Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 325 (2007) 438–458
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Uniform attractors for non-autonomous motion equations of viscoelastic medium ☆

Dmitry A. Vorotnikov *, Victor G. Zvyagin

Department of Mathematics, Voronezh State University, Universitetskaya Pl. 1, 394006 Voronezh, Russia

Received 25 July 2005

Available online 28 February 2006

Submitted by M.C. Nucci

Abstract

Sufficient conditions for existence of minimal uniform trajectory attractors and uniform global attractors of non-autonomous evolution equations in Banach spaces are obtained. It is not assumed that the symbol space of an equation is a compact metric space and that the family of trajectory spaces corresponding to this symbol space is translation-coordinated or closed in any sense. Using these results, existence of minimal uniform trajectory attractors and uniform global attractors for weak solutions of the boundary value problem for motion equations of an incompressible viscoelastic medium with the Jeffreys constitutive law is shown.
© 2006 Elsevier Inc. All rights reserved.

Keywords: Uniform global attractor; Minimal uniform trajectory attractor; Non-autonomous evolution equation; Symbol space; Motion of viscoelastic medium

0. Introduction

The study of attractors occupies an important place in the modern mathematical methods of natural sciences. Attractors for systems of differential equations or for dynamical systems are the sets to which the solutions of an equation or trajectories of a system are eventually attracted. As a rule, to the condition of attraction one adds the conditions of strict invariance, minimality and compactness.

☆ The work was partially supported by grants 04-01-00081 of Russian Foundation of Basic Research and VZ-010-0 of the Ministry of Education and Science of Russia and CRDF.

* Corresponding author.
E-mail addresses: mitvorot@math.vsu.ru (D.A. Vorotnikov), zvg@main.vsu.ru (V.G. Zvyagin).

0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.01.078

O.A. Ladyzenskaya [8] proposed to investigate attractors as the basic object of the turbulence theory for evolution systems with dissipation and for equations of hydrodynamics, in particular. In that work she also proved existence and investigated properties of the global attractor for the two-dimensional autonomous Navier–Stokes system. The further study of this attractor was carried out by various authors (see [1,14]).

Recently there appeared a lot of works on the attractors of non-autonomous equations and dynamical systems (see [2,4,6]). The uniform attractors of non-autonomous two-dimensional Navier–Stokes system were constructed and investigated (see [9]). Research of the three-dimensional case was restricted by the difficulty that in this case existence of global in time strong solutions for the basic initial-boundary value problem and uniqueness of weak solutions of this problem are not proved till now. In this connection there was developed the theory of trajectory attractors (see, e.g., [4,15]), which allows to construct a global attractor for various evolution equations without uniqueness of solutions of the corresponding Cauchy problem. In particular, there was proved existence of uniform trajectory attractors and uniform global attractors for weak solutions of the three-dimensional Navier–Stokes system.

However, it turned out that this theory cannot be directly applied for the following boundary value problem for the motion equations of a viscoelastic medium with the Jeffreys constitutive law:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p = \operatorname{Div} \sigma + f, \tag{0.1}$$

$$\sigma + \lambda_1 \left(\frac{\partial \sigma}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i} \right) = 2\eta \left(\mathcal{E} + \lambda_2 \left(\frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \mathcal{E}}{\partial x_i} \right) \right), \tag{0.2}$$

$$\operatorname{div} u = 0, \tag{0.3}$$

$$u|_{\partial\Omega} = 0. \tag{0.4}$$

Here Ω is an arbitrary bounded domain in $\mathbb{R}^n$ ($n = 2, 3$), u is an unknown vector of velocity of points of the medium, p is an unknown function of pressure, σ is an unknown deviator of the stress tensor, f is the given body force (all of them depend on a point $x \in \Omega$ and a moment of time t), $\mathcal{E} = \mathcal{E}(u) = (\mathcal{E}_{ij})$, $\mathcal{E}_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the strain velocity tensor, $\eta > 0$ is the viscosity of the medium, λ_1 is the relaxation time, λ_2 is the retardation time, $0 < \lambda_2 < \lambda_1$. The gradient grad and the divergence div are taken with respect to the variable x. The divergence Div of a tensor is the vector with the coordinates $(\operatorname{Div} \sigma)_j = \sum_{i=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_i}$. The density of the medium is considered to be equal to one.

Equation (0.1) is the general motion equation in the form of Cauchy [5]. Equation (0.2) is the Jeffreys constitutive law. This relation describes materials like solutions of polymers, bitumens, concrete, the earth's crust [10]. Equation (0.3) is the equation of continuity [5]. Equation (0.4) is the non-slip condition on the boundary of the domain Ω.

The attractor of two-dimensional problem (0.1), (0.3), (0.4) with (0.2) substituted by the linearized constitutive relation and with the body force f independent on time t (autonomous case) was studied in [7].

The difficulties in application of the theory of trajectory attractors to Jeffreys' problem (0.1)–(0.4) come from two points which are essential in this theory. The first point is the assumption that the family of the corresponding trajectory spaces are translation-coordinated [4]. The second point is the requirement of constructing an attracting set which is contained in the united

trajectory space [15], which may be replaced [4] by the assumption that the family of trajectory spaces is closed in some sense. But for Jeffreys' problem the classes of existence of solutions for the corresponding initial-boundary value problem (which are determined by a function space and an energy inequality) generate families of trajectory spaces which are either not translation-coordinated (and not closed in the appropriate sense) or too wide to provide the existence of an attractor. The construction of an attracting set contained in the united trajectory space is also a problem.

Existence of weak solutions for problem (0.1)–(0.4) for $n = 2, 3$ is proved in [16] using the approximating-topological method [17]. The problem of uniqueness of these solutions is open even at $n = 2$.

In this paper we use a way of construction of a uniform trajectory attractor which is different from the earlier ones. This way of proof of existence of attractors does not require the assumptions that the family of the corresponding trajectory spaces are translation-coordinated, that the family of trajectory spaces is closed in any sense and that the symbol space of an equation is a compact metric space. Construction of an attracting set contained in the united trajectory space is not necessary. This approach allows to show existence of uniform trajectory attractors and of uniform global attractors for wider classes of equations, in particular, at the situations when these conditions are not fulfilled.

The plan of this paper is the following. In Section 1, we give necessary notations, most of which are standard. In Section 2, we introduce the concept of minimal uniform trajectory attractor generalizing the concept of uniform trajectory attractor from [15] for a trajectory space of an abstract non-autonomous evolution equation. With the help of this concept it is possible to construct the uniform global attractor. We do not assume that the symbol space of the equation is a compact metric space and that the family of trajectory spaces corresponding to this symbol space is translation-coordinated. We do not assume also that the family of trajectory spaces is closed in any sense. We obtain theorems, which give sufficient conditions for existence of minimal uniform trajectory attractors and uniform global attractors. These conditions look like existence of an attracting set possessing some additional properties (like compactness, boundedness, semi-invariance). Note that we do not require for such attracting set to be contained in the trajectory space. We also prove that a uniform trajectory attractor generates a uniform global attractor.

In Section 3, we show that using the methods of [16] one can construct a solution to the initial-boundary value problem for system (0.1)–(0.3), which satisfies a suitable energy estimate. In Section 4, with the help of the obtained results we prove existence of minimal uniform trajectory attractors and uniform global attractors for weak solutions of the boundary value problem (0.1)–(0.4) for motion equations of a viscoelastic medium at $n = 2, 3$.

1. Notations

We shall use the following notations. Most of them are standard.

Denote by $\mathbb{R}^{n\times n}$ the space of matrices of the order $n \times n$ with the following scalar product: for $A = (A_{ij})$, $B = (B_{ij})$,

$$(A, B)_{\mathbb{R}^{n\times n}} = \sum_{i,j=1}^{n} A_{ij} B_{ij}$$

and by $\mathbb{R}^{n\times n}_S$ its subspace of symmetric matrices.

Denote by $\mathbb{R}^{n\times n\times n}$ the space of ordered collections of n matrices of the order $n\times n$ with the following scalar product: for $A=(A_1,\dots,A_n)$, $B=(B_1,\dots,B_n)$,

$$(A,B)_{\mathbb{R}^{n\times n\times n}}=\sum_{i=1}^{n}(A_i,B_i)_{\mathbb{R}^{n\times n}}.$$

The symbol ∇u stands for the Jacobi matrix of a vector function $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$. The symbol $\nabla\tau$ denotes the ordered collection of the Jacobi matrices of the columns of a matrix function $\tau:\Omega\subset\mathbb{R}^n\to\mathbb{R}^{n\times n}$.

The symbol $K(\cdot,\dots,\cdot)$ stands for positive constants, depending continuously on arguments, which will be enumerated.

Below F stands for one of the spaces $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n}_S$, $\mathbb{R}^{n\times n\times n}$.

We use the standard notations $L_p(\Omega,F)$, $W^\beta_p(\Omega,F)$, $H^\beta(\Omega,F)=W^\beta_2(\Omega,F)$ $(\beta\in\mathbb{R})$, $H^\beta_0(\Omega,F)=\mathring{W}{}^\beta_2(\Omega,F)$ $(\beta>0)$ for Lebesgue and Sobolev spaces of functions with values in F. Sometimes we shall write simply L_p instead of $L_p(\Omega,F)$, etc., if it is clear from the context which space F is used.

Parentheses denote the following bilinear form:

$$(u,v)=\int_\Omega\big(u(x),v(x)\big)_F\,dx.$$

The Euclid norm in F is denoted as $|\cdot|$ and in L_2 as $\|\cdot\|$. We shall also use the notation $\|v\|_1=\|\nabla v\|$, $v\in H^1$.

By the symbol $C^\infty_0(\Omega,F)$ we denote the space of smooth functions with compact support in Ω and with values in F.

For brevity we denote by C^∞_0 the space $C^\infty_0(\Omega,\mathbb{R}^{n\times n}_S)$.

Let $\mathcal{V}=\{u\in C^\infty_0(\Omega,\mathbb{R}^n),\ \operatorname{div}u=0\}$.

