

Decomposition formulas for some triple hypergeometric functions[☆]

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Abstract

With the help of some techniques based upon certain inverse pairs of symbolic operators, the authors investigate several decomposition formulas associated with Srivastava's hypergeometric functions H_A , H_B and H_C in three variables. Many operator identities involving these pairs of symbolic operators are first constructed for this purpose. By means of these operator identities, as many as 15 decomposition formulas are then found, which express the aforementioned triple hypergeometric functions in terms of such simpler functions as the products of the Gauss and Appell hypergeometric functions. Other closely-related results are also considered briefly.

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1. Introduction

A great interest in the theory of multiple hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [25, p. 47 et seq., Section 1.7]; see also the recent works [14,15] and the references cited therein). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [11]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well (cf. [13,24]). Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [7].

We note that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions of several variables [8]. In investigation of the boundary-value problems for these partial differential equations, we need decompositions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of (for example) the Gauss and Appell types.

The familiar operator method of Burchinal and Chaundy (cf. [2,3]; see also [4]) has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable. In our present investigation, we construct decompositions for each of Srivastava's triple hypergeometric functions H_A , H_B and H_C (see [21,22]) with the help of the Burchinal–Chaundy method. These decompositions involve such simpler hypergeometric functions as the Appell and Gauss functions. By means of the decompositions obtained by us, we also deduce some definite integrals associated with the aforementioned triple hypergeometric functions H_A , H_B and H_C defined by

$$H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \\ (|x| := r < 1; |y| := s < 1; |z| := t < (1-r)(1-s)), \quad (1.1)$$

$$H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \\ (r := |x|; s := |y|; t := |z|; r + s + t + 2\sqrt{rst} < 1) \quad (1.2)$$

and

$$H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \\ (r := |x| < 1; s := |y| < 1; t := |z| < 1; r + s + t - 2\sqrt{(1-r)(1-s)(1-t)} < 2), \quad (1.3)$$

which were introduced and investigated, over four decades ago, by Srivastava (see, for details, [21,22]; see also [25, p. 43] and [26, pp. 68–69]). Here, and in what follows

$$(\lambda)_\mu := \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)}$$

denotes the Pochhammer symbol (or the *shifted* factorial) for all admissible (real or complex) values of λ and μ .

2. The main pairs of symbolic operators

Over six decades ago, Burchnall and Chaundy [2,3] and Chaundy [4] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla_{xy}(h) := \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(h)_k k!}, \quad (2.1)$$

$$\begin{aligned} \Delta_{xy}(h) &:= \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)}{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-\delta_1)_k (-\delta_2)_k}{(h+k-1)_k (\delta_1+h)_k (\delta_2+h)_k k!} \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \nabla_{xy}(h)\Delta_{xy}(g) &:= \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + h)} \frac{\Gamma(\delta_1 + g)\Gamma(\delta_2 + g)}{\Gamma(g)\Gamma(\delta_1 + \delta_2 + g)} \\ &= \sum_{k=0}^{\infty} \frac{(g-h)_k (g)_{2k} (-\delta_1)_k (-\delta_2)_k}{(g+k-1)_k (\delta_1+g)_k (\delta_2+g)_k k!} \\ &= \sum_{k=0}^{\infty} \frac{(h-g)_k (-\delta_1)_k (-\delta_2)_k}{(h)_k (1-g-\delta_1-\delta_2)_k k!} \\ &\left(\delta_1 := x \frac{\partial}{\partial x}; \delta_2 := y \frac{\partial}{\partial y} \right). \end{aligned} \quad (2.3)$$

Indeed, as already observed by Srivastava and Karlsson [25, pp. 332–333], the aforementioned method of Burchnall and Chaundy (cf. [2,3]; see also [4]) was subsequently applied *mutatis mutandis* by Pandey [16] and Srivastava [22] in order to derive the corresponding expansion and decomposition formulas for the triple hypergeometric functions

$$F_A^{(3)}, F_E, F_K, F_M, F_N, F_P \quad \text{and} \quad F_T, H_A, H_C,$$

respectively (see, for definitions, [25, Section 1.5] and [26, p. 66 et seq.]), and by Singhal and Bhati [20] for deriving analogous multiple-series expansions associated with several *multivariable* hypergeometric functions. Subsequently, by making use of the Laplace and inverse Laplace transform techniques in conjunction with the principle of multidimensional mathematical induction, Srivastava [23] established several general families of expansion and decomposition formulas for Kampé de Fériet's double hypergeometric function

$$F_{u:v:v'}^{p:q:q'}$$

and for Srivastava's general triple hypergeometric function

$$F^{(3)}[x, y, z]$$

(see also [25, p. 333, Theorem 1; p. 335, Theorem 2]). Some closely-related results involving Kampé de Fériet's double hypergeometric function can also be found in the works by Ragab [18] and Verma [27].

