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Volterra's realization of the KM-system ☆

M.A. Agrotis *, P.A. Damianou

Department of Mathematics and Statistics, University of Cyprus, Nicosia 1678, Cyprus

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Abstract

We construct a symplectic realization of the KM-system and obtain the higher order Poisson tensors and commuting flows via the use of a recursion operator. This is achieved by doubling the number of variables through Volterra's coordinate transformation. An application of Oevel's theorem yields master symmetries, invariants and deformation relations.

Keywords: KM-system; Recursion operator; Symplectic structure

1. Introduction

The Kac–van Moerbeke system (KM-system), also known as the Volterra system, is defined by

$$\dot{u}_i = u_i(u_{i+1} - u_{i-1}), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $u_0 = u_{n+1} = 0$. It has been used as a model for predator–prey evolution systems [12], as well as a discretization of the Korteweg–de Vries equation. Its integrability was established in [7,9]. In [7], Kac and van Moerbeke formulated the inverse scattering technique in a discrete setting and applied it on Eqs. (1) to produce explicit solutions. Moser using a different method, namely continued fractions, has also integrated the model.

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* Corresponding author.
E-mail addresses: agrotis@ucy.ac.cy, marenges@yahoo.com (M.A. Agrotis).

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A diffeomorphism is established between the KM-system (1) and the celebrated Toda lattice equations

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i), \quad i = 1, \dots, n-1, \\ \dot{b}_i &= 2\big(a_i^2 - a_{i-1}^2\big), \quad i = 1, \dots, n, \end{aligned}$$

via a transformation of Hénon

$$\begin{aligned} a_i &= -\frac{1}{2}\sqrt{u_{2i}u_{2i-1}}, \quad i = 1, \dots, n-1, \\ b_i &= \frac{1}{2}(u_{2i-1} + u_{2i-2}), \quad i = 1, \dots, n. \end{aligned}$$

Thus, the hierarchy of Poisson tensors, Hamiltonian functions, constants of motion and master symmetries known for the Toda lattice and expressed in Flaschka's coordinates (a, b), can be mapped to the corresponding ones for the KM-system in u-coordinates [2]. We note that the number of variables for the Toda lattice is odd and therefore we restrict our attention to the KM-system with an odd number of variables.

A Hamiltonian description of the KM-system can be found in the book of Fadeev and Takhtajan [4]. Later on, in [2] two polynomial Poisson tensors of degree two and three are considered and placed in an infinite sequence of Poisson tensors that satisfy Lenard type relations. The quadratic Poisson bracket, π_2, is defined by the formulas

$$\{u_i, u_{i+1}\} = u_i u_{i+1}, \tag{2}$$

and all other brackets are zero. Using $H = \sum_{i=1}^{2n-1} u_i$ as the Hamiltonian and the Poisson bracket π_2, the Volterra equations are written in Poisson form, $\dot{u}_i = \{u_i, H\}$.

We will follow [2] and use the Lax pair of that reference. It has the advantage of making the equations homogeneous, polynomial. The Lax pair is given by

$$\dot{L} = [B, L], \tag{3}$$

where

$$L = \begin{pmatrix} u_1 & 0 & \sqrt{u_1u_2} & 0 & \dots & & 0 \\ 0 & u_1+u_2 & 0 & \sqrt{u_2u_3} & & & \vdots \\ \sqrt{u_1u_2} & 0 & u_2+u_3 & & \ddots & & \\ 0 & \sqrt{u_2u_3} & & & & & \\ \vdots & & \dots & & & & \sqrt{u_{2n-2}u_{2n-1}} \\ & & & & & u_{2n-2}+u_{2n-1} & 0 \\ & & & & \sqrt{u_{2n-2}u_{2n-1}} & 0 & u_{2n-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{u_1u_2} & 0 & \dots & & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{u_2u_3} & & & \vdots \\ -\frac{1}{2}\sqrt{u_1u_2} & 0 & 0 & & \ddots & & \\ 0 & -\frac{1}{2}\sqrt{u_2u_3} & & & & & \\ \vdots & & \dots & & & & \frac{1}{2}\sqrt{u_{2n-2}u_{2n-1}} \\ & & & & & 0 & 0 \\ & & & & -\frac{1}{2}\sqrt{u_{2n-2}u_{2n-1}} & 0 & 0 \end{pmatrix}.$$

This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{Tr} L^i$ are constants of motion.