Let the symbols H, V, V_δ $(\delta\in(0,1])$ denote the closures of $\mathcal{V}$ in $L_2(\Omega,\mathbb{R}^n)$, $H^1(\Omega,\mathbb{R}^n)$, $H^\delta(\Omega,\mathbb{R}^n)$, respectively.

Following [13], we identify the space H and its conjugate space H^*. Therefore we have the embedding

$$V_\delta\subset H\equiv H^*\subset V^*_\delta.$$

The value of a functional from H^{-m} $(m\in\mathbb{N})$, V^*, V^*_δ on an element from H^m_0, V, V_δ, respectively, is denoted by brackets $\langle\cdot,\cdot\rangle$.

Since Ω is bounded, the inequality

$$\|v\|\leqslant K_0(\Omega)\|v\|_1,\quad v\in V,\tag{1.1}$$

is valid.

We consider V to be equipped with the norm $\|\cdot\|_1$ and V^* to be equipped with the corresponding norm of a conjugate space.

The symbols $C(J;X)$, $C_w(J;X)$, $L_2(J;X)$, etc., denote the spaces of continuous, weakly continuous, quadratically integrable, etc., functions on an interval $J\subset\mathbb{R}$ (which may be unbounded) with values in a Banach space X.

Let us remind that a pre-norm in the Fréchet space $C([0,+\infty);X)$ may be defined by the formula

$$\|v\|_{C([0,+\infty);X)}=\sum_{i=0}^{+\infty}2^{-i}\frac{\|v\|_{C([0,i];X)}}{1+\|v\|_{C([0,i];X)}},$$

and in the space $C((-\infty,+\infty);X)$ by the formula

$$\|v\|_{C((-\infty,+\infty);X)}=\sum_{i=0}^{+\infty}2^{-i}\frac{\|v\|_{C([-i,i];X)}}{1+\|v\|_{C([-i,i];X)}}.$$

We shall also use the space $\mathcal{X}$ that consists of all elements from $L_{2,\mathrm{loc}}(0,+\infty;V^*)$, for which the norm

$$\|\phi\|_{\mathcal{X}}=\sup_{t\geqslant 0}\|\phi\|_{L_2(t,t+1;V^*)}$$

is finite.

2. Existence of attractors of non-autonomous equations

Let E and E_0 be Banach spaces, $E\subset E_0$. Following [3,4], we write an abstract non-autonomous evolution differential equation in the form

$$u'(t)=A_{\sigma(t)}\big(u(t)\big),\quad u(t)\in E. \tag{2.1}$$

Here σ is a functional parameter, which is called the time symbol of Eq. (2.1). Assume that σ belongs to some fixed parameter set Σ, which is called the symbol space and is usually a subset of some space of time-dependent functions. In applications, a function $\sigma(t)$ consists of all time-dependent coefficients, terms and right-hand sides of a considered equation.

We investigate attractors of solutions of Eq. (2.1) which belong to the space $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$.

It is supposed that the space E is reflexive. Then $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)\subset C_w([0,+\infty);E)$ (see [13, Chapter III, Lemma 1.4], and also [12, p. 232] where it is shown that without the condition of reflexivity of E there may be no such embedding). Hence, the values of functions from $C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ belong to E at every $t\geqslant 0$.

Consider the translation (shift) operators $T(h)$, $h\geqslant 0$,

$$T(h)(u)(t)=u(t+h),$$

$u\in C([0,+\infty);E_0)$, $L_\infty(0,+\infty;E)$, $C((-\infty,+\infty);E_0)$, $L_\infty(-\infty,+\infty;E)$ or $\mathcal{X}$.

For any fixed $h\geqslant 0$ the operators $T(h)$ are continuous bounded mappings of the spaces $C([0,+\infty);E_0)$, $L_\infty(0,+\infty;E)$, $\mathcal{X}$ into themselves.

Assume that for every $\sigma\in\Sigma$ we have a fixed set

$$\mathcal{H}_\sigma^+\subset C\big([0,+\infty);E_0\big)\cap L_\infty(0,+\infty;E)$$

of solutions (strong, weak, etc.) for Eq. (2.1), defined on the positive axis $t\geqslant 0$. The sets $\mathcal{H}_\sigma^+$ are called trajectory spaces and their elements are called trajectories.

Remark 2.1. We do not assume that the semigroup $T(t)$ acts on Σ and the family of trajectory spaces $\mathcal{H}_\sigma^+$, $\sigma\in\Sigma$ is translation-coordinated, i.e., $T(h)\mathcal{H}_\sigma^+\subset\mathcal{H}_{T(h)\sigma}^+$ for $h\geqslant 0$ (see [4]).

Consider the united trajectory space $\mathcal{H}_\Sigma^+=\bigcup_{\sigma\in\Sigma}\mathcal{H}_\sigma^+$.

Definition 2.1. A set $P\subset C([0,+\infty);E_0)\cap L_\infty(0,+\infty;E)$ is called uniformly (with respect to $\sigma\in\Sigma$) attracting (for Eq. (2.1)) if for any bounded in $L_\infty(0,+\infty;E)$ set $B\subset\mathcal{H}_\Sigma^+$ one has

$$\sup_{u\in B}\inf_{v\in P}\big\|T(h)u-v\big\|_{C([0,+\infty);E_0)}\xrightarrow[h\to\infty]{}0.$$

Definition 2.2. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called uniformly absorbing if for any bounded in $L_\infty(0,+\infty; E)$ set $B \subset \mathcal{H}^+_\Sigma$ there is $h \geqslant 0$ such that for all $t \geqslant h$,

$$T(t)B \subset P.$$

It is easy to see that any uniformly absorbing set is uniformly attracting.

Definition 2.3. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a uniform trajectory semiattractor (for Eq. (2.1)) if

(i) P is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$;
(ii) $T(t)P \subset P$ for any $t \geqslant 0$;
(iii) P is uniformly attracting in the sense of Definition 2.1.

Definition 2.4. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a uniform trajectory quasiattractor (for Eq. (2.1)) if it satisfies conditions (i), (iii) of Definition 2.3 and

(ii′) $T(t)P \supset P$ for any $t \geqslant 0$.

Definition 2.5. A set $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ is called a uniform (with respect to $\sigma \in \Sigma$) trajectory attractor (for Eq. (2.1)) if it is a uniform trajectory semiattractor and a uniform trajectory quasiattractor (i.e. $T(t)P = P$). A uniform trajectory attractor $\mathcal{U}$ is called minimal if it is contained in any other uniform trajectory attractor.

Definition 2.6. A set $\mathcal{A} \subset E$ is called a uniform (with respect to $\sigma \in \Sigma$) global attractor (in E_0) for Eq. (2.1) if

(i) $\mathcal{A}$ is compact in E_0 and bounded in E;
(ii) for any bounded in $L_\infty(0,+\infty; E)$ set $B \subset \mathcal{H}^+_\Sigma$ the attraction property is fulfilled:

$$\sup_{u\in B} \inf_{v\in\mathcal{A}} \|u(t) - v\|_{E_0} \xrightarrow[t\to\infty]{} 0;$$

(iii) $\mathcal{A}$ is the minimal set satisfying conditions (i) and (ii) (that is $\mathcal{A}$ is contained in every set satisfying conditions (i) and (ii)).

Remark 2.2. Definition 2.6 is customary for equations without uniqueness of the corresponding Cauchy problem (see, e.g., [15, p. 211]).

Remark 2.3. If a uniform trajectory attractor is contained in $\mathcal{H}^+_\Sigma$, then it is minimal. It follows from Lemma 2.7 (see below). Under the additional assumptions that

(a) the family of trajectory spaces is translation-coordinated and
(b) Σ is a compact metric space,

in [15] there were considered only uniform trajectory attractors (in the sense of Definition 2.5), which are contained in $\mathcal{H}^+_\Sigma$. Nevertheless, the (more general) concept of minimal uniform trajectory attractor used by us has many usual properties of trajectory attractors. In particular, a minimal uniform trajectory attractor always generates a uniform global attractor (see below, Theorem 2.2).

Remark 2.4. Under the same assumptions (a) and (b) and the assumption that

(c) the family $\{\mathcal{H}_\sigma^+\}$ is closed in the sense that for any sequence of symbols $\sigma_m \to \sigma_0$ and for any converging (in the topology of $\mathcal{H}_\Sigma^+$, i.e., in $C([0,+\infty);E_0)$ in our situation) sequence of trajectories $u_m \in \mathcal{H}_{\sigma_m}^+$ one has $\lim_{m\to+\infty} u_m \in \mathcal{H}_{\sigma_0}^+$,

in [4] there was used a concept of uniform trajectory attractor, which is a little bit different from our concept of minimal uniform trajectory attractor and from the concept from [15] described above. However in the applications to the Navier–Stokes system considered in [4,15] all three definitions give the same set.

Remark 2.5. The minimal uniform trajectory attractor and the uniform global attractor depend on the symbol space Σ. It is easy to observe that if we have two symbol spaces $\Sigma_1 \subset \Sigma_2$, then for corresponding minimal uniform trajectory attractors and uniform global attractors one has $\mathcal{U}_{\Sigma_1} \subset \mathcal{U}_{\Sigma_2}$, $\mathcal{A}_{\Sigma_1} \subset \mathcal{A}_{\Sigma_2}$.

In order to prove the sufficient conditions for existence of uniform attractors we need the following two lemmas on properties of attracting sets.

Lemma 2.1.

(a) *Let sets* $P_1, P_2 \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ *satisfy conditions* (i) *or* (ii) *of Definition* 2.3. *Then* $P_1 \cap P_2$ *also satisfies a corresponding condition.*
(b) *If* P_1, P_2 *are compact in* $C([0,+\infty);E_0)$ *and satisfy condition* (iii) *of Definition* 2.3*, then* $P_1 \cap P_2$ *also satisfies condition* (iii).