We now recall here the following multivariable analogues of the Burchnell–Chaundy symbolic operators $\nabla_{xy}(h)$ and $\Delta_{xy}(h)$ defined by (2.1) and (2.2), respectively (cf. [9, pp. 115–116] and [20, p. 240]; see also [22, p. 113] for the case when $r = 3$):

$$\begin{aligned}\tilde{\nabla}_{x_1; x_2 \dots x_r}(h) &:= \frac{\Gamma(h)\Gamma(\delta_1 + \dots + \delta_r + h)}{\Gamma(\delta_1 + h)\Gamma(\delta_2 + \dots + \delta_r + h)} \\ &= \sum_{m_2, \dots, m_r=0}^{\infty} \frac{(-\delta_1)_{m_2+\dots+m_r}(-\delta_2)_{m_2} \dots (-\delta_r)_{m_r}}{m_2! \dots m_r! (h)_{m_2+\dots+m_r}} \\ &\quad \left(\delta_j := x_j \frac{\partial}{\partial x_j}; j = 1, \dots, r \right)\end{aligned}\quad (2.4)$$

and

$$\begin{aligned}\tilde{\Delta}_{x_1; x_2 \dots x_r}(h) &:= \frac{\Gamma(\delta_1 + h)\Gamma(\delta_2 + \dots + \delta_r + h)}{\Gamma(h)\Gamma(\delta_1 + \dots + \delta_r + h)} \\ &= \sum_{m_2, \dots, m_r=0}^{\infty} \frac{(-\delta_1)_{m_2+\dots+m_r}(-\delta_2)_{m_2} \dots (-\delta_r)_{m_r}}{m_2! \dots m_r! (1-h-\delta_1-\dots-\delta_r)_{m_2+\dots+m_r}} \\ &= \sum_{m_2, \dots, m_r=0}^{\infty} \frac{(-1)^{m_2+\dots+m_r}}{m_2! \dots m_r!} \frac{(h)_{2(m_2+\dots+m_r)}}{(h+m_2+\dots+m_r-1)_{m_2+\dots+m_r}} \\ &\quad \cdot \frac{(-\delta_1)_{m_2+\dots+m_r}(-\delta_2)_{m_2} \dots (-\delta_r)_{m_r}}{(\delta_1+h)_{m_2+\dots+m_r}(\delta_2+\dots+\delta_r+h)_{m_2+\dots+m_r}} \\ &\quad \left(\delta_j := x_j \frac{\partial}{\partial x_j}; j = 1, \dots, r \right),\end{aligned}\quad (2.5)$$

where we have applied such known multiple hypergeometric summation formulas as (cf. [10]; see also [1, p. 117])

$$\begin{aligned}F_D^{(r)}[a, b_1, \dots, b_r; c; 1, \dots, 1] &= \frac{\Gamma(c)\Gamma(c-a-b_1-\dots-b_r)}{\Gamma(c-a)\Gamma(c-b_1-\dots-b_r)} \\ &\quad (\Re(c-a-b_1-\dots-b_r) > 0; c \notin \mathbb{Z}_0^- := \{0, -1, -2, -3, \dots\})\end{aligned}\quad (2.6)$$

for the Lauricella function $F_D^{(r)}$ in r variables, defined by (cf. [10]; see also [25, p. 33, Eq. 1.4(4)])

$$\begin{aligned}F_D^{(r)}[a, b_1, \dots, b_r; c; x_1, \dots, x_r] &:= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{m_1+\dots+m_r} (b_1)_{m_1} \dots (b_r)_{m_r}}{(c)_{m_1+\dots+m_r}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_r^{m_r}}{m_r!} \\ &\quad (\max\{|x_1|, \dots, |x_r|\} < 1; c \notin \mathbb{Z}_0^-).\end{aligned}\quad (2.7)$$

3. A set of operator identities

By applying the pairs of symbolic operators in (2.1) to (2.5), we find the following set of operator identities involving the Gauss function ${}_2F_1$, the Appell functions $F_1, \dots, F_4$, and Srivastava's triple hypergeometric functions H_A , H_B and H_C defined by (1.1), (1.2) and (1.3), respectively:

$$\begin{aligned} H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) {}_2F_1(\alpha, \beta_1; \gamma_1; x) F_1(\beta_2, \beta_1, \alpha; \gamma_2; y, z); \end{aligned} \quad (3.1)$$

$$\begin{aligned} H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma, \gamma; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\gamma) {}_2F_1(\alpha, \beta_1; \gamma_1; x) F_1(\beta_2, \beta_1, \alpha; \gamma; y, z); \end{aligned} \quad (3.2)$$

$$\begin{aligned} H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \tilde{\Delta}_{x;yz}(\gamma) {}_2F_1(\alpha, \beta_1; \gamma; x) F_1(\beta_2, \beta_1, \alpha; \gamma; y, z); \end{aligned} \quad (3.3)$$

$$\begin{aligned} H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \Delta_{yz}(\gamma_2) {}_2F_1(\alpha, \beta_1; \gamma_1; x) F_2(\beta_2, \beta_1, \alpha; \gamma_2, \gamma_2; y, z); \end{aligned} \quad (3.4)$$

$$\begin{aligned} H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) {}_2F_1(\alpha, \beta_1; \gamma_1; x) F_2(\beta_2, \beta_1, \alpha; \gamma_2, \gamma_3; y, z); \end{aligned} \quad (3.5)$$

$$\begin{aligned} H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\gamma) \tilde{\Delta}_{x;yz}(\gamma) {}_2F_1(\alpha, \beta_1; \gamma; x) F_2(\beta_2, \beta_1, \alpha; \gamma, \gamma; y, z); \end{aligned} \quad (3.6)$$

$$\begin{aligned} H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\beta_2) {}_2F_1(\alpha, \beta_1; \gamma_1; x) F_3(\beta_1, \beta_2, \beta_2, \alpha; \gamma_2; y, z); \end{aligned} \quad (3.7)$$

$$\begin{aligned} H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma, \gamma; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\beta_2) \nabla_{yz}(\gamma) {}_2F_1(\alpha, \beta_1; \gamma_1; x) F_3(\beta_2, \alpha, \beta_1, \beta_2; \gamma; y, z); \end{aligned} \quad (3.8)$$

