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Compatible maps and invariant approximations

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Abstract

The existence of invariant best approximations for compatible maps is proved. Our results unify, and generalize various known results to a more general class of noncommuting mappings.

Keywords: Common fixed point; Condition (C); Diminishing orbital diameters; Compatible maps; C_q-commuting maps; Invariant approximation

1. Introduction and preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|\cdot\|)$. The set $P_M(u) = \{x \in M\colon \|x - u\| = \operatorname{dist}(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $\operatorname{dist}(u, M) = \inf\{\|y - u\|\colon y \in M\}$. We denote by $\Im_0$ the class of closed convex subsets of X containing 0 [1,16]. For $M \in \Im_0$, we define $M_u = \{x \in M\colon \|x\| \leqslant 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$. We shall use N to denote the set of positive integers, $\operatorname{cl}(S)$ to denote the closure of a set S and $\operatorname{wcl}(S)$ to denote the weak closure of a set S. The diameter of M is denoted and defined by $\delta(M) = \sup\{\|x - y\|\colon x, y \in M\}$. A mapping $f : X \to X$ has diminishing orbital diameters (d.o.d.) [8] if for each $x \in X$, $\delta(O(x)) < \infty$ and whenever $\delta(O(x)) > 0$, there exists $n = n_x \in N$ such that $\delta(O(x)) > \delta(O(f^n(x)))$, where $O(x) = \{f^k(x)\colon k \in N \cup \{0\}\}$, is the orbit of f at x and $O(f^n(x)) = \{f^k(x)\colon k \in N \cup \{0\}$ and $k \geqslant n\}$ is the orbit of f at $f^n(x)$

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for $n \in N \cup \{0\}$. Let f be a self-map of a topological space X. The orbit $O(x)$ of f at x is proper if and only if $O(x) = \{x\}$ or there exists $n = n_x \in N$ such that $\mathrm{cl}(O(f^n(x)))$ is a proper subset of $\mathrm{cl}(O(x))$. If $O(x)$ is proper for each $x \in M \subset X$, we shall say that f has proper orbits on M. Observe that in metric space (X, d) if g has d.o.d. on X, then g has proper orbits. Let $f : M \to M$ be a mapping. A mapping $T : M \to M$ is called an f-contraction if, for any $x, y \in M$, there exists $0 \leqslant k < 1$ such that $\|Tx - Ty\| \leqslant k\|fx - fy\|$. If $k = 1$, then T is called f-nonexpansive. The set of fixed points of T (respectively f) is denoted by $F(T)$ (respectively $F(f)$). A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$. The pair $\{f, T\}$ is called

(1) commuting if $Tfx = fTx$ for all $x \in M$,
(2) R-weakly commuting [6,14] if for all $x \in M$, there exists $R > 0$ such that $\|fTx - Tfx\| \leqslant R\|fx - Tx\|$. If $R = 1$, then the maps are called weakly commuting [7];
(3) compatible [7,8] if $\lim_n \|Tfx_n - fTx_n\| = 0$ when $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M;
(4) weakly compatible [2,9] if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

If f and T are compatible and do have a coincidence point, f and T are called [3,8] nontrivially compatible. The set M is called q-starshaped with $q \in M$, if the segment $[q, x] = \{(1 - k)q + kx\colon 0 \leqslant k \leqslant 1\}$ joining q to x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called

(5) C_q-commuting [2] if $fTx = Tfx$ for all $x \in C_q(f, T)$, where $C_q(f, T) = \bigcup\{C(f, T_k)\colon 0 \leqslant k \leqslant 1\}$, where $T_k x = (1 - k)q + kTx$;
(6) R-subweakly commuting on M (see [14]) if for all $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \leqslant R\,\mathrm{dist}(fx, [q, Tx])$.

It is well known that R-subweakly commuting maps are R-weakly commuting and R-weakly commuting maps are compatible but not conversely in general (see [7,8,14]). C_q-commuting maps are weakly compatible but not conversely in general and R-subweakly commuting maps are C_q-commuting but the converse does not hold in general (see, for example, [2]).

In 1963, Meinardus [11] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. In 1977, Subrahmanyam [19] proved the following extension of the result of Meinardus.