Proof. Statement (a) is clear. Let us show (b). Let P_1, P_2 be compact in $C([0,+\infty);E_0)$ and satisfy condition (iii). We have to show that $P_1 \cap P_2$ is a uniformly attracting set. If not, then for some $\delta > 0$ and some bounded in $L_\infty(0,+\infty;E)$ set $B \subset \mathcal{H}_\Sigma^+$ there is a sequence $h_m \to \infty$ such that

$$\sup_{u\in B} \inf_{v\in P_1\cap P_2} \bigl\| T(h_m)u - v \bigr\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in B$ such that

$$\inf_{v\in P_1\cap P_2} \bigl\| T(h_m)u_m - v \bigr\|_{C([0,+\infty);E_0)} > \delta. \tag{2.2}$$

On the other hand, since P_1 and P_2 are attracting sets, for any natural number k there exist a number m_k and elements $v_k^1 \in P_1$, $v_k^2 \in P_2$ such that

$$\bigl\| T(h_{m_k})u_{m_k} - v_k^1 \bigr\|_{C([0,+\infty);E_0)} < \frac{1}{k},$$
$$\bigl\| T(h_{m_k})u_{m_k} - v_k^2 \bigr\|_{C([0,+\infty);E_0)} < \frac{1}{k}.$$

Since P_1 is compact in $C([0,+\infty);E_0)$, without loss of generality we may assume that the sequence v_k^1 converges to an element v_0 as $k \to \infty$. Then sequences $T(h_{m_k})u_{m_k}$ and v_k^2 also converge to v_0. Thus, $v_0 \in P_1 \cap P_2$ and $\|T(h_{m_k})u_{m_k} - v_0\|_{C([0,+\infty);E_0)} \xrightarrow[k\to\infty]{} 0$, which contradicts (2.2). □

Lemma 2.2. *Let a set* $P \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$ *satisfy one of the conditions* (i), (ii), (ii′) *or* (iii) *of Definitions* 2.3, 2.4. *Then* $T(h)P$ *also satisfies a corresponding condition for all* $h \geqslant 0$.

Proof. The statement of the lemma concerning conditions (i), (ii), (ii′) is obvious. Let P satisfy condition (iii), that is it is uniformly attracting. Then for any bounded in $L_\infty(0,+\infty; E)$ set $B \subset \mathcal{H}^+_\Sigma$ one has

$$\begin{aligned}
&\sup_{u\in B} \inf_{v\in T(h)P} \big\| T(t)u - v \big\|_{C([0,+\infty);E_0)} \\
&\quad = \sup_{u\in B} \inf_{v\in P} \big\| T(h)\big(T(t-h)u - v\big) \big\|_{C([0,+\infty);E_0)} \\
&\quad \leqslant \big\| T(h) \big\|_{C([0,+\infty);E_0)\to C([0,+\infty);E_0)} \sup_{u\in B} \inf_{v\in P} \big\| T(t-h)u - v \big\|_{C([0,+\infty);E_0)} \xrightarrow[t\to\infty]{} 0,
\end{aligned}$$

that is $T(h)P$ is also uniformly attracting. □

We shall also use the following lemma on compact sets.

Lemma 2.3. *Let* (X,ρ) *be a metric space and* $\{K_\alpha\}_{\alpha\in\Xi}$ *be a system of non-empty compact sets in* X. *Assume that for any* $\alpha_1, \alpha_2 \in \Xi$ *there is* $\alpha_3 \in \Xi$ *such that* $K_{\alpha_1} \cap K_{\alpha_2} = K_{\alpha_3}$. *Then* $K_0 = \bigcap_{\alpha\in\Xi} K_\alpha \neq \emptyset$ *and for any* $\epsilon > 0$ *there is* $\alpha_\epsilon \in \Xi$ *such that for any* $y \in K_{\alpha_\epsilon}$:

$$\inf_{x\in K_0} \rho(x,y) < \epsilon.$$

Proof. By induction one easily proves that an intersection of any finite number of sets from the system $\{K_\alpha\}$ belongs to this system.

Consider the set

$$K_\epsilon = \Big\{ y \in X \;\Big|\; \inf_{x\in K_0} \rho(x,y) < \epsilon \Big\}.$$

It is clear that K_ϵ is open and $K_0 \subset K_\epsilon$. Note that if $K_\epsilon \neq \emptyset$, then $K_0 \neq \emptyset$. Thus, to prove the lemma, it suffices to find a set $K_{\alpha_\epsilon} \subset K_\epsilon$ in the system $\{K_\alpha\}$.

Let $\alpha_0 \in \Xi$. Then $K_{\alpha_0}\backslash K_\epsilon$ is compact and $\{X\backslash K_\alpha\}_{\alpha\in\Xi}$ is an open cover for $K_{\alpha_0}\backslash K_\epsilon$. One can choose its finite subcover $X\backslash K_{\alpha_1}, X\backslash K_{\alpha_2}, \ldots, X\backslash K_{\alpha_m}$. Thus, $K_{\alpha_0}\backslash K_\epsilon \subset X\backslash\bigcap_{i=1}^m K_{\alpha_i}$. Therefore $\bigcap_{i=1}^m K_{\alpha_i} \subset X\backslash(K_{\alpha_0}\backslash K_\epsilon)$. It implies that the set $\bigcap_{i=0}^m K_{\alpha_i}$ is contained in K_ϵ, so it can be taken as the required set K_{α_ϵ}. □

By analogy to the concept of minimal uniform trajectory attractor it is possible to introduce the concept of minimal uniform trajectory semiattractor as a uniform trajectory semiattractor contained in any other uniform trajectory semiattractor. However we can prove the following statement.

Lemma 2.4. *A minimal uniform trajectory semiattractor is always a minimal uniform trajectory attractor.*

Remark 2.6. We shall prove below the inverse statement (Lemma 2.5).

Proof. Let $\mathcal{U}$ be a minimal uniform trajectory semiattractor, that is it is a uniform trajectory semiattractor and it is contained in any other uniform trajectory semiattractor. By Lemma 2.2, $T(h)\mathcal{U}$ is a uniform trajectory semiattractor for all $h \geqslant 0$, therefore $\mathcal{U} \subset T(h)\mathcal{U}$. Thus, $\mathcal{U}$ is a uniform trajectory semiattractor and a uniform trajectory quasiattractor, that is it is a (minimal) uniform trajectory attractor. □

Now we can formulate and prove the main results of this section.

Theorem 2.1. *Assume that there exists a uniform trajectory semiattractor P for Eq.* (2.1). *Then there exists a minimal uniform trajectory attractor $\mathcal{U} \subset P$ for Eq.* (2.1).

Proof. Take in Lemma 2.3 $X = C([0, +\infty); E_0)$ and let $\{K_\alpha\}_{\alpha\in\Xi}$ be the system of all uniform trajectory semiattractors. Denote by $\mathcal{U}$ the intersection of all uniform trajectory semiattractors. By Lemma 2.1 the intersection of any two sets from the system $\{K_\alpha\}$ belongs to this system. Thus, for the system of semiattractors $\{K_\alpha\}$ the conditions of Lemma 2.3 hold.

Let us show that $\mathcal{U}$ is a trajectory semiattractor. Clearly, $\mathcal{U}$ satisfies conditions (i) and (ii) of Definition 2.3. We shall show now that $\mathcal{U}$ satisfies condition (iii), that is it is uniformly attracting. Fix $\epsilon > 0$ and a bounded in $L_\infty(0, +\infty; E)$ set $B \subset \mathcal{H}^+_\Sigma$. By Lemma 2.3 there is a semiattractor P_ϵ such that for any $v \in P_\epsilon$:

$$\inf_{w\in\mathcal{U}} \|w - v\|_{C([0,+\infty);E_0)} < \frac{\epsilon}{2}.$$

Since P_ϵ is an attracting set, there exists $t \geqslant 0$ such that for $h \geqslant t$:

$$\sup_{u\in B} \inf_{v\in P_\epsilon} \big\|T(h)u - v\big\|_{C([0,+\infty);E_0)} < \frac{\epsilon}{2}.$$

Therefore for every $u \in B$ there exists $v(u) \in P_\epsilon$ such that $\|T(h)u - v(u)\|_{C([0,+\infty);E_0)} < \frac{\epsilon}{2}$. We have:

$$\begin{aligned}
&\sup_{u\in B} \inf_{w\in\mathcal{U}} \big\|T(h)u - w\big\|_{C([0,+\infty);E_0)} \\
&\quad\leqslant \sup_{u\in B}\Big(\big\|T(h)u - v(u)\big\|_{C([0,+\infty);E_0)} + \inf_{w\in\mathcal{U}} \big\|v(u) - w\big\|_{C([0,+\infty);E_0)}\Big) \\
&\quad\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, $\mathcal{U}$ is a uniform trajectory semiattractor. It is clear that it is minimal. By Lemma 2.4, $\mathcal{U}$ is a minimal uniform trajectory attractor. □

Lemma 2.5. *A minimal uniform trajectory attractor is always a minimal uniform trajectory semiattractor.*

Proof. Let $\mathcal{U}$ be a minimal uniform trajectory attractor and let P be a uniform trajectory semiattractor. Since the minimal uniform trajectory attractor is unique, by Theorem 2.1, $\mathcal{U} \subset P$. Thus, $\mathcal{U}$ is contained in any uniform trajectory semiattractor, so it is a minimal uniform trajectory semiattractor. □

Consider the sections of a uniform trajectory attractor at fixed $t \geqslant 0$: $\mathcal{U}(t) = \{v(t) \mid v \in \mathcal{U}\}$. It is easy to see that these sets are contained in E (see the beginning of Section 2).

Theorem 2.2. *If there exists a minimal uniform trajectory attractor $\mathcal{U}$ for Eq.* (2.1)*, then there is a uniform global attractor $\mathcal{A}$ for Eq.* (2.1) *and for all $t \geqslant 0$ one has $\mathcal{A} = \mathcal{U}(t)$.*

Proof. Observe first that since $T(t)\mathcal{U} = \mathcal{U}$, $t \geqslant 0$, the set $\mathcal{A} = \mathcal{U}(t)$ does not depend on t. The set $\mathcal{U}$ is compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$. Then its section $\mathcal{A} = \mathcal{U}(0)$ is compact in E_0 and bounded in E.