The cubic Poisson bracket, which corresponds to the second KdV bracket in the continuum limit, is defined by

$$\{u_i, u_{i+1}\} = u_i u_{i+1}(u_i + u_{i+1}),$$
$$\{u_i, u_{i+2}\} = u_i u_{i+1} u_{i+2},$$

and all other brackets are zero. We denote this bracket by π_3. The Lenard relations take the form

$$\pi_3 \nabla H_i = \pi_2 \nabla H_{i+1}.$$

The higher order Poisson brackets are constructed using a sequence of master symmetries Y_i, $i = 0, 1, \ldots$. We define Y_0 to be the Euler vector field

$$Y_0 = \sum_{i=1}^{2n-1} u_i \frac{\partial}{\partial u_i},$$

and Y_1 the master symmetry

$$Y_1 = \sum_{i=1}^{2n-1} U_i \frac{\partial}{\partial u_i},$$

where

$$U_i = (i+1)u_i u_{i+1} + u_i^2 + (2-i)u_{i-1}u_i.$$

One can verify that the bracket π_3 is obtained from π_2 by taking the Lie derivative in the direction of Y_1.

The brackets π_2 and π_3 are just the beginning of an infinite family constructed in [2] using master symmetries. We quote the result:

Theorem 1. *There exists a sequence of Poisson tensors π_j and a sequence of master symmetries Y_j such that*:

(i) *π_j are all Poisson.*
(ii) *The functions H_i are in involution with respect to all of the π_j.*
(iii) $Y_i(H_j) = (i+j)H_{i+j}$.
(iv) $L_{Y_i}\pi_j = (j-i-2)\pi_{i+j}$.
(v) $[Y_i, Y_j] = (j-i)Y_{i+j}$.
(vi) $\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}$, *where π_j denotes the Poisson matrix of the tensor π_j.*

The KM-system is a special case of the more general Lotka–Volterra equations, which have the form

$$\dot{u}_i = \sum_{k=1}^{N} a_{ik} u_i u_k, \quad i = 1, 2, \ldots, N, \tag{4}$$

where (a_{ij}) is a fixed matrix.

In the early work on (4), Volterra introduced a transformation from $\mathbb{R}^{2N}$ to $\mathbb{R}^N$, in his attempt to provide a Hamiltonian formulation; see, for example, [5]. Specifically he doubled the number of variables by defining

$$q_i(t) = \int_0^t u_i(\tau)\,d\tau, \tag{5}$$

$$p_i(t) = \ln(\dot{q}_i) - \frac{1}{2}\sum_{k=1}^{N} a_{ik}q_k, \tag{6}$$

$i = 1, \dots, N$, for a skew-symmetric (a_{ij}).

The explicit form of Volterra's transformation from $\mathbb{R}^{2N}$ to $\mathbb{R}^N$ is

$$u_i = e^{p_i + \frac{1}{2}\sum_{k=1}^{N} a_{ik}q_k}, \quad i = 1, 2, \dots, N. \tag{7}$$

The Hamiltonian function is given by

$$H = \sum_{i=1}^{N} \dot{q}_i = \sum_{i=1}^{N} u_i, \tag{8}$$

which takes the form

$$H = \sum_{i=1}^{N} e^{p_i + \frac{1}{2}\sum_{k=1}^{N} a_{ik}q_k}. \tag{9}$$

System (4) can then be expressed in the following form:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}, \tag{10}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}, \tag{11}$$

where the Poisson bracket in (q, p) coordinates in $\mathbb{R}^{2N}$ is the canonical one. We note that for the KM-system in u-space both Poisson tensors π_2 and π_3 are degenerate. Therefore, an application of the theory of recursion operators is hindered.

In this paper we consider the KM-system in $\mathbb{R}^{2n-1}$ and obtain a symplectic realization of the system by increasing the dimension of the space. Namely, the number of variables is doubled through Volterra's coordinate transformation. We produce the higher order Poisson tensors and flows for the system via the use of a recursion operator. We define a conformal symmetry in symplectic space and apply Oevel's theorem to produce deformation relations, which can then be projected to give the deformation relations for the Volterra system in $\mathbb{R}^{2n-1}$.

