r)) \geqslant 0$. By Lemma 2.1, we have

$$-\int_0^1 \left[\int_s^\eta \phi_p^{-1}\left(A_x - \int_r^1 q(\tau) f\big(\tau, x(\tau)\big)\, d\tau \right) dr \right] ds \geqslant 0,$$

which means that $(Tx)(1) \geqslant 0$. By the concavity of Tx, it is obvious that $(Tx)(t) \geqslant 0$, $t \in [0,1]$.

From above all, we conclude $TP \subseteq P$.

Secondly, we show that $T : P \to P$ is completely continuous. The continuity of T is obvious since we have proved that A_x is continuous with respect to x in P in Lemma 2.2. Now, we prove that T is compact. Let $\Omega \subset P$ be a bounded set. Then there exists R such that $\Omega \subset \{x \in P \mid \|x\| \leqslant R\}$. For any $x \in \Omega$, we have $0 \leqslant \int_0^1 q(s) f(s, x(s))\, ds \leqslant \max_{s \in [0,1], u \in [0,R]} f(s,u) \int_0^1 q(s)\, ds =: M$. From Remark 1, we get

$$|A_x| \leqslant \frac{\phi_p\big(\sum_{i=1}^n \beta_i\big) M}{1 - \phi_p\big(\sum_{i=1}^n \beta_i\big)}.$$

Therefore,

$$\big\|(Tx)\big\| \leqslant \frac{\eta\big(1 - \sum_{i=1}^m \alpha_i + \sum_{i=1}^m \alpha_i \xi_i\big)\phi_p^{-1}(M)}{\big(1 - \sum_{i=1}^m \alpha_i\big)\phi_p^{-1}\big(1 - \phi_p\big(\sum_{i=1}^n \beta_i\big)\big)},$$

$$\big\|(Tx)'\big\| \leqslant \frac{\eta \phi_p^{-1}(M)}{\phi_p^{-1}\big(1 - \phi_p\big(\sum_{i=1}^n \beta_i\big)\big)}.$$

The Arzela–Ascoli theorem guarantees that $T\Omega$ is relatively compact in $C[0,1]$, which means that T is compact.

At last, we show that Tx is nondecreasing with respect to x on P if $f(t,x)$ is nondecreasing with respect to x on $[0,1] \times [0, +\infty)$. For any $x_i \in P$ $(i = 1,2)$ with $x_1 \geqslant x_2$, let A_i $(i = 1,2)$ be the unique constant decided in Eq. (5) corresponding to x_i $(i = 1,2)$. From the definition of Tx, we get, for any $t \in [0,1]$,

$$\begin{aligned}
(Tx_1)(t) - (Tx_2)(t) = {} & -\frac{1}{1 - \sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_t^\eta \left[\phi_p^{-1}\left(A_1 - \int_r^1 q(\tau) f\big(\tau, x_1(\tau)\big)\, d\tau \right) \right. \\
& \left. - \phi_p^{-1}\left(A_2 - \int_r^1 q(\tau) f\big(\tau, x_2(\tau)\big)\, d\tau \right) \right] dr\, dt \\
& - \int_0^t \int_s^\eta \left[\phi_p^{-1}\left(A_1 - \int_r^1 q(\tau) f\big(\tau, x_1(\tau)\big)\, d\tau \right) \right. \\
& \left. - \phi_p^{-1}\left(A_2 - \int_r^1 q(\tau) f\big(\tau, x_2(\tau)\big)\, d\tau \right) \right] dr\, ds. \qquad (13)
\end{aligned}$$

By Lemma 2.2, we know

$$A_1 - \int_r^1 q(\tau) f\big(\tau, x_1(\tau)\big)\, d\tau \leqslant A_2 - \int_r^1 q(\tau) f\big(\tau, x_2(\tau)\big)\, d\tau \leqslant 0.$$

Thus, the first part on the right-hand side of Eq. (13) is obviously nonnegative. As for the second part, let, for $r \in [0, 1]$,

$$g(r) = \phi_p^{-1}\left(A_1 - \int_r^1 q(\tau) f\big(\tau, x_1(\tau)\big)\, d\tau\right) - \phi_p^{-1}\left(A_2 - \int_r^1 q(\tau) f\big(\tau, x_2(\tau)\big)\, d\tau\right).$$