Let a set $B \subset \mathcal{H}^+_\Sigma$ be bounded in $L_\infty(0,+\infty; E)$. Since $\mathcal{U}$ is an attracting set,

$$\sup_{u\in B} \inf_{v\in\mathcal{U}} \big\| T(h)u - v \big\|_{C([0,+\infty);E_0)} \xrightarrow[h\to\infty]{} 0.$$

It yields the pointwise convergence:

$$\sup_{u\in B} \inf_{v\in\mathcal{U}} \big\| \big(T(h)u - v\big)(t) \big\|_{E_0} \xrightarrow[h\to\infty]{} 0, \quad t \geqslant 0.$$

At $t = 0$ we get

$$\sup_{u\in B} \inf_{v\in\mathcal{A}=\mathcal{U}(0)} \big\| u(h) - v \big\|_{E_0} \xrightarrow[h\to\infty]{} 0.$$

It remains to show that $\mathcal{A}$ is contained in every set $\mathcal{A}_0$ which is compact in E_0, bounded in E and possesses the attraction property

$$\sup_{u\in B} \inf_{v\in\mathcal{A}_0} \big\| u(t) - v \big\|_{E_0} \xrightarrow[t\to\infty]{} 0 \tag{2.3}$$

for every bounded in $L_\infty(0,+\infty; E)$ set $B \subset \mathcal{H}^+_\Sigma$. Let $\mathcal{U}_0 = \{u \in \mathcal{U} \mid u(t) \in \mathcal{A}_0 \ \forall t \geqslant 0\}$. It suffices to show that $\mathcal{U} \subset \mathcal{U}_0$. By Lemma 2.5, $\mathcal{U}$ is contained in every trajectory semiattractor. Hence, it is enough to show that $\mathcal{U}_0$ is a semiattractor. Since $\mathcal{U}_0 \subset \mathcal{U}$, $\mathcal{U}_0$ is relatively compact in $C([0,+\infty); E_0)$ and is bounded in $L_\infty(0,+\infty; E)$. For any converging in $C([0,+\infty); E_0)$ sequence $\{u_m\} \subset \mathcal{U}_0$ its limit u_0 belongs to the (closed in $C([0,+\infty); E_0)$) set $\mathcal{U}$. The convergence in $C([0,+\infty); E_0)$ yields the pointwise convergence: $u_m(t) \to u_0(t)$ in E_0, $t \geqslant 0$. Since $\mathcal{A}_0$ is compact in E_0, $u_0(t) \in \mathcal{A}_0$, $t \geqslant 0$. Thus, $\mathcal{U}_0$ is closed and compact in $C([0,+\infty); E_0)$. Since $T(t)\mathcal{U} \subset \mathcal{U}$, one has $T(t)\mathcal{U}_0 \subset \mathcal{U}_0$, $t \geqslant 0$. It remains to show that $\mathcal{U}_0$ is a uniformly attracting set. If not, for some $\delta > 0$ and some bounded in $L_\infty(0,+\infty; E)$ set $B \subset \mathcal{H}^+_\Sigma$ there exists a sequence $h_m \to \infty$ such that

$$\sup_{u\in B} \inf_{v\in\mathcal{U}_0} \big\| T(h_m)u - v \big\|_{C([0,+\infty);E_0)} > \delta.$$

Then there are elements $u_m \in B$ such that

$$\inf_{v\in\mathcal{U}_0} \big\| T(h_m)u_m - v \big\|_{C([0,+\infty);E_0)} > \delta. \tag{2.4}$$

On the other hand, since $\mathcal{U}$ is a uniformly attracting set, for every natural k there exist a number m_k and an element $v_k \in \mathcal{U}$ such that

$$\big\| T(h_{m_k})u_{m_k} - v_k \big\|_{C([0,+\infty);E_0)} < \frac{1}{k}.$$

Since $\mathcal{U}$ is compact in $C([0,+\infty); E_0)$, without loss of generality the sequence v_k converges to an element $v_0 \in \mathcal{U}$ as $k \to \infty$. Then

$$\big\| T(h_{m_k})u_{m_k} - v_0 \big\|_{C([0,+\infty);E_0)} \xrightarrow[k\to\infty]{} 0. \tag{2.5}$$

Now (2.5) together with (2.4) yield $v_0 \notin \mathcal{U}_0$, that is $v_0(t_0) \notin \mathcal{A}_0$ for some $t_0 \geqslant 0$. But from (2.3) it follows that

$$\inf_{v\in\mathcal{A}_0} \left\| T(h_{m_k})u_{m_k}(t_0) - v \right\|_{E_0} \xrightarrow[k\to\infty]{} 0.$$

Then there is a sequence $\{v_k^*\} \subset \mathcal{A}_0$ such that

$$\left\| T(h_{m_k})u_{m_k}(t_0) - v_k^* \right\|_{E_0} \xrightarrow[k\to\infty]{} 0.$$

Since $\mathcal{A}_0$ is compact, without loss of generality v_k^* converges to some element v^*. But (2.5) gives:

$$\left\| T(h_{m_k})u_{m_k}(t_0) - v_0(t_0) \right\|_{E_0} \xrightarrow[k\to\infty]{} 0.$$

Therefore $v_0(t_0) = v^* \in \mathcal{A}_0$, and we have a contradiction. □

Theorems 2.1 and 2.2 give existence of a uniform global attractor under the conditions of Theorem 2.1. It appears that existence of a uniform global attractor may be proved also under weaker assumptions.

Theorem 2.3. *Assume that there exists a compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$ uniformly attracting set P for Eq.* (2.1)*. Then there exists a uniform global attractor $\mathcal{A}$ for Eq.* (2.1)*.*

Proof. Take in Lemma 2.3 $X = C([0,+\infty); E_0)$ and let $\{K_\alpha\}_{\alpha\in\Xi}$ be the system of all compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$ uniformly attracting sets. Denote by $\mathcal{U}$ the intersection of all such sets. By Lemma 2.1 the intersection of any two sets from the system $\{K_\alpha\}$ belongs to this system. Thus, for system $\{K_\alpha\}$ the conditions of Lemma 2.3 hold.

Clearly, $\mathcal{U}$ satisfies condition (i) of Definition 2.3. As in the proof of Theorem 2.1 one shows that $\mathcal{U}$ is a uniformly attracting set. By Lemma 2.2, $T(h)\mathcal{U}$ is a compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$ attracting set for all $h \geqslant 0$, therefore $\mathcal{U} \subset T(h)\mathcal{U}$. Thus, $\mathcal{U}$ is a trajectory quasiattractor.

Let us show that the set $\mathcal{A} = \mathcal{U}(t)$ does not depend on $t \geqslant 0$ and is a uniform global attractor for Eq. (2.1).

As in the proof of Theorem 2.2 one shows that $\mathcal{U}(0)$ satisfies conditions (i), (ii) of Definition 2.6. Since $\mathcal{U} \subset T(t)\mathcal{U}$, one has $\mathcal{U}(0) \subset \mathcal{U}(t)$ for all $t \geqslant 0$. It remains to show that $\mathcal{U}(t)$ is contained in any set $\mathcal{A}_0$ which satisfies conditions (i), (ii) of Definition 2.6 (in particular, in $\mathcal{U}(0)$). Denote $\mathcal{U}_0 = \{u \in \mathcal{U} \mid u(t) \in \mathcal{A}_0\ \forall t \geqslant 0\}$. It suffices to show that $\mathcal{U} \subset \mathcal{U}_0$. Since $\mathcal{U}$ is contained in every set from the system $\{K_\alpha\}$, it is enough to show that $\mathcal{U}_0$ belongs to this system, that is it is a compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$ attracting set. It may be realized in the same way as the check of the corresponding statements in the proof of Theorem 2.2. □

Remark 2.7. For facilitation of check of conditions of Theorems 2.1 and 2.3 the following simple statement may be used.

Lemma 2.6. *Let P be a relatively compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$ uniformly attracting set for Eq.* (2.1)*. Then its closure $\bar{P}$ in $C([0,+\infty); E_0)$ is a compact in $C([0,+\infty); E_0)$ and bounded in $L_\infty(0,+\infty; E)$ uniformly attracting set. If in addition $T(t)P \subset P$ for any $t \geqslant 0$, then $\bar{P}$ is a uniform trajectory semiattractor.*

Proof. Since P is bounded in $L_\infty(0,+\infty;E)$ we may assume that there is a constant C such that for any function $u \in P$ one has $\|u(t)\|_E \leqslant C$ for $t \geqslant 0$. Let $u_0 \in \bar{P}$. Then in P there is a sequence $u_m \to u_0$ in $C([0,+\infty);E_0)$. Hence, $u_m(t) \to u_0(t)$ in E_0 for all $t \geqslant 0$. Since $\|u_m(t)\|_E \leqslant C$ and the space E is reflexive, $u_m(t) \rightharpoonup u_0(t)$ weakly in E and $\|u_0(t)\|_E \leqslant C$. Thus, $\bar{P}$ is bounded in $L_\infty(0,+\infty;E)$. It is easy to see that $\bar{P}$ is a compact in $C([0,+\infty);E_0)$ attracting set.

Let $T(h)P \subset P$ for any $t \geqslant 0$. We shall show now that $T(h)\bar{P} \subset \bar{P}$. Let $u \in T(h)\bar{P}$. Then there are $u_0 \in \bar{P}$ and a sequence $u_m \in P$ such that $u_m \to u_0$, $u = T(h)u_0$. Then $T(h)u_m \to T(h)u_0 = u$, that is $u \in \overline{T(h)P} \subset \bar{P}$. Thus, $\bar{P}$ is a semiattractor. □

Uniform trajectory attractors which are contained in $\mathcal{H}^+_\Sigma$ have additional properties (see also Remark 2.3).