$$\begin{aligned} H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\beta_2) \tilde{\Delta}_{x;yz}(\gamma) {}_2F_1(\alpha, \beta_1; \gamma; x) F_3(\beta_2, \beta_2, \beta_1, \alpha; \gamma; y, z); \end{aligned} \quad (3.9)$$

$$\begin{aligned} H_A(\alpha, \alpha, \beta_2; \gamma_1, \gamma_2; x, y, z) \\ = \nabla_{xy}(\alpha) \nabla_{xz}(\alpha) \Delta_{yz}(\alpha) \Delta_{yz}(\gamma_2) {}_2F_1(\alpha, \alpha; \gamma_1; x) F_4(\alpha, \beta_2; \gamma_2, \gamma_2; y, z); \end{aligned} \quad (3.10)$$

$$\begin{aligned} H_B(\alpha, \alpha, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\alpha) \Delta_{yz}(\alpha) {}_2F_1(\alpha, \alpha; \gamma_1; x) F_4(\alpha, \beta_2; \gamma_2, \gamma_3; y, z); \end{aligned} \quad (3.11)$$

$$\begin{aligned} H_C(\alpha, \alpha, \beta_2; \gamma; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\alpha) \Delta_{yz}(\alpha) \Delta_{yz}(\gamma) \tilde{\Delta}_{x;yz}(\gamma) {}_2F_1(\alpha, \alpha; \gamma; x) F_4(\alpha, \beta_2; \gamma, \gamma; y, z); \end{aligned} \quad (3.12)$$

$$\begin{aligned} H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\beta_2) \Delta_{yz}(\gamma_2) \\ \cdot {}_2F_1(\alpha, \beta_1; \gamma_1; x) {}_2F_1(\beta_1, \beta_2; \gamma_2; y) {}_2F_1(\alpha, \beta_2; \gamma_2; z); \end{aligned} \quad (3.13)$$

$$\begin{aligned} H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ = \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\beta_2) {}_2F_1(\alpha, \beta_1; \gamma_1; x) {}_2F_1(\beta_1, \beta_2; \gamma_2; y) {}_2F_1(\alpha, \beta_2; \gamma_3; z); \end{aligned} \quad (3.14)$$

$$\begin{aligned}
H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
= \nabla_{xz}(\alpha) \nabla_{xy}(\beta_1) \nabla_{yz}(\beta_2) \Delta_{yz}(\gamma) \tilde{\Delta}_{x;yz}(\gamma) \\
\cdot {}_2F_1(\alpha, \beta_1; \gamma; x) {}_2F_1(\beta_1, \beta_2; \gamma; y) {}_2F_1(\alpha, \beta_2; \gamma; z).
\end{aligned} \quad (3.15)$$

In view of the known Mellin–Barnes contour integral representations for the Gauss function ${}_2F_1$, the Appell functions $F_1, \dots, F_4$, and Srivastava's triple hypergeometric functions H_A, H_B and H_C , it is not difficult to give *alternative* proofs of the operator identities (3.1) to (3.15) above by using the Mellin and the inverse Mellin transformations (see, for example, [1,12,22, 25]). The details involved in these *alternative* derivations of the operator identities (3.1) to (3.15) are being omitted here.

4. Decompositions for Srivastava's triple hypergeometric functions H_A, H_B and H_C

Making use of the principle of superposition of operators, from the operator identities (3.1) to (3.15) we can derive the following decomposition formulas for Srivastava's triple hypergeometric functions H_A, H_B and H_C :

$$\begin{aligned}
H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
= \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}(\beta_1)_{i+j}(\beta_2)_{i+j}}{(\gamma_1)_{i+j}(\gamma_2)_{i+j}i!j!} x^{i+j} y^j z^i \\
\cdot {}_2F_1(\alpha + i + j, \beta_1 + i + j; \gamma_1 + i + j; x) \\
\cdot F_1(\beta_2 + i + j, \beta_1 + i + j, \alpha + i; \gamma_2 + i + j; y, z);
\end{aligned} \quad (4.1)$$

$$\begin{aligned}
H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma, \gamma; x, y, z) \\
= \sum_{i,j,k=0}^{\infty} \frac{(\alpha)_{i+j+k}(\beta_1)_{i+j+k}(\beta_2)_{2i+j+k}}{(\gamma_1)_{j+k}(\gamma)_i(\gamma)_{2i+j+k}i!j!k!} x^{j+k} y^{i+j} z^{i+k} \\
\cdot {}_2F_1(\alpha + i + j + k, \beta_1 + i + j + k; \gamma_1 + j + k; x) \\
\cdot F_1(\beta_2 + 2i + j + k, \beta_1 + i + j, \alpha + i + j + k; \gamma + 2i + j + k; y, z);
\end{aligned} \quad (4.2)$$

$$\begin{aligned}
H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
= \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j} \frac{(\alpha)_{i+2j+k+l}(\beta_1)_{2i+j+k+l}(\beta_2)_{i+j+k+l}(\gamma)_{2i+2j}}{(\gamma + i + j - 1)_{i+j}[(\gamma)_{2i+2j+k+l}]^2 i!j!k!l!} x^{i+j+k+l} y^{i+k} z^{j+l} \\
\cdot {}_2F_1(\alpha + i + 2j + k + l, \beta_1 + 2i + j + k + l; \gamma + 2i + 2j + k + l; x) \\
\cdot F_1(\beta_2 + i + j + k + l, \beta_1 + 2i + j + k, \alpha + i + 2j + k + l; \\
\gamma + 2i + 2j + k + l; y, z);
\end{aligned} \quad (4.3)$$