Theorem 1.1. *Let T be a nonexpansive self-mapping of a normed space X, M be a finite-dimensional T-invariant subspace of X and $u \in F(T)$. Then $P_M(u) \cap F(T) \neq \emptyset$.*

Smoluk [18] found that Theorem 1.1 remains true if finite dimensionality of M is replaced by the requirement "$\mathrm{cl}(T(D))$ is compact for every bounded $D \subset M$ and T is linear." Subsequently, Habiniak [4] observed that Smoluk's result remains valid if the linearity of T is dropped. In 1988, Sahab, Khan and Sessa [13] established the following result.

Theorem 1.2. *Let I and T be selfmaps of a normed space X with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial M) \subset M$, and $q \in F(I)$. If $D = P_M(u)$ is compact and q-starshaped, $I(D) = D$, I is* *continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on* $D \cup \{u\}$, *then* $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.

Further generalizations of the result of Meinardus were obtained by Hicks and Humphries [5], Jungck and Sessa [10], and Singh [17]. Recently, Al-Thagafi [1] extended Theorem 1.2 and proved some results on invariant approximations for commuting maps. Shahzad [14,16], Hussain and Jungck [6] and O'Regan and Shahzad [12] extended the work of Al-Thagafi [1] for R-subweakly commuting maps.

Naturally, one may raise the question: Do the above mentioned results remain valid for the more general class of compatible maps? In this paper, we give a partial answer to this question. Thus we unify and extend most of the known results to the class of compatible maps by utilizing the following recent result of Grinc and Snoha [3].

Theorem 1.3. *Let X be a Hausdorff topological space, and f and g be continuous and nontrivially compatible self-maps of X. Then there exists a point z in X such that $fz = gz = z$, provided g satisfies following condition*:

(C) *$A \cap F(g) \neq \emptyset$ for any nonempty g-invariant closed set $A \subset X$.*

The next theorem gives conditions under which condition (C) is satisfied.

Theorem 1.4. [8, Theorem 3.1] *Let X be a Hausdorff topological space, and g be a continuous self-map of X. If g has relatively compact proper orbits, then g satisfies condition* (C).

2. Main results

Definition. Let X be a set and $f, T : X \to X$. Let x be point in X such that $fx = Tx$. The point x is a coincidence point of f and T (see the definition in Section 1), and we shall call the point fx or Tx a point of coincidence of f and T (see the example below).

The following result extends and improves Theorem 2.1 of [2,16] and Lemma 2.1 of [12].

Theorem 2.1. *Let M be a nonempty subset of a metric space (X, d), and f and g be weakly compatible self-maps of M. Assume that* $\operatorname{cl} g(M) \subset f(M)$, $\operatorname{cl} g(M)$ *is complete, and f and g satisfy for all $x, y \in M$ and $0 \leqslant h < 1$,*

$$d(gx, gy) \leqslant h \max\bigl\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\bigr\}. \tag{2.1}$$

Then there is a point $w \in M$ which is the unique point of coincidence and the unique common fixed point of f and g.

Proof. As $g(M) \subset f(M)$, one can choose x_n in M, for $n \in N$, such that $gx_n = fx_{n+1}$. Set $y_n = gx_n$ and let $O(y_k; n) = \{y_k, y_{k+1}, \ldots, y_{k+n}\}$. Then following the arguments of [12, Lemma 2.1], we infer that $\{y_n\} = \{gx_n\}$ is a Cauchy sequence. It follows from the completeness of $\operatorname{cl} g(M)$ that $gx_n \to w$ for some $w \in M$ and hence $fx_n \to w$ as $n \to \infty$. Consequently, $\lim_n fx_n = \lim_n gx_n = w \in \operatorname{cl} g(M) \subset f(M)$. Thus $w = fy$ for some $y \in M$. Notice that for all $n \geqslant 1$, we have

$$\begin{aligned} d(w, gy) &\leqslant d(w, gx_n) + d(gx_n, gy) \\ &\leqslant d(w, gx_n) \\ &\quad + h\max\bigl\{d(fx_n, fy), d(gx_n, fx_n), d(gy, fy), d(gy, fx_n), d(gx_n, fy)\bigr\} \end{aligned}$$