Lemma 2.7. *Let $\mathcal{U} \subset \mathcal{H}^+_\Sigma$ be a uniform trajectory quasiattractor. Then*

(a) *$\mathcal{U}$ is contained in any compact in $C([0,+\infty);E_0)$ uniformly attracting set P;*
(b) *in $\mathcal{H}^+_\Sigma$ there is no trajectory quasiattractors different from $\mathcal{U}$;*
(c) *if in addition it is known that $T(h)\mathcal{U} \subset \mathcal{H}^+_\Sigma$ for all $h \geqslant 0$, then $\mathcal{U}$ is a minimal uniform trajectory attractor.*

Proof. (a) Since $\mathcal{U} \subset \mathcal{H}^+_\Sigma$ is bounded in $L_\infty(0,+\infty;E)$, for any neighborhood P_ϵ of the attracting set P in $C([0,+\infty);E_0)$ one has $\mathcal{U} \subset T(h)\mathcal{U} \subset P_\epsilon$ at $h \geqslant 0$ large enough. Since P is compact in $C([0,+\infty);E_0)$, it yields $\mathcal{U} \subset P$.

(b) If $\mathcal{U}_1 \subset \mathcal{H}^+_\Sigma$ is another trajectory quasiattractor, then by point (a) one has $\mathcal{U}_1 \subset \mathcal{U}, \mathcal{U} \subset \mathcal{U}_1$.

(c) By Lemma 2.2, $T(h)\mathcal{U}$ is a trajectory quasiattractor for all $h \geqslant 0$. If $T(h)\mathcal{U} \subset \mathcal{H}^+_\Sigma$, then by point (b) $T(h)\mathcal{U} = \mathcal{U}$ and $\mathcal{U}$ is a trajectory attractor. By point (a) it is minimal. □

Theorem 2.4. *Assume that there exists a compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$ uniformly attracting set P for Eq.* (2.1). *Let $T(h)P \subset \mathcal{H}^+_\Sigma$ for all $h \geqslant 0$. Then there exist a minimal uniform trajectory attractor $\mathcal{U}$ and a global attractor $\mathcal{A} = \mathcal{U}(t)$, $t \geqslant 0$ for Eq.* (2.1).

Proof. During the proof of Theorem 2.3 we have shown that there exists a uniform trajectory quasiattractor $\mathcal{U} \subset P$ for Eq. (2.1) such that the set $\mathcal{A} = \mathcal{U}(t)$ does not depend on $t \geqslant 0$ and is a uniform global attractor for Eq. (2.1). Since $T(h)\mathcal{U} \subset T(h)P \subset \mathcal{H}^+_\Sigma$, by Lemma 2.7(c) the set $\mathcal{U}$ is a minimal uniform trajectory attractor. □

Remark 2.8. In [15, Theorem 4.1, p. 210] this theorem was proved under additional assumptions that the family of trajectory spaces is translation-coordinated (see Remark 2.1) and that the symbol space Σ is a compact metric space.

3. Weak solutions of boundary value problem for the motion equations of viscoelastic medium

In this section we describe the weak setting of problem (0.1)–(0.4) and show that for given initial data one can construct a weak solution to this problem, which satisfies an energy estimate, suitable for construction of attractors.

Definition 3.1. Let $f \in \mathcal{X}$ and $T > 0$. A weak solution of problem (0.1)–(0.4) on an interval $(0, T)$ is a pair of functions (u, σ),

$$u \in L_2(0, T; V) \cap C_w\big([0, T]; H\big), \quad \frac{du}{dt} \in L_1\big(0, T; V^*\big),$$
$$\sigma \in L_2\big(0, T; L_2\big(\Omega, \mathbb{R}^{N\times N}_S\big)\big) \cap C_w\big([0, T]; H^{-1}\big(\Omega, \mathbb{R}^{N\times N}_S\big)\big) \tag{3.1}$$

satisfying identities

$$\frac{d}{dt}(u, \varphi) + (\sigma, \nabla\varphi) - \sum_{i=1}^{n}\left(u_i u, \frac{\partial\varphi}{\partial x_i}\right) = \big\langle f(t), \varphi\big\rangle, \tag{3.2}$$

$$(\sigma, \Phi) + \lambda_1\frac{d}{dt}(\sigma, \Phi) - \lambda_1\sum_{i=1}^{n}\left(u_i\sigma, \frac{\partial\Phi}{\partial x_i}\right)$$
$$= -2\eta(u, \operatorname{Div}\Phi) - 2\eta\lambda_2\left(\frac{d}{dt}(u, \operatorname{Div}\Phi) + \sum_{i=1}^{n}\left(u_i\mathcal{E}(u), \frac{\partial\Phi}{\partial x_i}\right)\right) \tag{3.3}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$ in the sense of distributions on $(0, T)$.

Let us introduce the notations

$$\mu_1 = \eta\frac{\lambda_2}{\lambda_1}, \qquad \mu_2 = \frac{\eta - \mu_1}{\lambda_1}, \qquad \tau = \sigma - 2\mu_1\mathcal{E}. \tag{3.4}$$

With the help of these notations we can rewrite (3.3) and (3.2) in the following form:

$$\frac{d}{dt}(\tau, \Phi) + \frac{1}{\lambda_1}(\tau, \Phi) - \sum_{i=1}^{n}\left(u_i\tau, \frac{\partial\Phi}{\partial x_i}\right) + 2\mu_2(u, \operatorname{Div}\Phi) = 0, \tag{3.5}$$

$$\frac{d}{dt}(u, \varphi) - \sum_{i=1}^{n}\left(u_i u, \frac{\partial\varphi}{\partial x_i}\right) + \mu_1(\nabla u, \nabla\varphi) + (\tau, \nabla\varphi) = \big\langle f(t), \varphi\big\rangle \tag{3.6}$$

for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$.

For research of attractors for weak solutions of problem (0.1)–(0.4) it appears to be convenient to pass to the variables (u, τ) and to investigate attractors of problem (3.5)–(3.6).

Now we state a theorem of existence of solutions for these problems.

Theorem 3.1. *Let $f \in \mathcal{X}$. Given $a \in H$, $\tau_0 \in L_2(\Omega, \mathbb{R}^{n\times n}_S)$, there is a pair of functions (u, τ) which*

(i) *belongs to the class*

$$u \in L_2(0, T; V) \cap L_\infty(0, T; H) \cap C_w\big([0, T]; H\big),$$
$$u' \in L_{4/3}\big(0, T; V^*\big),$$
$$\tau \in L_\infty(0, T; L_2) \cap C_w\big([0, T]; L_2\big), \quad \tau' \in L_2\big(0, T; H^{-2}\big); \tag{3.7}$$

(ii) *satisfies the initial condition*

$$u|_{t=0} = a, \qquad \tau|_{t=0} = \tau_0; \tag{3.8}$$

(iii) *satisfies identities* (3.5), (3.6) *a.e. on* $(0,T)$ *for all* $\varphi \in \mathcal{V}$ *and* $\Phi \in C_0^\infty$;
(iv) *satisfies the energy inequality*:

$$\frac{1}{2}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{4\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)} \leqslant e^{-2\gamma t}\left(\|a\|^2 + \frac{1}{2\mu_2}\|\tau_0\|^2\right) + \frac{2e^{4\gamma}+e^{2\gamma}-1}{2\mu_1(e^{2\gamma}-1)}\|f\|^2_{\mathcal{X}} \tag{3.9}$$

for $t \in [0, T-1]$. *Here* $\gamma = \min(\frac{1}{\lambda_1}, \frac{\mu_1}{2K_0(\Omega)^2})$, *where* K_0 *is the constant from inequality* (1.1).

Remark 3.1. It is clear that under the conditions of Theorem 3.1 the pair (u,σ), $\sigma = \tau + 2\mu_1\mathcal{E}(u)$ is a weak solution of problem (0.1)–(0.4) in the sense of Definition 3.1.

For the proof of the theorem, consider an auxiliary problem:

$$\frac{d}{dt}(\tau,\Phi) + \frac{1}{\lambda_1}(\tau,\Phi) - \sum_{i=1}^n\left(\frac{u_i\tau}{1+\frac{1}{m}\left(\frac{|\tau|^2}{2\mu_2}+|u|^2\right)}, \frac{\partial\Phi}{\partial x_i}\right) + 2\mu_2(u,\operatorname{Div}\Phi) + \frac{1}{k\lambda_1}(\nabla\tau,\nabla\Phi) = 0, \tag{3.10}$$

$$\frac{d}{dt}(u,\varphi) - \sum_{i=0}^n\left(\frac{u_iu}{1+\frac{1}{m}\left(\frac{|\tau|^2}{2\mu_2}+|u|^2\right)}, \frac{\partial\varphi}{\partial x_i}\right) + \mu_1(\nabla u,\nabla\varphi) + (\tau,\nabla\varphi) = \langle f,\varphi\rangle \tag{3.11}$$

for all $\varphi \in V$, $\Phi \in H_0^1$ and a.e. $t \in (0,T)$ with initial condition (3.8). Natural numbers m, k are parameters.

Below we need two auxiliary statements.