Then, $g(r) \leqslant 0$ and

$$\begin{aligned} g'(r) &= \big(\phi_p^{-1}\big)'\left(A_1 - \int_r^1 q(\tau) f\big(\tau, x_1(\tau)\big)\, d\tau\right) q(r) f\big(r, x_1(r)\big) \\ &\quad - \big(\phi_p^{-1}\big)'\left(A_2 - \int_r^1 q(\tau) f\big(\tau, x_2(\tau)\big)\, d\tau\right) q(r) f\big(r, x_2(r)\big). \end{aligned}$$

Since $1 < p \leqslant 2$, we have that $(\phi_p^{-1})'(s)$ is nonnegative and nonincreasing on $(-\infty, 0)$. Thus,

$$\big(\phi_p^{-1}\big)'\left(A_1 - \int_r^1 q(\tau) f\big(\tau, x_1(\tau)\big)\, d\tau\right) \geqslant \big(\phi_p^{-1}\big)'\left(A_2 - \int_r^1 q(\tau) f\big(\tau, x_2(\tau)\big)\, d\tau\right) \geqslant 0,$$

combining $f(\tau, x_1(\tau)) \geqslant f(\tau, x_2(\tau)) \geqslant 0$, we obtain $g'(r) \geqslant 0$. Lemma 2.1 guarantees that the second part on the right-side of Eq. (13) is nonnegative. Thus, for any $t \in [0, 1]$, $(Tx_1)(t) \geqslant (Tx_2)(t)$, which means that Tx is nondecreasing with respect to x in P. □

3. Existence and iteration of solution for (1)

Define, for $t \in [0, \min\{\xi_1, 1 - \eta\}]$,

$$y(t) = \frac{\sum_{i=1}^m \alpha_i \int_t^{\xi_i} \big[\int_s^\eta \phi_p^{-1}\big(\int_r^{1-t} q(\tau)\, d\tau\big)\, dr\big]\, ds}{1 - \sum_{i=1}^m \alpha_i} + \int_t^\eta \left[\int_s^\eta \phi_p^{-1}\left(\int_r^{1-t} q(\tau)\, d\tau\right) dr\right] ds.$$

Then by (H3), $y(t) > 0$ is continuous on $[0, \min\{\xi_1, 1 - \eta\}]$.

Denote

$$\begin{aligned} A &= \frac{\big(1 - \sum_{i=1}^m \alpha_i\big)\phi_p^{-1}\big(1 - \phi_p\big(\sum_{i=1}^n \beta_i\big)\big)}{\big[\sum_{i=1}^m \alpha_i \xi_i \big(\eta - \frac{\xi_i}{2}\big) + \frac{\eta^2}{2}\big(1 - \sum_{i=1}^m \alpha_i\big)\big]\phi_p^{-1}\big(\int_0^1 q(s)\, ds\big)} > 0, \\ B &= \frac{1}{\min_{t \in [0, \min\{\xi_1, 1-\eta\}]} y(t)} > 0. \end{aligned} \tag{14}$$

Theorem 3.1. *Assume* (H1)–(H3) *hold. If there exist a constant* $\delta \in (0, \min\{\xi_1, 1 - \eta\})$ *and two positive numbers* $b < a$, *such that*

(H4) $f:[0,a]\to[0,+\infty)$ *is nondecreasing*;
(H5) $\sup_{t\in[0,1]} f(t,a)\leqslant (aA)^{p-1}$, $\inf_{t\in[\delta,1-\delta]} f(t,\frac{\delta}{\eta}b)\geqslant (bB)^{p-1}$,

then problem (1) *has at least two solutions* $w^*, v^*\in P$ *with*

$$b\leqslant \|w^*\|\leqslant a \quad \text{and} \quad \lim_{n\to+\infty} T^n w_0 = w^*, \quad \text{where } w_0(t)=a,\ t\in[0,1],$$

$$b\leqslant \|v^*\|\leqslant a \quad \text{and} \quad \lim_{n\to+\infty} T^n v_0 = v^*, \quad \text{where } v_0(t)=b\min\left\{\frac{t}{\eta}, \frac{1-t}{1-\eta}\right\},\ t\in[0,1].$$

Proof. We denote $P[b,a]=\{w\in P:\ b\leqslant\|w\|\leqslant a\}$. In what follows, we first prove $TP[b,a]\subset P[b,a]$.