$$\begin{aligned}
H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
= \sum_{i,j,k=0}^{\infty} (-1)^k \frac{(\gamma_2)_{2k}(\alpha)_{i+j+k}(\beta_1)_{i+j+k}(\beta_2)_{i+j+2k}}{(\gamma_2 + k - 1)_k(\gamma_1)_{i+j}(\gamma_2)_{i+2k}(\gamma_2)_{j+2k}i!j!k!} x^{i+j} y^{j+k} z^{i+k} \\
\cdot {}_2F_1(\alpha + i + j + k, \beta_1 + i + j + k; \gamma_1 + i + j; x) \\
\cdot F_2(\beta_2 + i + j + 2k, \beta_1 + j + k, \alpha + i + j + k; \gamma_2 + j + 2k, \gamma_2 + i + 2k; y, z);
\end{aligned} \quad (4.4)$$

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\
&= \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}(\beta_1)_{i+j}(\beta_2)_{i+j}}{(\gamma_1)_{i+j}(\gamma_2)_j(\gamma_3)_i i! j!} x^{i+j} y^j z^i \\
&\quad \cdot {}_2F_1(\alpha + i + j, \beta_1 + i + j; \gamma_1 + i + j; x) \\
&\quad \cdot F_2(\beta_2 + i + j, \beta_1 + i + j, \alpha + i; \gamma_2 + j, \gamma_3 + i; y, z); \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
& H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
&= \sum_{i,j,k,l,r=0}^{\infty} (-1)^{i+j+r} \frac{(\alpha)_{i+2j+k+l+r}(\beta_1)_{2i+j+k+l}(\beta_1)_{2i+j+k+r}(\beta_2)_{i+j+k+l+2r}}{(\gamma + i + j - 1)_{i+j}(\gamma + 2i + 2j + k + l + r - 1)_r(\beta_1)_{2i+j+k}} \\
&\quad \cdot \frac{(\gamma)_{2i+2j}}{(\gamma)_{2i+2j+k+l}(\gamma)_{2i+2j+k+l+2r} i! j! k! l! r!} x^{i+j+k+l} y^{i+k+r} z^{j+l+r} \\
&\quad \cdot {}_2F_1(\alpha + i + 2j + k + l, \beta_1 + 2i + j + k + l; \gamma + 2i + 2j + k + l; x) \\
&\quad \cdot F_2(\beta_2 + i + j + k + l + 2r, \beta_1 + 2i + j + k + r, \alpha + i + 2j + k + l + r; \\
&\quad \quad \gamma + 2i + 2j + k + l + 2r, \gamma + 2i + 2j + k + l + 2r; y, z); \tag{4.6}
\end{aligned}$$

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
&= \sum_{i,j,k=0}^{\infty} \frac{(\alpha)_{i+j+k}(\beta_1)_{i+j+k}(\beta_2)_{i+j}(\beta_2)_{i+k}}{(\beta_2)_i(\gamma_1)_{j+k}(\gamma_2)_{2i+j+k} i! j! k!} x^{j+k} y^{i+j} z^{i+k} \\
&\quad \cdot {}_2F_1(\alpha + i + j + k, \beta_1 + i + j + k; \gamma_1 + j + k; x) \\
&\quad \cdot F_3(\beta_1 + i + j, \beta_2 + i + k, \beta_2 + i + j, \alpha + i + j + k; \gamma_2 + 2i + j + k; y, z); \tag{4.7}
\end{aligned}$$

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma, \gamma; x, y, z) \\
&= \sum_{i,j,k,l=0}^{\infty} \frac{(\alpha)_{j+2k+l}(\beta_1)_{i+j+k+l}(\beta_2)_{2i+j+k}(\beta_2)_{2i+j+l}}{(\beta_2)_{2i+j}(\gamma)_i(\gamma)_{2i+2j+k+l}(\gamma_1)_{k+l} i! j! k! l!} x^{k+l} y^{i+j+k} z^{i+j+l} \\
&\quad \cdot {}_2F_1(\alpha + j + 2k + l, \beta_1 + i + j + k + l; \gamma_1 + k + l; x) \\
&\quad \cdot F_3(\beta_2 + 2i + j + k, \alpha + i + j + k + l, \beta_1 + i + j + k, \beta_2 + 2i + j + l; \\
&\quad \quad \gamma + 2i + 2j + k + l; y, z); \tag{4.8}
\end{aligned}$$