Lemma 3.1. (The proof may be found in [16].) *For* $u \in V$, $\tau \in H_0^1$ *the following identities take place*:

$$(\tau,\nabla u) + (u,\operatorname{Div}\tau) = 0, \tag{3.12}$$

$$\sum_{i=1}^n\left(\frac{u_iu}{1+\frac{1}{m}\left(\frac{|\tau|^2}{2\mu_2}+|u|^2\right)}, \frac{\partial u}{\partial x_i}\right) + \frac{1}{2\mu_2}\sum_{i=1}^n\left(\frac{u_i\tau}{1+\frac{1}{m}\left(\frac{|\tau|^2}{2\mu_2}+|u|^2\right)}, \frac{\partial\tau}{\partial x_i}\right) = 0. \tag{3.13}$$

Lemma 3.2. *Let* $f \in \mathcal{X}$. *Assume that a pair* (u,τ) (*where* $u \in L_2(0,T;V) \cap L_\infty(0,T;H)$, $\tau \in L_\infty(0,T;L_2)$) *satisfies either identities* (3.10), (3.11) *or identities* (3.5), (3.6) *a.e. on* $(0,T)$ *for all* $\varphi \in \mathcal{V}$ *and* $\Phi \in C_0^\infty$. *Then for all* $t \in [0,T]$ *the following estimate is valid*:

$$\|u'\|_{L_{4/3}(t,T;V^*)} + \|\tau'\|_{L_2(t,T;H^{-2})} \leqslant K_1\big(\|u\|_{L_\infty(t,T;H)}, \|u\|_{L_2(t,T;V)}, \|\tau\|_{L_\infty(t,T;L_2)}, \|f\|_{\mathcal{X}}, T-t, \Omega\big). \tag{3.14}$$

Proof. Let a pair (u,τ) from the class $u \in L_2(0,T;V) \cap L_\infty(0,T;H)$, $\tau \in L_\infty(0,T;L_2)$ satisfy identities (3.10), (3.11) a.e. on $(0,T)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C_0^\infty$. From inequalities

$$\big\|u(t)\big\|_{L_4} \leqslant 2^{1/4}\big\|u(t)\big\|^{\frac{1}{2}}\big\|u(t)\big\|_1^{\frac{1}{2}}, \quad n=2,$$

$$\big\|u(t)\big\|_{L_4} \leqslant 2^{1/2}\big\|u(t)\big\|^{\frac{1}{4}}\big\|u(t)\big\|_1^{\frac{3}{4}}, \quad n=3$$

(see [13, Lemmas III.3.3 and III.3.5]) and from (1.1) it follows that

$$\big\|u(t)\big\|_{L_4} \leqslant K(\Omega)\big\|u(t)\big\|^{\frac{1}{4}}\big\|u(t)\big\|_1^{\frac{3}{4}}, \quad n = 2, 3,$$

and

$$\|u\|_{L_{8/3}(t,T;L_4)} \leqslant K(\Omega)\|u\|^{\frac{1}{4}}_{L_\infty(t,T;H)}\|u\|^{\frac{3}{4}}_{L_2(t,T;V)}.$$

We have from (3.11)

$$\begin{aligned}\big\|\langle u', \varphi\rangle\big\|_{L_{4/3}(t,T)} &= \left\|\frac{d}{dt}(u,\varphi)\right\|_{L_{4/3}(t,T)}\\ &\leqslant \sum_{i=1}^n\left\|\left(\frac{u_i u}{1+\frac{1}{m}\left(\frac{|\tau|^2}{2\mu_2}+|u|^2\right)}, \frac{\partial\varphi}{\partial x_i}\right)\right\|_{L_{4/3}(t,T)}\\ &\quad + \big\|\mu_1(\nabla u, \nabla\varphi)\big\|_{L_{4/3}(t,T)} + \big\|(\tau, \nabla\varphi)\big\|_{L_{4/3}(t,T)} + \big\|\langle f, \varphi\rangle\big\|_{L_{4/3}(t,T)}.\end{aligned}$$

Applying Hölder's inequality and the inequality $1/(1+\frac{1}{m}(\frac{|\tau|^2}{2\mu_2}+|u|^2)) \leqslant 1$ we get that the right-hand side does not exceed

$$\begin{aligned}&\|\varphi\|_V\big(\|u\|^2_{L_{8/3}(t,T;L_4)} + \mu_1\|u\|_{L_{4/3}(t,T;V)} + \|\tau\|_{L_{4/3}(t,T;L_2)} + \|f\|_{L_{4/3}(t,T;V^*)}\big)\\ &\quad\leqslant \|\varphi\|_V K\big(\|u\|_{L_\infty(t,T;L_2)}, \|u\|_{L_2(t,T;V)}, \|\tau\|_{L_\infty(t,T;L_2)}, \|f\|_{\mathcal{X}}, T-t, \Omega\big),\end{aligned}$$

and it gives the estimate of the first term in the left-hand side of (3.14).

Taking into account the embedding $H_0^2 \subset W_4^1$ for $n = 2, 3$, from (3.10) we obtain the estimate of the second term in the left-hand side of (3.14):

$$\begin{aligned}\big\|\langle\tau', \Phi\rangle\big\|_{L_2(t,T)} &= \left\|\frac{d}{dt}(\tau,\Phi)\right\|_{L_2(t,T)} \leqslant \left\|\frac{1}{\lambda_1}(\tau,\Phi)\right\|_{L_2(t,T)}\\ &\quad + \frac{1}{k\lambda_1}\big\|(\tau, \Delta\Phi)\big\|_{L_2(t,T)} + \left\|\sum_{i=1}^n\left(\frac{u_i\tau}{1+\frac{1}{m}\left(\frac{|\tau|^2}{2\mu_2}+|u|^2\right)}, \frac{\partial\Phi}{\partial x_i}\right)\right\|_{L_2(t,T)}\\ &\quad + \big\|2\mu_2(u, \operatorname{Div}\Phi)\big\|_{L_2(t,T)}\\ &\leqslant \frac{1}{\lambda_1}\|\tau\|_{L_2(t,T;L_2)}\|\Phi\|_{L_2} + \frac{1}{k\lambda_1}\|\Phi\|_{H_0^2}\|\tau\|_{L_2(t,T;L_2)}\\ &\quad + \|u\|_{L_2(t,T;L_4)}\|\tau\|_{L_\infty(t,T;L_2)}\left\|\frac{\partial\Phi}{\partial x_i}\right\|_{L_4} + 2\mu_2\|u\|_{L_2(t,T;L_2)}\|\operatorname{Div}\Phi\|_{L_2}\\ &\leqslant \|\Phi\|_{H_0^2}K\big(\|u\|_{L_\infty(t,T;L_2)}, \|u\|_{L_2(t,T;V)}, \|\tau\|_{L_\infty(t,T;L_2)}, T-t, \Omega\big).\end{aligned}$$

If (u,τ) satisfies identities (3.5), (3.6) instead of (3.10), (3.11), then the proof of (3.14) is carried out in the same way. □

Now we are ready to turn to the proof of Theorem 3.1.

In [16, Theorem 3.2] it was proved that there exist solutions $(u_{m,k}, \tau_{m,k})$ (where m, k are natural parameters in (3.10), (3.11)) of problem (3.10), (3.11), (3.8) in the class

$$\begin{aligned}&u_{m,k} \in L_2(0,T;V), \quad \frac{du_{m,k}}{dt} \in L_2\big(0,T;V^*\big)\\ &\tau_{m,k} \in L_2\big(0,T;H_0^1\big), \quad \frac{d\tau_{m,k}}{dt} \in L_2\big(0,T;H^{-1}\big).\end{aligned}$$

Take in (3.10) $\Phi = \frac{\tau_{m,k}(t)}{2\mu_2}$ and in (3.11) $\varphi = u_{m,k}(t)$ for almost all $t \in (0, T)$, add the results. Taking into account (3.12) and (3.13), we get:

$$\frac{1}{2}\frac{d}{dt}(u_{m,k}, u_{m,k}) + \frac{1}{4\mu_2}\frac{d}{dt}(\tau_{m,k}, \tau_{m,k}) + \mu_1(\nabla u_{m,k}, \nabla u_{m,k}) + \frac{1}{2\lambda_1\mu_2}(\tau_{m,k}, \tau_{m,k}) + \frac{1}{2k\lambda_1\mu_2}(\nabla\tau_{m,k}, \nabla\tau_{m,k}) = \langle f, u_{m,k}\rangle. \tag{3.15}$$

Let $\bar{u}_{m,k}(t) = e^{\gamma t}u_{m,k}(t)$, $\bar{\tau}_{m,k}(t) = e^{\gamma t}\tau_{m,k}(t)$ where $t \in [0, T]$. We have from (3.15):

$$\frac{1}{2}\frac{d}{dt}e^{-2\gamma t}\|\bar{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}e^{-2\gamma t}\|\bar{\tau}_{m,k}\|^2 + \mu_1 e^{-2\gamma t}\|\bar{u}_{m,k}\|_1^2 + \frac{1}{2\lambda_1\mu_2}e^{-2\gamma t}\|\bar{\tau}_{m,k}\|^2 + \frac{1}{2k\lambda_1\mu_2}e^{-2\gamma t}\|\bar{\tau}_{m,k}\|_1^2 = \big\langle f(t), e^{-\gamma t}\bar{u}_{m,k}\big\rangle.$$

Applying the formula of derivative of a product and multiplying by $e^{2\gamma t}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}_{m,k}\|^2 - \gamma\|\bar{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}_{m,k}\|^2 - \frac{\gamma}{2\mu_2}\|\bar{\tau}_{m,k}\|^2 + \mu_1\|\bar{u}_{m,k}\|_1^2 + \frac{1}{2\lambda_1\mu_2}\|\bar{\tau}_{m,k}\|^2 + \frac{1}{2k\lambda_1\mu_2}\|\bar{\tau}_{m,k}\|_1^2 = \big\langle e^{\gamma t}f, \bar{u}_{m,k}\big\rangle. \tag{3.16}$$

Since $\|\bar{u}_{m,k}\| \leqslant K_0(\Omega)\|\bar{u}_{m,k}\|_1$, from (3.16) it follows that

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}_{m,k}\|^2 + \frac{\mu_1}{2}\|\bar{u}_{m,k}\|_1^2 \leqslant e^{\gamma t}\big\|f(t)\big\|_{V^*}\|\bar{u}_{m,k}\|_1.$$

Applying Cauchy's inequality we get

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}_{m,k}\|^2 + \frac{\mu_1}{2}\|\bar{u}_{m,k}\|_1^2 \leqslant \frac{e^{2\gamma t}}{2\mu_1}\|f\|_{V^*}^2 + \frac{\mu_1}{2}\|\bar{u}_{m,k}\|_1^2,$$

that is

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}_{m,k}\|^2 \leqslant \frac{e^{2\gamma t}}{2\mu_1}\big\|f(t)\big\|_{V^*}^2.$$

Integrating from 0 to t we obtain

$$\frac{1}{2}\|\bar{u}_{m,k}\|^2 + \frac{1}{4\mu_2}\|\bar{\tau}_{m,k}\|^2 \leqslant \frac{1}{2}\|a\|^2 + \frac{1}{4\mu_2}\|\tau_0\|^2 + \frac{1}{2\mu_1}\int_0^t e^{2\gamma s}\big\|f(s)\big\|_{V^*}^2\,ds.$$