2. Master symmetries and recursion operators

Let us consider the differential equation $\dot{x} = \mathcal{X}(x)$ on a manifold M defined by the Hamiltonian vector field $\mathcal{X}$. Below we give the definition of master symmetries, due to Fokas and Fuchssteiner [6], and briefly mention their basic properties. A vector field Z is a symmetry of the equation if $[Z, \mathcal{X}] = 0$. In the case that $Z = Z(t, x)$, Z is a time-dependent symmetry if

$$\frac{\partial Z}{\partial t} + [Z, \mathcal{X}] = 0.$$

A more general definition is that of a generator of symmetries. Z is called a generator of degree zero if $[Z, \mathcal{X}] = 0$, and a generator of degree one if $[[Z, \mathcal{X}], \mathcal{X}] = 0$. A generator of degree k is the one that satisfies $[\dots[Z,\mathcal{X}],\dots]=0$, where there are $k+1$ nested Lie brackets. We remark that if Z is a generator of degree k then $[Z,\mathcal{X}]$ is a generator of degree $k-1$. Also, if Z is a generator of degree k then Z is a generator of degree $i \geqslant k$. A symmetry is a generator of degree zero. A generator of degree one that is not a generator of degree zero is called a master symmetry. Oevel's theorem provides a useful method for constructing master symmetries.

Suppose that we have a bi-Hamiltonian system [8] with a symplectic Poisson tensor. Namely, a pair of Poisson tensors P_0 and P_1, with P_0 symplectic and a pair of Hamiltonian functions H_1, H_2 that give rise to the same system, i.e.,

$$P_0\nabla H_2 = P_1\nabla H_1. \tag{12}$$

Then a recursion operator $\mathcal{R}$ is defined by $\mathcal{R}=P_1P_0^{-1}$, and gives rise to a family of Hamiltonian vector fields that are defined recursively as

$$\mathcal{X}_i=\mathcal{R}^{i-1}\mathcal{X}_1,$$

and higher order Poisson tensors

$$P_i=\mathcal{R}^iP_0. \tag{13}$$

The Hamiltonians H_i corresponding to the vector fields $\mathcal{X}_i$ are given by $\nabla H_i=(\mathcal{R}^*)^i\nabla H_0$. These higher order flows have a multi-Hamiltonian formulation,

$$\mathcal{X}_{i+j}=P_i\nabla H_j. \tag{14}$$

Magri's theorem [8] states that the flows $\mathcal{X}_i$ pairwise commute. Also the functions H_i are constants of motion for each flow and commute with respect to all higher order Poisson tensors. We thus have an infinite sequence of involutive Hamiltonian flows. Furthermore, Oevel's theorem provides a method for constructing master symmetries [10]. We quote the theorem.

Theorem 2. *Suppose that X_0 is a conformal symmetry for both π_1, π_2 and H_1, i.e., for some scalars λ, μ, and ν we have*

$$\mathcal{L}_{X_0}\pi_1=\lambda\pi_1,\qquad \mathcal{L}_{X_0}\pi_2=\mu\pi_2,\qquad \mathcal{L}_{X_0}H_1=\nu H_1. \tag{15}$$

Then the vector fields $X_i=\mathcal{R}^iX_0$ are master symmetries and we have

(a) $\mathcal{L}_{X_i}H_j=(\nu+(j-1+i)(\mu-\lambda))H_{i+j}$;
(b) $\mathcal{L}_{X_i}\pi_j=(\mu+(j-i-2)(\mu-\lambda))\pi_{i+j}$;
(c) $[X_i,X_j]=(\mu-\lambda)(j-i)X_{i+j}$.

As a corollary to Oevel's theorem we have the existence of the following time-dependent symmetries for each flow in the hierarchy:

$$Y_{\mathcal{X}_i}=X_i+t\big(\mu+\nu+(j-1)(\mu-\lambda)\big)\mathcal{X}_{i+j},\quad i,j=1,2,\dots. \tag{16}$$

In the next section we will formulate the bi-Hamiltonian Volterra system in a symplectic setting so that we can apply the theory described in this section and obtain the results stemming out of the theorems of Magri and Oevel.