Let $w\in P[b,a]$, then

$$0\leqslant w(t)\leqslant\|w\|\leqslant a.$$

Since $\delta\in(0,\min\{\xi_1,1-\eta\})$, by (1) we have:

$$\min_{t\in[\delta,1-\delta]} w(t)\geqslant\|w\|\min\left\{\frac{\delta}{\eta},\frac{\delta}{1-\eta}\right\}=\frac{\delta}{\eta}\|w\|\geqslant\frac{\delta}{\eta}b.$$

So, by assumptions (H4) and (H5), we have

$$0\leqslant f\big(t,w(t)\big)\leqslant f(t,a)\leqslant\sup_{t\in[0,1]} f(t,a)\leqslant (aA)^{p-1},\quad t\in[0,1]; \tag{15}$$

$$f\big(t,w(t)\big)\geqslant f\left(t,\frac{\delta}{\eta}b\right)\geqslant\inf_{t\in[\delta,1-\delta]} f\left(t,\frac{\delta}{\eta}b\right)\geqslant (bB)^{p-1},\quad t\in[\delta,1-\delta]. \tag{16}$$

For any $w(t)\in P[b,a]$, by Lemma 2.4 we know that $Tw\in P$ and, as a result,

$$\begin{aligned}\|Tw\|&=(Tw)(\eta)\\&=-\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\left[\int_t^\eta\phi_p^{-1}\left(A_w-\int_s^1 q(\tau)f\big(\tau,w(\tau)\big)\,d\tau\right)ds\right]dt\\&\quad-\int_0^\eta\left[\int_s^\eta\phi_p^{-1}\left(A_w-\int_r^1 q(\tau)f\big(\tau,w(\tau)\big)\,d\tau\right)dr\right]ds.\end{aligned}$$

Since we have proved in Lemma 2.2 that

$$A_w\in\left[-\frac{\phi_p\big(\sum_{i=1}^n\beta_i\big)}{1-\phi_p\big(\sum_{i=1}^n\beta_i\big)}\int_0^1 q(s)f\big(s,w(s)\big)\,ds,0\right],$$

therefore, by (15) and (16),

$$\begin{aligned}\|Tw\|\leqslant\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_0^{\xi_i}\left[\int_t^\eta\phi_p^{-1}\left(\frac{\phi_p\big(\sum_{i=1}^n\beta_i\big)}{1-\phi_p\big(\sum_{i=1}^n\beta_i\big)}\int_0^1 q(s)f\big(s,w(s)\big)\,ds\right.\right.\\\left.\left.+\int_s^1 q(\tau)f\big(\tau,w(\tau)\big)\,d\tau\right)ds\right]dt\end{aligned}$$

$$+\int_0^\eta\left[\int_s^\eta \phi_p^{-1}\left(\frac{\phi_p(\sum_{i=1}^n \beta_i)}{1-\phi_p(\sum_{i=1}^n \beta_i)}\int_0^1 q(s)f(s,w(s))\,ds\right.\right.$$

$$\left.\left.+\int_r^1 q(\tau)f(\tau,w(\tau))\,d\tau\right)dr\right]ds$$

$$\leqslant \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i\int_0^{\xi_i}\left[\int_t^\eta \phi_p^{-1}\left(\frac{\phi_p(\sum_{i=1}^n \beta_i)}{1-\phi_p(\sum_{i=1}^n \beta_i)}\int_0^1 q(s)f(s,w(s))\,ds\right.\right.$$

$$\left.\left.+\int_0^1 q(\tau)f(\tau,w(\tau))\,d\tau\right)ds\right]dt$$

$$+\int_0^\eta\left[\int_s^\eta \phi_p^{-1}\left(\frac{\phi_p(\sum_{i=1}^n \beta_i)}{1-\phi_p(\sum_{i=1}^n \beta_i)}\int_0^1 q(s)f(s,w(s))\,ds\right.\right.$$

$$\left.\left.+\int_0^1 q(\tau)f(\tau,w(\tau))\,d\tau\right)dr\right]ds$$

$$=\frac{\sum_{i=1}^m \alpha_i\xi_i(\eta-\frac{\xi_i}{2})+\frac{\eta^2}{2}(1-\sum_{i=1}^m \alpha_i)}{(1-\sum_{i=1}^m \alpha_i)\phi_p^{-1}(1-\phi_p(\sum_{i=1}^n \beta_i))}\phi_p^{-1}\left(\int_0^1 q(s)f(s,w(s))\,ds\right)$$