Using the following simple inequality (see the proof in Appendix A)

$$\int_0^t a^s\phi(s)\,ds \leqslant \frac{a^{t+2}}{a-1}\sup_{s\in[0,t-1]}\int_s^{s+1}\big|\phi(\zeta)\big|\,d\zeta, \tag{3.17}$$

where $a > 1$ and $\phi(s)$ is a scalar function, we get:

$$\frac{1}{2}\left\|e^{\gamma t}u_{m,k}\right\|^2+\frac{1}{4\mu_2}\left\|e^{\gamma t}\tau_{m,k}\right\|^2$$
$$\leqslant\frac{1}{2}\|a\|^2+\frac{1}{4\mu_2}\|\tau_0\|^2+\frac{e^{2\gamma(t+2)}}{2\mu_1(e^{2\gamma}-1)}\sup_{s\in[0,t-1]}\int_s^{s+1}\left\|f(\zeta)\right\|_{V^*}^2 d\zeta.$$

Multiplying by $2e^{-2\gamma t}$ and taking the maximum along the interval $(t,t+1)$ we conclude

$$\max_{s\in(t,t+1)}\left\|u_{m,k}(s)\right\|^2+\frac{1}{2\mu_2}\max_{s\in(t,t+1)}\left\|\tau_{m,k}(s)\right\|^2$$
$$\leqslant e^{-2\gamma t}\left(\|a\|^2+\frac{1}{2\mu_2}\|\tau_0\|^2\right)+\frac{e^{4\gamma}}{\mu_1(e^{2\gamma}-1)}\|f\|_{\mathcal{X}}^2. \tag{3.18}$$

Applying to the right-hand side of (3.15) Cauchy's inequality we get:

$$\frac{1}{2}\frac{d}{dt}\|u_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\tau_{m,k}\|^2+\mu_1\|u_{m,k}\|_1^2+\frac{1}{2\lambda_1\mu_2}\|\tau_{m,k}\|^2$$
$$+\frac{1}{2k\lambda_1\mu_2}\|\tau_{m,k}\|_1^2\leqslant\frac{1}{2\mu_1}\|f\|_{V^*}^2+\frac{\mu_1}{2}\|u_{m,k}\|_1^2.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u_{m,k}\|^2+\frac{1}{4\mu_2}\frac{d}{dt}\|\tau_{m,k}\|^2+\frac{\mu_1}{2}\|u_{m,k}\|_1^2+\frac{1}{2k\lambda_1\mu_2}\|\tau_{m,k}\|_1^2\leqslant\frac{1}{2\mu_1}\left\|f(t)\right\|_{V^*}^2.$$

Integrating from t to $t+1$ we obtain

$$\frac{1}{2}\left\|u_{m,k}(t+1)\right\|^2-\frac{1}{2}\left\|u_{m,k}(t)\right\|^2+\frac{1}{4\mu_2}\left\|\tau_{m,k}(t+1)\right\|^2-\frac{1}{4\mu_2}\left\|\tau_{m,k}(t)\right\|^2$$
$$+\frac{\mu_1}{2}\int_t^{t+1}\left\|u_{m,k}(s)\right\|_1^2 ds+\frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\left\|\tau_{m,k}(s)\right\|_1^2 ds\leqslant\frac{1}{2\mu_1}\|f\|_{\mathcal{X}}^2. \tag{3.19}$$

Adding (3.18) and (3.19) and ignoring some positive terms in the left-hand side we arrive at

$$\frac{1}{2}\max_{s\in(t,t+1)}\left\|u_{m,k}(s)\right\|^2+\frac{1}{4\mu_2}\max_{s\in(t,t+1)}\left\|\tau_{m,k}(s)\right\|^2+\frac{\mu_1}{2}\int_t^{t+1}\left\|u_{m,k}(s)\right\|_1^2 ds$$
$$+\frac{1}{2k\lambda_1\mu_2}\int_t^{t+1}\left\|\tau_{m,k}(s)\right\|_1^2 ds$$
$$\leqslant e^{-2\gamma t}\left(\|a\|^2+\frac{1}{2\mu_2}\|\tau_0\|^2\right)+\frac{2e^{4\gamma}+e^{2\gamma}-1}{2\mu_1(e^{2\gamma}-1)}\|f\|_{\mathcal{X}}^2,\quad 0\leqslant t\leqslant T-1. \tag{3.20}$$

This estimate enables to assume without loss of generality that there exist

(i) the limit $u=\lim_{k\to\infty}\lim_{m\to\infty}u_{m,k}$, which is weak in $L_2(0,T;V)$ and $*$-weak in $L_\infty(0,T;H)$,
(ii) the limit $\tau=\lim_{k\to\infty}\lim_{m\to\infty}\tau_{m,k}$, which is $*$-weak in $L_\infty(0,T;L_2)$.

From (3.20) it follows that the pair (u,τ) satisfies inequality (3.9). From the results of [16] it follows that the estimate (3.20) together with Lemma 3.2 ensures convergence of all terms in identities (3.10), (3.11) with $(u_{m,k},\tau_{m,k})$ substituted there to the corresponding terms in (3.5), (3.6) at least in the sense of scalar distributions on $(0,T)$. Therefore the pair (u,τ) satisfies identities (3.5), (3.6) for all $\varphi\in\mathcal{V}$ and $\Phi\in C_0^\infty$. From the results of [16] and Lemma 3.2 it follows that the pair (u,τ) belongs to class (3.7) and satisfies condition (3.8).

4. Uniform attractors for weak solutions of boundary value problem for the motion equations of viscoelastic medium

In this section we construct the minimal uniform trajectory attractor and the uniform global attractor for problem (3.5), (3.6). We choose $H\times L_2(\Omega,\mathbb{R}^{n\times n}_S)$ as the space E and the space $V_\delta^*\times H^{-\delta}(\Omega,\mathbb{R}^{n\times n}_S)$ as the space E_0, where $\delta\in(0,1]$ is a fixed number.

Fix some $f\in\mathcal{X}$. As it was pointed out in [4, pp. 914, 917] , an appropriate symbol space must contain the set $\Sigma_0=\bigcup_{t\geqslant 0}T(h)f$. In [4] the closure of this set in the weak topology of the space $L_{2,\mathrm{loc}}(0,+\infty;V^*)$ was chosen as the symbol space for the Navier–Stokes problem. This was done in order to provide compactness of the symbol space and to have a closed family of trajectory spaces (i.e., to have the property (c) from Remark 2.4 in our paper). In our paper these assumptions are not required, so we can choose the symbol space Σ, for instance, from the following variants:

(a) Σ_0;
(b) the closure of Σ_0 in the strong topology of $\mathcal{X}$;
(c) the closure of Σ_0 in the weak topology of $\mathcal{X}$;
(d) any other subset of $\mathcal{X}$ which contains Σ_0 and satisfies the property $\|\sigma\|_{\mathcal{X}}\leqslant\|f\|_{\mathcal{X}}$ for all $\sigma\in\Sigma$.

As it was mentioned above (see Remark 2.5), the uniform attractors depend on the choice of the symbol space.

Fix a symbol space Σ, given by one of the statements (a)–(d). Define the trajectory space $\mathcal{H}^+_\sigma$ for a symbol $\sigma\in\Sigma$ as the set of pairs of functions (u,τ) which

(i) belong to the class

$$\begin{aligned}&u\in L_{2,\mathrm{loc}}(0,+\infty;V)\cap L_\infty(0,+\infty;H)\cap C_w\big([0,+\infty);H\big),\\&\tau\in L_\infty(0,+\infty;L_2)\cap C_w\big([0,+\infty);L_2\big);\end{aligned}\tag{4.1}$$

(ii) satisfy identity (3.5) and the identity

$$\begin{aligned}&\frac{d}{dt}\big(u(t),\varphi\big)-\sum_{i=1}^n\left(u_i(t)u(t),\frac{\partial\varphi}{\partial x_i}\right)+\mu_1\big(\nabla u(t),\nabla\varphi\big)+\big(\tau(t),\nabla\varphi\big)\\&\quad=\big\langle\sigma(t),\varphi\big\rangle\end{aligned}\tag{4.2}$$

a.e. on $(0,+\infty)$ for all $\varphi\in\mathcal{V}$ and $\Phi\in C_0^\infty$;

(iii) satisfy the energy inequality:

$$\frac{1}{2}\|u\|^2_{L_\infty(t,t+1;H)}+\frac{1}{4\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)}+\frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)}$$

$$\leqslant e^{-2\gamma t}\left(\|u\|^2_{L_\infty(0,+\infty;H)}+\frac{1}{2\mu_2}\|\tau\|^2_{L_\infty(0,+\infty;L_2)}\right)+\frac{2e^{4\gamma}+e^{2\gamma}-1}{2\mu_1(e^{2\gamma}-1)}\|f\|^2_{\mathcal{X}} \quad (4.3)$$

for all $t \geqslant 0$ where γ is as in Theorem 3.1.

Remark 4.1. In Section 2 it was supposed that $\mathcal{H}^+_\sigma \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$. Let us show that this condition holds for the Jeffreys model. In fact, every pair (u,τ) from $\mathcal{H}^+_\sigma$ belongs to $L_\infty(0,+\infty; H \times L_2) = L_\infty(0,+\infty; E)$. For any $T \geqslant 0$ the function $u \in L_2(0,T;V)$. By Lemma 3.2 $u' \in L_{4/3}(0,T;V^*)$, $\tau' \in L_2(0,T;H^{-2})$. Since Ω is bounded, $H^\delta_0 \subset L_2$ compactly (see, e.g., [13, Theorem 1.1 of Chapter II]). Therefore $V_\delta \subset H$ compactly, $L_2 \subset H^{-\delta}$ compactly, $H \subset V^*_\delta$ compactly. By the Aubin–Simon compactness theorem (see [11, Corollary 4]) one has: $u \in C([0,T]; V^*_\delta)$, $\tau \in C([0,T]; H^{-\delta})$, that is $(u,\tau) \in C([0,T]; E_0)$ for any $T \geqslant 0$.