$$\begin{aligned}
& H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
&= \sum_{i,j,k,l,r=0}^{\infty} (-1)^{i+j} \\
&\quad \cdot \frac{(\alpha)_{i+2j+k+l}(\alpha)_{i+2j+k+r}(\beta_1)_{2i+j+k+l+r}(\beta_2)_{i+j+k+l+r}(\gamma)_{2i+2j}}{(\gamma + i + j - 1)_{i+j}(\alpha)_{i+2j+k}(\gamma)_{2i+2j+k+l}(\gamma)_{2i+2j+k+l+2r} i! j! k! l! r!} \\
&\quad \cdot x^{i+j+k+l} y^{i+l+r} z^{j+k+r} \\
&\quad \cdot {}_2F_1(\alpha + i + 2j + k + l, \beta_1 + 2i + j + k + l; \gamma + 2i + 2j + k + l; x) \\
&\quad \cdot F_3(\beta_2 + i + j + k + l + r, \beta_2 + i + j + k + l + r, \beta_1 + 2i + j + k + l + r, \\
&\quad \quad \alpha + i + 2j + k + r; \gamma + 2i + 2j + k + l + 2r; y, z); \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
&= \sum_{i,j,k=0}^{\infty} \frac{(\alpha)_{i+j}(\alpha)_{i+k}(\beta_1)_{i+j+k}(\beta_2)_{i+j+k}(\gamma_2 - \beta_2)_k}{(\gamma_2 + i + j + k - 1)_k(\alpha)_i(\gamma_1)_{i+j}(\gamma_2)_{i+j+2k}i!j!k!} x^{i+j} y^{j+k} z^{i+k} \\
&\quad \cdot {}_2F_1(\alpha + i + j, \beta_1 + i + j; \gamma_1 + i + j; x) \\
&\quad \cdot {}_2F_1(\beta_2 + i + j + k, \beta_1 + i + j + k; \gamma_2 + i + j + 2k; y) \\
&\quad \cdot {}_2F_1(\beta_2 + i + j + k, \alpha + i + k; \gamma_2 + i + j + 2k; z); \tag{4.10}
\end{aligned}$$

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\
&= \sum_{i,j,k=0}^{\infty} \frac{(\alpha)_{i+j+k}(\beta_1)_{i+j+k}(\beta_2)_{i+j}(\beta_2)_{j+k}}{(\beta_2)_j(\gamma_1)_{i+k}(\gamma_2)_{i+j}(\gamma_3)_{j+k}i!j!k!} x^{i+k} y^{i+j} z^{j+k} \\
&\quad \cdot {}_2F_1(\alpha + i + j + k, \beta_1 + i + j + k; \gamma_1 + i + k; x) \\
&\quad \cdot {}_2F_1(\beta_1 + i + j, \beta_2 + i + j; \gamma_2 + i + j; y) \\
&\quad \cdot {}_2F_1(\beta_2 + j + k, \alpha + i + j + k; \gamma_3 + j + k; z); \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
& H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
&= \sum_{i,j,k,l,r=0}^{\infty} (-1)^{i+j} \frac{(\alpha)_{i+2j+k+l+r}(\beta_1)_{2i+j+k+l}(\beta_1)_{2i+j+k+r}(\beta_2)_{i+j+k+l+r}}{(\gamma + i + j - 1)_{i+j}(\gamma + 2i + 2j + k + l + r - 1)_r(\beta_1)_{2i+j+k}} \\
&\quad \cdot \frac{(\gamma)_{2i+2j}(\gamma - \beta_2)_{i+j+r}}{(\gamma - \beta_2)_{i+j}(\gamma)_{2i+2j+k+l}(\gamma)_{2i+2j+k+l+2r}i!j!k!l!r!} x^{i+j+k+l} y^{i+k+r} z^{j+l+r} \\
&\quad \cdot {}_2F_1(\alpha + i + 2j + k + l, \beta_1 + 2i + j + k + l; \gamma + 2i + 2j + k + l; x) \\
&\quad \cdot {}_2F_1(\beta_2 + i + j + k + l + r, \beta_1 + 2i + j + k + r; \gamma + 2i + 2j + k + l + 2r; y) \\
&\quad \cdot {}_2F_1(\beta_2 + i + j + k + l + r, \alpha + i + 2j + k + l + r; \gamma + 2i + 2j + k + l + 2r; z); \tag{4.12}
\end{aligned}$$

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, y) \\
&= \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}(\beta_1)_{i+j}(\beta_2)_{i+j}}{(\gamma_1)_{i+j}(\gamma_2)_{i+j}i!j!} x^{i+j} y^{i+j} \\
&\quad \cdot {}_2F_1(\alpha + i + j, \beta_1 + i + j; \gamma_1 + i + j; x) \\
&\quad \cdot {}_2F_1(\beta_2 + i + j, \alpha + \beta_1 + 2i + j; \gamma_2 + i + j; y); \tag{4.13}
\end{aligned}$$

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma, \gamma; x, y, y) \\
&= \sum_{i,j,k=0}^{\infty} \frac{(\alpha)_{i+j+k}(\beta_1)_{i+j+k}(\beta_2)_{2i+j+k}}{(\gamma_1)_{j+k}(\gamma)_i(\gamma)_{2i+j+k}i!j!k!} x^{j+k} y^{2i+j+k} \\
&\quad \cdot {}_2F_1(\alpha + i + j + k, \beta_1 + i + j + k; \gamma_1 + j + k; x) \\
&\quad \cdot {}_2F_1(\beta_2 + 2i + j + k, \alpha + \beta_1 + 2i + 2j + k; \gamma + 2i + j + k; y); \tag{4.14}
\end{aligned}$$

$$\begin{aligned}
& H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, y) \\
&= \sum_{i,j,k,l=0}^{\infty} (-1)^{i+j} \frac{(\alpha)_{i+2j+k+l} (\beta_1)_{2i+j+k+l} (\beta_2)_{i+j+k+l} (\gamma)_{2i+2j}}{(\gamma+i+j-1)_{i+j} [(\gamma)_{2i+2j+k+l}]^2 i! j! k! l!} x^{i+j+k+l} y^{i+j+k+l} \\
&\quad \cdot {}_2F_1(\alpha+i+2j+k+l, \beta_1+2i+j+k+l; \gamma+2i+2j+k+l; x) \\
&\quad \cdot {}_2F_1(\beta_2+i+j+k+l, \alpha+\beta_1+3i+3j+2k+l; \gamma+2i+2j+k+l; y).
\end{aligned} \tag{4.15}$$