3. Symplectic setting

We consider the Volterra map

$$\begin{aligned}
&\Psi : \mathbb{R}^{2(2n-1)} \mapsto \mathbb{R}^{2n-1},\\
&u_i = e^{p_i + \frac{1}{2}(q_{i+1}-q_{i-1})}, \quad i = 1, \dots, 2n-1,
\end{aligned} \tag{17}$$

where $q_0 = q_{2n} = 0$. We note that $u_0 = u_{2n} = 0$. The Hamiltonian in (q, p) coordinates is given by

$$h_1 = \sum_{i=1}^{2n-1} e^{p_i + \frac{1}{2}(q_{i+1}-q_{i-1})}, \tag{18}$$

and together with the standard, symplectic Poisson tensor

$$P_2 = \begin{pmatrix} 0 & I_{(2n-1)} \\ -I_{(2n-1)} & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix, corresponds to the KM-system (1) under the mapping (17). To avoid confusion, a Poisson tensor is symplectic (by definition) if it is invertible. We call it standard symplectic since its inverse is the standard, canonical, symplectic two-form in $\mathbb{R}^{2(2n-1)}$. In particular, the degenerate quadratic Poisson tensor π_2, defined in Section 1, is lifted to P_2 via transformation (17). To find the pre-image of the cubic bracket π_3, we will lift the master symmetry Y_1 of Section 1 from the u-space in $\mathbb{R}^{2n-1}$ to a master symmetry X_1 in the symplectic space $(q, p) \in \mathbb{R}^{2(2n-1)}$. In fact, $\mathcal{L}_{X_1} P_2 = P_3$, where X_1 projects to Y_1 using the Volterra map. One possible definition for X_1 is the following:

$$X_1 = \sum_{i=1}^{2n-1} A_i \frac{\partial}{\partial q_i} + \sum_{i=1}^{2n-1} B_i \frac{\partial}{\partial p_i}, \tag{19}$$

where

$$\begin{aligned}
A_i &= \sum_{j=1}^{2n-1} c_{j,i} e^{p_j + \frac{1}{2}(q_{j+1}-q_{j-1})},\\
B_i &= (i+1)e^{p_{i+1} + \frac{1}{2}(q_{i+2}-q_i)} + e^{p_i + \frac{1}{2}(q_{i+1}-q_{i-1})} + (2-i)e^{p_{i-1} + \frac{1}{2}(q_i - q_{i-2})}\\
&\quad + \frac{1}{2} \sum_{j=1}^{2n-1} (c_{j,i-1} - c_{j,i+1}) e^{p_j + \frac{1}{2}(q_{j+1}-q_{j-1})},
\end{aligned}$$

for $i = 1, 2, \dots, 2n-1$. The constants $c_{i,j}$ are given by

$$\begin{aligned}
&c_{i,j} = 0, && i = 1, \dots, 2n-2,\ j > i,\\
&c_{i,j} = -1, && i = 2, \dots, 2n-1,\ j < i,\\
&c_{i,i} = i-1, && i = 1, \dots, 2n-1.
\end{aligned}$$

We note that $c_{j,0} = c_{j,2n} = 0$. The constant matrix $C := (c_{i,j})$ takes the form

$$C := \begin{pmatrix}
0 & 0 & 0 & 0 & \dots & 0\\
-1 & 1 & 0 & 0 & \dots & 0\\
-1 & -1 & 2 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
-1 & \dots & \dots & -1 & 2n-3 & 0\\
-1 & \dots & \dots & \dots & -1 & 2n-2
\end{pmatrix}.$$

Taking the Lie derivative of the symplectic bracket P_2 in the direction of X_1 we obtain the Poisson bracket P_3,