$$\leqslant \frac{[\sum_{i=1}^m \alpha_i\xi_i(\eta-\frac{\xi_i}{2})+\frac{\eta^2}{2}(1-\sum_{i=1}^m \alpha_i)]\phi_p^{-1}(\int_0^1 q(s)\,ds)}{(1-\sum_{i=1}^m \alpha_i)\phi_p^{-1}(1-\phi_p(\sum_{i=1}^n \beta_i))}Aa=a$$

and

$$\|Tw\|\geqslant \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i\int_0^{\xi_i}\left[\int_t^\eta \phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,x(\tau))\,d\tau\right)ds\right]dt$$

$$+\int_0^\eta\left[\int_s^\eta \phi_p^{-1}\left(\int_r^1 q(\tau)f(\tau,x(\tau))\,d\tau\right)dr\right]ds$$

$$\geqslant \frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i\int_\delta^{\xi_i}\left[\int_t^\eta \phi_p^{-1}\left(\int_s^{1-\delta} q(\tau)f(\tau,x(\tau))\,d\tau\right)ds\right]dt$$

$$+\int_\delta^\eta\left[\int_s^\eta \phi_p^{-1}\left(\int_r^{1-\delta} q(\tau)f(\tau,x(\tau))\,d\tau\right)dr\right]ds$$

$$\geqslant Bb\left[\frac{1}{1-\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i\int_\delta^{\xi_i}\left[\int_t^\eta \phi_p^{-1}\left(\int_s^{1-\delta} q(\tau)\,d\tau\right)ds\right]dt\right.$$

$$
\begin{aligned}
&+\int_{\delta}^{\eta}\left[\int_{s}^{\eta}\phi_p^{-1}\left(\int_{r}^{1-\delta}q(\tau)\,d\tau\right)dr\right]ds\Bigg] \\
&= Bby(\delta) \geqslant b.
\end{aligned}
$$

Thus, we get $b \leqslant \|Tw\| \leqslant a$, which means $TP[b,a] \subset P[b,a]$.

Let $w_0(t) \equiv a$, $t \in [0,1]$, then $w_0(t) \in P[b,a]$. Let $w_1 = Tw_0$, then $w_1 \in P[b,a]$. We denote

$$
w_{n+1} = Tw_n = T^{n+1}w_0, \quad n = 0,1,2,\ldots. \tag{17}
$$

Since $TP[b,a] \subset P[b,a]$, we have $w_n \in P[b,a]$, $n = 0,1,2,\ldots$. From Lemma 2.4, T is compact, we assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in P[b,a]$ such that $w_{n_k} \to w^*$.

Now, since $w_1 \in P[b,a] \subset P$, we have

$$
0 \leqslant w_1(t) \leqslant \|w\| \leqslant a = w_0(t).
$$

By Lemma 2.4, we know that $Tw_1 \leqslant Tw_0$, which means $w_2(t) \leqslant w_1(t)$, $0 \leqslant t \leqslant 1$.

By induction, then $w_{n+1}(t) \leqslant w_n(t)$, $0 \leqslant t \leqslant 1$ $(n = 0,1,2,\ldots)$. Hence, we assert that $w_n \to w^*$. Let $n \to \infty$ in (17) to obtain $Tw^* = w^*$ since T is continuous. Since $\|w^*\| \geqslant b > 0$ and w^* is a nonnegative concave function on [0,1], we conclude that $w^*(t) > 0$, $t \in (0,1)$.

It is well known that the fixed point of operator T is a solution of problem (1). Therefore, w^* is a positive, concave solution of (1).

Let $v_0(t) = b\min\{\frac{t}{\eta}, \frac{1-t}{1-\eta}\}$, $t \in [0,1]$, then $\|v_0\| = b$, and $v_0 \in P[b,a]$. Let $v_1 = Tv_0$, then $v_1 \in P[b,a]$. We denote

$$
v_{n+1} = Tv_n = T^{n+1}v_0, \quad n = 0,1,2,\ldots. \tag{18}
$$

Similarly to $\{w_n\}_{n=1}^{\infty}$, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in P[b,a]$ such that $v_{n_k} \to v^*$.

Now, since $v_1 \in P[b,a]$, we have by (3):

$$
v_1(t) \geqslant \|v_1\|\min\left\{\frac{t}{\eta}, \frac{1-t}{1-\eta}\right\} \geqslant b\min\left\{\frac{t}{\eta}, \frac{1-t}{1-\eta}\right\} = v_0(t), \quad t \in [0,1].
$$

By Lemma 2.4, we know that $Tv_1 \geqslant Tv_0$, which means $v_2(t) \geqslant v_1(t)$, $0 \leqslant t \leqslant 1$.