Our operational derivations of the decomposition formulas (4.1) to (4.15) would indeed run parallel to those presented in the earlier works which we have already cited in the preceding sections. In addition to the various operator expressions and operator identities listed in Sections 2 and 3, we also make use of the following operator identities [17, p. 93]:

$$\begin{aligned}
(\delta + \alpha)_n \{f(\xi)\} &= \xi^{1-\alpha} \frac{d^n}{d\xi^n} \{\xi^{\alpha+n-1} f(\xi)\} \\
\left(\delta &:= \xi \frac{d}{d\xi}; \alpha \in \mathbb{C}; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, 3, \dots\} \right)
\end{aligned} \tag{4.16}$$

and

$$(-\delta)_n \{f(\xi)\} = (-\xi)^n \frac{d^n}{d\xi^n} \{f(\xi)\} \quad \left(\delta := \xi \frac{d}{d\xi}; n \in \mathbb{N}_0 \right) \tag{4.17}$$

for every *analytic* function $f(\xi)$.

5. Alternative derivations of the decomposition formulas (4.1) to (4.15)

First of all, we prove the decomposition formula (4.1) with the help of the following known integral representation for H_A [21, p. 100, Eq. (3.3)]:

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
&= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1-\beta_1)\Gamma(\gamma_2-\beta_2)} \\
&\quad \cdot \int_0^1 \int_0^1 \xi^{\beta_1-1} \eta^{\beta_2-1} (1-\xi)^{\gamma_1-\beta_1-1} (1-\eta)^{\gamma_2-\beta_2-1} (1-y\eta)^{\alpha-\beta_1} \\
&\quad \cdot [(1-y\eta)(1-z\eta) - x\xi]^{-\alpha} d\xi d\eta \\
& \quad (\Re(\gamma_1) > \Re(\beta_1) > 0; \Re(\gamma_2) > \Re(\beta_2) > 0).
\end{aligned} \tag{5.1}$$

Since

$$\begin{aligned}
& [(1-y\eta)(1-z\eta) - x\xi]^{-\alpha} \\
&= [(1-x\xi)(1-y\eta)(1-z\eta)]^{-\alpha} \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}}{i!j!} \sigma_1^i \sigma_2^j \\
& \quad \left(\sigma_1 := \frac{xz\xi\eta}{(1-x\xi)(1-y\eta)(1-z\eta)}; \sigma_2 := \frac{xy\xi\eta}{(1-x\xi)(1-y\eta)} \right),
\end{aligned} \tag{5.2}$$

upon substituting from (5.2) into the integral representation (5.1), we find that

$$\begin{aligned}
H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) &= \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}}{i!j!} x^{i+j} y^j z^i \\
&\cdot \frac{\Gamma(\gamma_1)}{\Gamma(\beta_1)\Gamma(\gamma_1 - \beta_1)} \int_0^1 \xi^{\beta_1+i+j-1} (1-\xi)^{\gamma_1-\beta_1-1} (1-x\xi)^{-\alpha-i-j} d\xi \\
&\cdot \frac{\Gamma(\gamma_2)}{\Gamma(\beta_2)\Gamma(\gamma_2 - \beta_2)} \int_0^1 \eta^{\beta_2+i+j-1} (1-\eta)^{\gamma_2-\beta_2-1} (1-y\eta)^{-\beta_1-i-j} (1-z\eta)^{-\alpha-i} d\eta,
\end{aligned} \tag{5.3}$$

which, by virtue of the following well-known integral representations:

$$\begin{aligned}
\int_0^1 \xi^{b-1} (1-\xi)^{c-b-1} (1-x\xi)^{-a} d\xi &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; x) \\
&(\Re(c) > \Re(b) > 0)
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
\int_0^1 \eta^{a-1} (1-\eta)^{c-a-1} (1-y\eta)^{-b_1} (1-z\eta)^{-b_2} d\eta \\
= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_1(a, b_1, b_2; c; y, z) \quad (\Re(c) > \Re(a) > 0),
\end{aligned} \tag{5.5}$$

yields the decomposition formula (4.1).

Next, in order to give an alternative derivation of the decomposition formula (4.3), we similarly apply the following known integral representation [22, p. 100, Eq. (1.4)]:

$$\begin{aligned}
H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\gamma - \alpha - \beta_1)} \\
&\cdot \int_0^1 \int_0^1 \xi^{\alpha-1} \eta^{\beta_1-1} (1-\xi)^{\gamma-\alpha-1} (1-\eta)^{\gamma-\alpha-\beta_1-1} (1-x\xi)^{\beta_2-\beta_1} \\
&\cdot (1-x\xi - y\eta - z\xi + y\xi\eta + zx\xi^2)^{-\beta_2} d\xi d\eta \\
&(\Re(\alpha) > 0; \Re(\beta_1) > 0; \Re(\gamma) > \Re(\alpha + \beta_1))
\end{aligned} \tag{5.6}$$

in conjunction with (5.4) and (5.5). In view of (5.4), Srivastava's result (5.6) is an immediate consequence of the following *single-integral* representation for H_C :

$$\begin{aligned}
H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \eta^{\alpha-1} (1-\eta)^{\gamma-\alpha-1} (1-x\eta)^{-\beta_1} (1-z\eta)^{-\beta_2}
\end{aligned}$$

$$\cdot {}_2F_1\left(\beta_1, \beta_2; \gamma - \alpha; \frac{(1-\eta)y}{(1-x\eta)(1-z\eta)}\right) d\eta$$

$$(\Re(\gamma) > \Re(\alpha) > 0). \quad (5.7)$$

In a similar manner, many of the other decomposition formulas of Section 4 can also be derived alternatively by means of some appropriate integral representations for the triple hypergeometric functions involved in them.