$$
\begin{aligned}
&\{q_i, q_j\} = e^{p_j+\frac{1}{2}(q_{j+1}-q_{j-1})}, && i<j,\\
&\{q_1, p_1\} = e^{p_1+\frac{1}{2}q_2} + \frac{1}{2}e^{p_2+\frac{1}{2}(q_3-q_1)}, &&\\
&\{q_i, p_i\} = \frac{1}{2}e^{p_i+\frac{1}{2}(q_{i+1}-q_{i-1})} + \frac{1}{2}e^{p_{i+1}+\frac{1}{2}(q_{i+2}-q_i)}, && i=2,\dots,2n-1,\\
&\{q_i, p_{i+1}\} = \frac{1}{2}e^{p_{i+2}+\frac{1}{2}(q_{i+3}-q_{i+1})}, && i=1,\dots,2n-2,\\
&\{q_2, p_1\} = e^{p_2+\frac{1}{2}(q_3-q_1)}, &&\\
&\{q_i, p_{i-1}\} = \frac{1}{2}e^{p_i+\frac{1}{2}(q_{i+1}-q_{i-1})}, && i=3,\dots,2n-1,\\
&\{q_i, p_j\} = -\frac{1}{2}e^{p_{j-1}+\frac{1}{2}(q_j-q_{j-2})} + \frac{1}{2}e^{p_{j+1}+\frac{1}{2}(q_{j+2}-q_j)}, && j\geqslant i+2,\\
&\{q_i, p_1\} = \frac{1}{2}e^{p_i+\frac{1}{2}(q_{i+1}-q_{i-1})}, && i=3,\dots,2n-1,\\
&\{p_1, p_2\} = \frac{1}{2}e^{p_1+\frac{1}{2}q_2} + \frac{1}{4}e^{p_2+\frac{1}{2}(q_3-q_1)} - \frac{1}{4}e^{p_3+\frac{1}{2}(q_4-q_2)}, &&\\
&\{p_i, p_{i+1}\} = \frac{1}{4}e^{p_i+\frac{1}{2}(q_{i+1}-q_{i-1})} + \frac{1}{4}e^{p_{i+1}+\frac{1}{2}(q_{i+2}-q_i)}, && i=2,\dots,2n-2,\\
&\{p_1, p_3\} = \frac{1}{2}e^{p_2+\frac{1}{2}(q_3-q_1)} - \frac{1}{4}e^{p_4+\frac{1}{2}(q_5-q_3)}, &&\\
&\{p_i, p_{i+2}\} = \frac{1}{4}e^{p_{i+1}+\frac{1}{2}(q_{i+2}-q_i)}, && i=2,\dots,2n-3,\\
&\{p_1, p_j\} = -\frac{1}{4}e^{p_{j+1}+\frac{1}{2}(q_{j+2}-q_j)} + \frac{1}{4}e^{p_{j-1}+\frac{1}{2}(q_j-q_{j-2})}, && j=4,\dots,2n-1,
\end{aligned}
\tag{20}
$$

and all other brackets are zero. We recall that $e^{p_{2n}+\frac{1}{2}(q_{2n+1}-q_{2n-1})} = u_{2n} = 0$. The Jacobi identity for the bracket P_3 can be rigorously checked by considering the following four cases: (a) three q, (b) three p, (c) two p and one q, and (d) two q and one p. For example, the Jacobi identity for q_i, q_j, q_k for $1 \leqslant i < j < k \leqslant 2n-1$ can be broken up to two subcases: (a1) $k = j+1$, and (a2) $k \geqslant j+2$. In a similar manner one can consider the other three cases. We remark that a result of Sergyeyev that appears in [11] provides an elegant alternative around the Jacobi identity. We quote the relevant theorem. Given a Poisson structure P its Lie derivative $L_X P$ along a vector field X defines another Poisson structure compatible with P if and only if $[L_X^2 P, P] = 0$. What is more, if P is a Poisson tensor of locally constant rank and $\dim P \leqslant 1$ then all Poisson structures compatible with P are of the form $L_X P$ where X is a vector field such that $L_X^2 P = L_{X'} P$ for some other vector field X'. In our particular case if we set $X = X_1$ and $P = P_2$ then a necessary and sufficient condition for P_3 to be Poisson is that $[L_{X_1}^2 P_2, P_2] = 0$. One finds that $L_{X_1}^2 P_2 = 0$ and thus the condition is satisfied and P_3 is indeed a Poisson structure.

Under the Volterra transformation, P_2 maps to π_2 and P_3 to π_3. The function

$$h_2 = \frac{1}{2}\sum_{i=1}^{2n-1} e^{2p_i+q_{i+1}-q_{i-1}} + \sum_{i=1}^{2n-2} e^{p_i+p_{i+1}+\frac{1}{2}(q_{i+2}+q_{i+1}-q_i-q_{i-1})}$$

corresponds under mapping (17) to a constant multiple of $H_2 = \frac{1}{2}\mathrm{Tr}(L)^2$. We recall that H_1, H_2 and π_2, π_3 constitute a bi-Hamiltonian pair

$$\pi_2 \nabla H_2 = \pi_3 \nabla H_1. \tag{21}$$

However, both Poisson tensors are degenerate. The Volterra map places this bi-Hamiltonian pair in a symplectic setting. That is,

$$P_2 \nabla h_2 = P_3 \nabla h_1, \tag{22}$$

and a recursion operator is defined as $\mathcal{R} = P_3 P_2^{-1}$. P_3 is by construction compatible with P_2 since it is generated from a master symmetry; see [3]. We note the absence of a negative recursion operator as in [1] using this method, since the matrix representing P_3 is not invertible.