Remark 4.2. It is clear that on account of inequality (4.3) the family of trajectory spaces $\mathcal{H}^+_\sigma$, $\sigma \in \Sigma$ is not translation-coordinated (see Remark 2.1).

Theorem 4.1. *Given $a \in H$, $\tau_0 \in L_2(\Omega, \mathbb{R}^{n\times n}_S)$, $\sigma \in \Sigma$, there is a pair of functions* (*a trajectory*) *$(u,\tau) \in \mathcal{H}^+_\sigma$, satisfying initial condition* (3.8).

Proof. Take an increasing sequence of positive numbers $T_m \to \infty$. Put in Theorem 3.1 σ instead of f. Then by Theorem 3.1 for every natural m there is a pair (u_m,τ_m) satisfying conditions (i)–(iv) from the statement of Theorem 3.1 on the segment $[0,T_m]$ (with σ instead of f). Denote by $\tilde{u}_m$ and $\tilde{\tau}_m$ the functions which are equal to u_m and τ_m in $[0,T_m]$ and are equal to zero on $(T_m,+\infty)$. All pairs (u_m,τ_m) satisfy inequality (3.9) with σ instead of f. Since $\|\sigma\|_{\mathcal{X}} \leqslant \|f\|_{\mathcal{X}}$, they satisfy it also with f. Thus, without loss of generality one may assume that there exist limits $u = \lim_{m\to\infty} \tilde{u}_m$, which is $*$-weak in $L_\infty(0,+\infty;H)$; $\tau = \lim_{m\to\infty} \tilde{\tau}_m$, which is $*$-weak in $L_\infty(0,+\infty;L_2)$.

Fix an arbitrary interval $[0,T]$. Then estimate (3.9) for all pairs (u_m,τ_m) yields uniform boundedness of the sequence $\{\tilde{u}_m\}$ in $L_2(0,T;V)$. Therefore without loss of generality we may consider that $\tilde{u}_m \rightharpoonup u$ weakly in $L_2(0,T;V)$. It is easy to see that the pair (u,τ) satisfies inequality (3.9) (and, hence, (4.3)) and for $T_m \geqslant T$ the functions $\tilde{u}_m$ and $\tilde{\tau}_m$ coincide with u_m and τ_m on the segment $[0,T]$.

As in the proof of Theorem 3.1, estimate (3.9) for all pairs (u_m,τ_m) and Lemma 3.2 ensure convergence of all terms in identities (3.5), (4.2) with (u_m,τ_m) substituted there, $T_m \geqslant T$, to the corresponding terms in (3.5), (4.2) in the sense of scalar distributions on $(0,T)$. Therefore the pair (u,τ) satisfies identities (3.5), (4.2) on $(0,T)$ for all $\varphi \in \mathcal{V}$ and $\Phi \in C^\infty_0$, belongs to class (3.7) and satisfies condition (3.8). Since T was arbitrary, the pair (u,τ) satisfies conditions (i)–(iii) of the definition of $\mathcal{H}^+_\sigma$. □

The main result of this section is

Theorem 4.2. *There exists a minimal uniform* (*with respect to $\sigma \in \Sigma$*) *trajectory attractor $\mathcal{U}_J$ for problem* (3.5), (3.6).

Proof. By Theorem 2.1 it suffices to show existence of a uniform trajectory semiattractor.

Consider the set P which consists of pairs

$$(u,\tau) \in C\big([0,+\infty); E_0\big) \cap L_\infty(0,+\infty; E)$$

satisfying inequalities

$$\frac{1}{2}\|u\|^2_{L_\infty(t,t+1;H)} + \frac{1}{4\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|u\|^2_{L_2(t,t+1;V)} \leqslant \frac{2e^{4\gamma}+e^{2\gamma}-1}{\mu_1(e^{2\gamma}-1)}\|f\|^2_{\mathcal{X}},$$

$$\|u'\|_{L_{4/3}(t,t+1;V^*)} + \|\tau'\|_{L_2(t,t+1;H^{-2})} \leqslant K_1\big(\|u\|_{L_\infty(t,t+1;H)}, \|u\|_{L_2(t,t+1;V)}, \|\tau\|_{L_\infty(t,t+1;L_2)}, \|f\|_{\mathcal{X}}, 1, \Omega\big)$$

for all $t \geqslant 0$, where K_1 is the constant from estimate (3.14).

The set P is bounded in $L_\infty(0,+\infty;E)$. Let us show that P is relatively compact in $C([0,+\infty);E_0)$.

Fix $\epsilon > 0$. Let a natural number M be such that $\frac{1}{2^{M-1}} < \epsilon$. From the Aubin–Simon compactness theorem [11, Corollary 4] it follows that the set $P_M = \{v = u|_{[0,M]} \mid u \in P\}$ is relatively compact in $C([0,M];E_0)$. Then there is $\frac{\epsilon}{4}$-net $\{u_1,\dots,u_k\} \subset P_M$ in $C([0,M];E_0)$ for the set P_M. Let $\tilde{u}_j(t) = u_j(t)$, $0 \leqslant t \leqslant M$, $\tilde{u}_j(t) = u_j(M)$, $t > M$, $j = 1,\dots,k$. Then for any $u \in P$:

$$\begin{aligned}
&\min_{j=1,\dots,k} \|u-\tilde{u}_j\|_{C([0,+\infty);E_0)} \\
&\quad= \min_{j=1,\dots,k} \sum_{i=0}^{+\infty} 2^{-i} \frac{\|u-\tilde{u}_j\|_{C([0,i];E_0)}}{1+\|u-\tilde{u}_j\|_{C([0,i];E_0)}} \\
&\quad\leqslant \min_{j=1,\dots,k} \sum_{i=0}^{M} 2^{-i}\|u-\tilde{u}_j\|_{C([0,i];E_0)} + \sum_{i=M+1}^{+\infty} 2^{-i} \leqslant 2\cdot\frac{\epsilon}{4}+\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, $\{\tilde{u}_1,\dots,\tilde{u}_k\}$ is ϵ-net for P. Hence, P is relatively compact in $C([0,+\infty);E_0)$.

From estimates (4.3) and (3.14) it follows that the set P is uniformly absorbing. Furthermore, it is clear that $T(h)P \subset P$ for all $h \geqslant 0$. By Lemma 2.6, $\bar{P}$ is a uniform semiattractor. □

Theorems 2.2 and 4.2 imply

Theorem 4.3. *In the space $H \times L_2$ there is a uniform (with respect to $\sigma \in \Sigma$) global attractor $\mathcal{A}_J$ (in $V_\delta^* \times H^{-\delta}$) for problem* (3.5), (3.6).

Appendix A

Proof of inequality (3.17). Without loss of generality we may assume that $\phi(s) = 0$ for $s > t$. Denote by $[t]$ the integer part of t. Then we have:

$$\begin{aligned}
\int_0^t a^s\phi(s)\,ds &\leqslant \sum_{i=1}^{[t]} \int_i^{i+1} a^s\big|\phi(s)\big|\,ds \\
&\leqslant \sum_{i=0}^{[t]} a^{i+1} \int_i^{i+1} \big|\phi(s)\big|\,ds \leqslant a\big(1+a+a^2+\cdots+a^{[t]}\big) \sup_{s\in[0,t]} \int_s^{s+1} \big|\phi(\zeta)\big|\,d\zeta \\
&= \frac{a(a^{[t]+1}-1)}{a-1} \sup_{s\in[0,t-1]} \int_s^{s+1} \big|\phi(\zeta)\big|\,d\zeta \leqslant \frac{a^{t+2}}{a-1} \sup_{s\in[0,t-1]} \int_s^{s+1} \big|\phi(\zeta)\big|\,d\zeta.
\end{aligned}$$

References

[1] A.V. Babin, M.I. Vishik, Attractors of Evolution Equations, Nauka, Moscow, 1989.
[2] D. Cheban, Global Attractors of Nonautonomous Dissipative Dynamical Systems, Interdiscip. Math. Sci., vol. 1, World Scientific, River Edge, NJ, 2004.
[3] V.V. Chepyzhov, M.I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, J. Math. Pures Appl. 73 (1994) 279–333.
[4] V.V. Chepyzhov, M.I. Vishik, Evolution equations and their trajectory attractors, J. Math. Pures Appl. 76 (1997) 913–964.
[5] I. Gyarmati, Non-Equilibrium Thermodynamics. Field Theory and Variational Principles, Springer, Heidelberg, 1970.
[6] A. Haraux, Systèmes dynamiques dissipatifs et applications, Masson, Paris, 1991.
[7] A.A. Kotsiolis, A.P. Oskolkov, On the limiting conditions and an attractor for the motion equations of Oldroyd's fluid, Zap. Nauchn. Sem. LOMI 152 (1986) 67–71.
[8] O.A. Ladyzenskaya, On the dynamic system generated by Navier–Stokes equations, Zap. Nauchn. Sem. LOMI 27 (1972) 91–115.
[9] A. Miranville, X. Wang, Attractors for nonautonomous nonhomogeneous Navier–Stokes equations, Nonlinearity 10 (1997) 1047–1061.
[10] M. Reiner, Reology, in: S. Flugge (Ed.), Handbuch der Physik, Bd. VI, Springer, 1958.
[11] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[12] J. Simon, On the existence of the pressure for solutions of the variational Navier–Stokes equation, J. Math. Fluid Mech. 3 (1999) 225–234.
[13] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, North-Holland, 1979.
[14] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, second ed., Springer, 1997.
[15] M.I. Vishik, V.V. Chepyzhov, Trajectory and global attractors for 3D Navier–Stokes system, Mat. Zametki 71 (2002) 194–213.
[16] D.A. Vorotnikov, V.G. Zvyagin, On the existence of weak solutions for the initial-boundary value problem in the Jeffreys model of motion of a viscoelastic medium, Abstr. Appl. Anal. 2004 (10) (2004) 815–829.
[17] V.G. Zvyagin, V.T. Dmitrienko, Approximating-Topological Approach to Investigation of Problems of Hydrodynamics. Navier–Stokes System, Editorial USSR, Moscow, 2004.