6. Integral representations via decomposition formulas

For each of his triple hypergeometric functions H_A , H_B and H_C , Srivastava [21,22] gave several ordinary as well as contour integral representations of the Eulerian, Laplace, Mellin–Barnes, and Pochhammer’s double-loop types. Here, in this section, we first observe that several known integral representations of the Eulerian type can be deduced also from the corresponding decomposition formulas of Section 4. For example, we have (cf. [21, p. 100, Eq. (3.3)]; see also Eq. (5.1) above)

$$H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z)$$

$$= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1 - \beta_1)\Gamma(\gamma_2 - \beta_2)}$$

$$\cdot \int_0^1 \int_0^1 \xi^{\beta_1-1} \eta^{\beta_2-1} (1-\xi)^{\gamma_1-\beta_1-1} (1-\eta)^{\gamma_2-\beta_2-1} (1-y\eta)^{-\beta_1} (1-x\xi-z\eta)^{-\alpha}$$

$$\cdot \left(1 - \frac{xy\xi\eta}{(1-y\eta)(1-x\xi-z\eta)}\right)^{-\alpha} d\xi d\eta$$

$$(\Re(\gamma_1) > \Re(\beta_1) > 0; \Re(\gamma_2) > \Re(\beta_2) > 0), \quad (6.1)$$

which Srivastava [21] deduced from his *single-integral* representation [21, p. 100]:

$$H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z)$$

$$= \frac{\Gamma(\gamma_2)}{\Gamma(\beta_2)\Gamma(\gamma_2 - \beta_2)} \int_0^1 \eta^{\beta_2-1} (1-\eta)^{\gamma_2-\beta_2-1} (1-y\eta)^{-\beta_1} (1-z\eta)^{-\alpha}$$

$$\cdot {}_2F_1\left(\alpha, \beta_1; \gamma_1; \frac{x}{(1-y\eta)(1-z\eta)}\right) d\eta$$

$$(\Re(\gamma_2) > \Re(\beta_2) > 0). \quad (6.2)$$

Next we turn to a set of known *double-integral* representations of the Laplace type for H_A , H_B and H_C , each of which was derived by Srivastava [22, p. 101] from the following rather elementary formula:

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t} t^{\lambda+n-1} dt \quad (\Re(\lambda) > 0; n \in \mathbb{N}_0). \quad (6.3)$$

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_0^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} {}_0F_1(-; \gamma_1; xst) {}_1F_1(\beta_2; \gamma_2; ys+zt) ds dt \\
& \quad (\min\{\Re(\alpha), \Re(\beta_1)\} > 0; \max\{\Re(y), \Re(z)\} < 1),
\end{aligned} \tag{6.4}$$

which, in view of the elementary integral formula:

$${}_1F_1(\lambda; \mu; z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t} t^{\lambda-1} {}_0F_1(-; \mu; zt) dt \quad (\Re(\lambda) > 0), \tag{6.5}$$

immediately yields the following *triple-integral* representation of the Laplace type for H_A :

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-s-t-u} t^{\alpha-1} s^{\beta_1-1} u^{\beta_2-1} \\
& \quad \cdot {}_0F_1(-; \gamma_1; xst) {}_0F_1(-; \gamma_2; yus+zut) ds dt du \\
& \quad (\min\{\Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0).
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_0^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} \\
& \quad \cdot {}_0F_1(-; \gamma_1; xst) \Psi_2(\beta_2; \gamma_2, \gamma_3; ys, zt) ds dt \\
& \quad (\min\{\Re(\alpha), \Re(\beta_1)\} > 0; \max\{\Re(y), \Re(z)\} < 1),
\end{aligned} \tag{6.7}$$

where Ψ_2 denotes one of Humbert's confluent hypergeometric functions of two variables [5, p. 225]:

$$\Psi_2(\alpha; \gamma_1, \gamma_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma_1)_m (\gamma_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{6.8}$$

Indeed, since

$$\begin{aligned}
& \Psi_2(\alpha; \gamma_1, \gamma_2; x, y) \\
&= \int_0^\infty e^{-t} t^{\alpha-1} {}_0F_1(-; \gamma_1; xt) {}_0F_1(-; \gamma_2; yt) dt \quad (\Re(\alpha) > 0),
\end{aligned} \tag{6.9}$$

which is easily derivable by combining (6.3) with the definition (6.8), we find from Srivastava's result (6.7) that

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-s-t-u} t^{\alpha-1} s^{\beta_1-1} u^{\beta_2-1} \\
& \quad \cdot {}_0F_1(-; \gamma_1; xst) {}_0F_1(-; \gamma_2; yus) {}_0F_1(-; \gamma_3; zut) ds dt du \\
& \quad (\min\{\Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0).
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
& H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_0^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} \Phi_3(\beta_2; \gamma; ys + zt, xst) ds dt \\
& \quad (\min\{\Re(\alpha), \Re(\beta_1)\} > 0; \max\{\Re(y), \Re(z)\} < 1),
\end{aligned} \tag{6.11}$$

where Φ_3 denotes another Humbert's confluent hypergeometric function of two variables [5, p. 225]:

$$\Phi_3(\alpha; \gamma; x, y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{6.12}$$