By induction, $v_{n+1}(t) \geqslant v_n(t)$, $0 \leqslant t \leqslant 1$ $(n = 0,1,2,\ldots)$. Hence, we assert that $v_n \to v^*$, $Tv^* = v^*$ and $v^*(t) > 0$, $t \in (0,1)$. Therefore, v^* is a positive, concave solution of (1). □

Remark 3.1. We can easily get that w^* and v^* are the maximal and minimal solutions of (1) in $P[b,a]$.

Remark 3.2. When $p = 2$, condition (H5) is weaker than the corresponding ones, namely, (C1), (C2) in Refs. [12,13]. Furthermore, we get not only the existence but also the iteration of positive solutions for a multi-point boundary value problem, while only the existence of a positive solution for a three-point boundary value problem is obtained in Refs. [12,13].

Corollary 3.2. *Assume* (H1)–(H3) *hold. If there exists a constant* $\delta \in (0, \min\{\xi_1, 1-\eta\})$ *such that*

(H6) $f : [0,+\infty) \to [0,+\infty)$ *is nondecreasing*;

(H7) $\overline{\lim}_{l\to 0}\inf_{t\in[\delta,1-\delta]}\frac{f(t,l)}{l^{p-1}} \geqslant (\frac{B\eta}{\delta})^{p-1}$ *and* $\underline{\lim}_{l\to+\infty}\sup_{t\in[0,1]}\frac{f(t,l)}{l^{p-1}} \leqslant (A)^{p-1}$ (*particularly,* $\overline{\lim}_{l\to 0}\inf_{t\in[\delta,1-\delta]}\frac{f(t,l)}{l^{p-1}} = +\infty$ *and* $\underline{\lim}_{l\to+\infty}\sup_{t\in[0,1]}\frac{f(t,l)}{l^{p-1}} = 0$), *where* A, B *are defined as in* (14).

Then, there exist two constants $a > 0$ *and* $b > 0$ *such that problem* (1) *has two positive, concave solutions* $w^*, v^* \in P$ *with*

$$b \leqslant \|w^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} T^n w_0 = w^*, \quad \text{where } w_0(t) = a,\ t \in [0,1],$$
$$b \leqslant \|v^*\| \leqslant a \quad \text{and} \quad \lim_{n\to+\infty} T^n v_0 = v^*, \quad \text{where } v_0(t) = b\min\left\{\frac{t}{\eta}, \frac{1-t}{1-\eta}\right\},\ t \in [0,1].$$

Proof. Since (H4) and (H5) can be obtained from (H6) and (H7), we omit the proof. □

Example. Suppose $0 < k, m < 1/2$ and consider the following differential:

$$\begin{cases} \left(|u''|^{-1/2}u''\right)'(t) = \dfrac{1}{t(1-t)}\left[\left(u(t)\right)^m + \ln\left(\left(u(t)\right)^k + 1\right)\right], & t \in [0,1], \\ u(0) = \displaystyle\sum_{i=1}^m \alpha_i u(\xi_i), \qquad u'(2/3) = 0, \qquad u''(1) = \sum_{i=1}^n \beta_i u(\theta_i). \end{cases} \tag{19}$$

Corresponding to (1), we have $p = 3/2$, $\eta = 2/3$, $f(t,u) = u^m + \ln(u^k + 1)$, and

$$\lim_{l\to 0}\inf_{t\in[0,1]}\frac{f(t,l)}{l^{p-1}} = \lim_{l\to 0}\inf_{t\in[0,1]}\frac{l^m + \ln(l^k+1)}{l^{1/2}} = +\infty,$$
$$\underline{\lim}_{l\to+\infty}\sup_{t\in[0,1]}\frac{f(t,l)}{l^{p-1}} = \lim_{l\to+\infty}\inf_{t\in[0,1]}\frac{l^m + \ln(l^k+1)}{l^{1/2}} = 0.$$

By Corollary 3.2, we can get not only the existence but also the iteration of two concave and positive solutions for problem (19) for any $\xi_i \in (0, 2/3)$ $(i = 1, 2, \ldots, m)$, $\theta_i \in (0,1)$ $(i = 1, 2, \ldots, n)$ and α_i, β_i satisfying (H1), (H2).

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