A multi-Hamiltonian structure of the form $\mathcal{X}_{i+j} = P_i \nabla h_j$ is provided by the higher order Poisson tensors and Hamiltonian vector fields

$$P_i = \mathcal{R}^{i-2} P_2, \quad i = 3, 4, \ldots, \tag{23}$$

$$\mathcal{X}_i = \mathcal{R}^{i-1} \mathcal{X}_1, \quad i = 2, 3, \ldots, \tag{24}$$

where $\mathcal{X}_i$ stands for $\mathcal{X}_{h_i}$.

Theorem 2 requires the existence of a conformal symmetry X_0 such that

$$\mathcal{L}_{X_0} P_2 = \lambda P_2, \qquad \mathcal{L}_{X_0} P_3 = \mu P_3, \qquad \mathcal{L}_{X_0}(h_1) = \nu h_1. \tag{25}$$

We define the conformal symmetry

$$X_0 = \sum_{i=1}^{2n-1} \frac{\partial}{\partial p_i}, \tag{26}$$

and one can check that relations (25) are satisfied with $\lambda = 0$, $\mu = 1$, $\nu = 1$. Therefore, in addition to the infinite family of commuting Hamiltonian flows, we have the following deformation relations:

$$[X_i, h_j] = (i + j) h_{i+j}, \tag{27}$$

$$L_{X_i} P_j = (j - i - 2) P_{i+j}, \tag{28}$$

$$[X_i, X_j] = (j - i) X_{i+j}. \tag{29}$$

Using the Volterra map we can project these to the u-space and provide an alternative proof of the statements of Theorem 1.

4. Discussion

A different symplectic realization for the KM-system has been achieved recently in [1] using the map

$$\begin{aligned} &\Phi : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n-1}, \\ &u_{2i-1} = -e^{p_i}, \quad i = 1, \ldots, n, \\ &u_{2i} = e^{q_{i+1} - q_i}, \quad i = 1, \ldots, n-1. \end{aligned} \tag{30}$$

The Hamiltonian is defined as

$$H = -\sum_{i=1}^{n} e^{p_i} + \sum_{i=1}^{n-1} e^{q_{i+1} - q_i} \tag{31}$$

and the standard symplectic bracket in (q, p)-space maps to the degenerate quadratic Poisson tensor π_2 via transformation (30). A second symplectic bracket is obtained by lifting the cubic bracket π_3.

In this paper we consider the Volterra map

$$\begin{aligned} &\Psi : \mathbf{R}^{2(2n-1)} \mapsto \mathbf{R}^{2n-1}, \\ &u_i = e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})}, \quad i = 1, \ldots, 2n-1, \end{aligned} \tag{32}$$

in order to lift the bi-Hamiltonian structure of the KM-system to a symplectic space in $\mathbb{R}^{2(2n-1)}$. The big difference between the dimensions of the source and the target space in (32) impedes the application of the methodology used in [1]. However, we are able to find a pair of Poisson tensors that consists of the standard symplectic bracket P_2 and a second bracket P_3 in $\mathbb{R}^{2(2n-1)}$ so that its image under mapping (32) is the cubic bracket π_3. Since P_2 is symplectic, a recursion operator is defined as $\mathcal{R} = P_3 P_2^{-1}$, and used to give rise to an infinite hierarchy of commuting Hamiltonian flows and Poisson tensors. The conformal symmetry of the KM-system in u-space is lifted to the symplectic (q, p)-space, and an application of Oevel's theorem leads to an infinite number of master symmetries, Poisson tensors and invariants. Note that P_3 is non-invertible, and hence the recursion operator $R = P_3 P_2^{-1}$ cannot be inverted as well, i.e., no negative recursion operator exists in this realization.

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