By appealing to the following rather straightforward consequence of (6.3) and the definition (6.12):

$$\Phi_3(\alpha; \gamma; x, y) = \int_0^\infty e^{-t} t^{\alpha-1} {}_0F_1(-; \gamma; xt + y) dt \quad (\Re(\alpha) > 0), \tag{6.13}$$

we can easily rewrite Srivastava's result (6.11) as a *triple-integral* representation of the Laplace type for H_C :

$$\begin{aligned}
& H_C(\alpha, \beta_1, \beta_2; \gamma; x, y, z) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-s-t-u} t^{\alpha-1} s^{\beta_1-1} u^{\beta_2-1} \\
& \quad \cdot {}_0F_1(-; \gamma; xst + yus + zut) ds dt du \\
& \quad (\min\{\Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0).
\end{aligned} \tag{6.14}$$

In each of the integral representations presented in this as well as the preceding sections, it is *tacitly* assumed that both sides of the result exist. Multiple-integral extensions of (6.10) and (6.14) for the hypergeometric functions $H_B^{(n)}$ and $H_C^{(n)}$ in n variables were recorded also by Srivastava and Karlsson [25, p. 325, Eqs. (198) and (199)].

7. Concluding remarks and observations

By suitably specializing the decomposition formulas (4.1) to (4.15), we can deduce a number of (known or new) decomposition formulas including those given by (for example) Burchnell and Chaundy [2,3]. For instance, for Appell's hypergeometric functions, we find the following (presumably new) results:

$$\begin{aligned}
& F_1(\alpha, \beta, \beta; \gamma; x, y) \\
&= \sum_{i,j=0}^{\infty} (-1)^{i+j} \frac{(\alpha)_{2i+2j} (\beta)_i (\beta)_{i+j} (\gamma)_{2i}}{(\gamma+i-1)_i [(\gamma)_{2i+j}]^2 i! j!} x^{i+j} y^{i+j} \\
& \quad \cdot F_4(\alpha + 2i + 2j, \beta + i + j; \gamma + 2i + j, \gamma + 2i + j; x, y)
\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}
& F_2(\alpha, \beta_1, \beta_2; \gamma, \gamma; x, y) \\
&= \sum_{i,j=0}^{\infty} \frac{(\alpha)_{2i+j}(\beta_1)_{i+j}(\beta_2)_{i+j}}{(\gamma)_i(\gamma)_{2i+2j}i!j!} x^{i+j} y^{i+j} \\
&\quad \cdot F_3(\alpha + 2i + j, \alpha + 2i + j, \beta_1 + i + j, \beta_2 + i + j; \gamma + 2i + 2j; x, y). \quad (7.2)
\end{aligned}$$

Furthermore, by making use of the decompositions (4.1) and (4.2), we can derive the following known reduction formulas for Srivastava's triple hypergeometric functions H_A and H_B [21, p. 103, Eq. (5.3)]:

$$\begin{aligned}
& H_A(\alpha, \beta_1, \beta_2; \gamma_1, \beta_2; x, y, z) \\
&= (1-y)^{-\beta_1} (1-z)^{-\alpha} {}_2F_1\left(\alpha, \beta_1; \gamma_1; \frac{x}{(1-y)(1-z)}\right) \quad (7.3)
\end{aligned}$$

and [21, p. 104, Eq. (5.6)]

$$\begin{aligned}
& H_B(\alpha, \beta_1, \beta_2; \gamma_1, \beta_2, \beta_2; x, y, z) \\
&= (1-y)^{-\beta_1} (1-z)^{-\alpha} {}_4F_4\left(\alpha, \beta_1; \gamma_1, \beta_2; \frac{x}{(1-y)(1-z)}, \frac{yz}{(1-y)(1-z)}\right). \quad (7.4)
\end{aligned}$$

Some of the most recent contributions in the theory of Srivastava's triple hypergeometric series H_A , H_B and H_C include a paper by Harold Exton (1928–2001) [6] and a paper by Rathie and Kim [19]. The work of Exton [6] made use of elementary series manipulation and some well-known analytic continuation formulas for the Gauss hypergeometric function in order to derive a fundamental set of nine solutions of the system of partial differential equations satisfied by the *symmetrical* function H_B . Rathie and Kim [19], on the other hand, presented several summation formulas for the functions H_A and H_C by applying Srivastava's result [21, p. 104]:

$$\begin{aligned}
& F_1(a, b, b'; a + b - b' + 1; 1, -1) \\
&= \frac{\Gamma(a + b - b' + 1)\Gamma(1 - b')\Gamma(\frac{1}{2}a + 1)}{\Gamma(a + 1)\Gamma(b - b' + 1)\Gamma(\frac{1}{2}a - b' + 1)} \quad (\Re(b') < 1). \quad (7.5)
\end{aligned}$$

However, in deriving many of their applications of the summation formula (7.5), Rathie and Kim [19] obviously violated the constraint $\Re(b') < 1$ associated with (7.5) at least in situations in which the hypergeometric series involved in their investigation would not terminate.

Finally, we note that, here in this paper, we have not applied such superpositions of operators as those provided by (for example)

$$\nabla_{xy}(\alpha)\nabla_{xz}(\alpha)\Delta_{yz}(\alpha)\Delta_{yz}(\gamma_2), \quad \nabla_{xz}(\alpha)\nabla_{xy}(\alpha)\Delta_{yz}(\alpha),$$

and

$$\nabla_{xz}(\alpha)\nabla_{xy}(\alpha)\Delta_{yz}(\alpha)\Delta_{yz}(\gamma)\tilde{\Delta}_{x;yz}(\gamma).$$

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