Letting $n \to \infty$, we obtain $fy = w = gy$. We now show that the point of coincidence gy is unique. Suppose that for some $z \in M$, $fz = gz$. Then by inequality (2.1),

$$\begin{aligned} d(fy, fz) = d(gy, gz) &\leqslant h\max\bigl\{d(fy, fz), d(fy, gy), d(fz, gz), d(fy, gz), d(fz, gy)\bigr\} \\ &\leqslant hd(fy, fz). \end{aligned}$$

Hence $fz = fy = gy$ as $h \in (0, 1)$. This implies that the point of coincidence $w = gy$ is unique. Since f and g are weakly compatible and $fy = gy$, we obtain $ggy = fgy = gfy$, thereby showing that ggy is a point of coincidence (and gy a coincidence point) of f and g. By the uniqueness of gy as a point of coincidence, we have $ggy = fgy = gy$; thus gy is a common fixed point of f and g. But any common fixed point of f and g is also a point of coincidence of f and g, and is therefore unique. □

Example. Let $X = R$ with usual norm and $M = \{0, 1, 2\}$. Define self-maps f and g of M by $g0 = g1 = g2 = f0 = f1 = 0$ and $f2 = 1$. Then $g0 = 0$ is the unique point of coincidence of f and g, whereas 0 and 1 are coincidence points of f and g. Thus $ga = fa$, then ga must be $0 = w$ and is therefore the unique point of coincidence of f and g, whereas the coincidence point a may be 0 or 1 and is therefore not unique. It is immediate that the hypothesis of Theorem 2.1 is satisfied.

The first part of the proof of Theorem 2.1 establishes the following corollary.

Corollary 2.2. *Let M be a nonempty subset of a metric space (X, d), and let f and g be self-mappings of M such that* $\operatorname{cl} g(M) \subset f(M)$. *Assume that* $\operatorname{cl} g(M)$ *is complete, g and f satisfy* (2.1). *Then f and g have a unique point of coincidence in M.*

The following result extends and improves Theorem 2.2 of [1], Theorem 2.2 of [12], Lemma 2.2 of [14], and Theorem 2.2 of [16].

Theorem 2.3. *Let M be a nonempty q-starshaped subset of a normed space X and f and g be continuous self-maps of M. Suppose that g satisfies condition* (C), *f is affine with $q \in F(f)$ and* $\operatorname{cl} g(M) \subset f(M)$. *If* $\operatorname{cl}(g(M))$ *is compact, the pair $\{f, g\}$ is compatible and satisfies, for all $x, y \in M$,*

$$\begin{aligned} \|gx - gy\| \leqslant \max\bigl\{&\|fx - fy\|, \operatorname{dist}\bigl(fx, [q, gx]\bigr), \operatorname{dist}\bigl(fy, [q, gy]\bigr), \\ &\operatorname{dist}\bigl(fx, [q, gy]\bigr), \operatorname{dist}\bigl(fy, [q, gx]\bigr)\bigr\}, \end{aligned} \tag{2.2}$$

then $M \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Define $g_n : M \to M$ by

$$g_n x = (1 - k_n)q + k_n gx$$

for all $x \in M$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1. Then, each g_n is a self-mapping of M and for each n, $\operatorname{cl} g_n(M) \subset f(M)$ since f is affine, $q \in F(f)$ and $\operatorname{cl} g(M) \subset f(M)$. Also by (2.2),

$$\begin{aligned}\|g_n x - g_n y\| &= k_n \|gx - gy\| \\ &\leqslant k_n \max\big\{\|fx - fy\|, \operatorname{dist}\big(fx, [q, gx]\big), \\ &\qquad \operatorname{dist}\big(fy, [q, gy]\big), \operatorname{dist}\big(fx, [q, gy]\big), \operatorname{dist}\big(fy, [q, gx]\big)\big\} \\ &\leqslant k_n \max\big\{\|fx - fy\|, \|fx - g_n x\|, \|fy - g_n y\|, \\ &\qquad \|fx - g_n y\|, \|fy - g_n x\|\big\},\end{aligned}$$

for each $x, y \in M$ and $0 < k_n < 1$. By Corollary 2.2, for each $n \geqslant 1$, there exists $x_n \in M$ such that $fx_n = g_n x_n$. The compactness of $\operatorname{cl}(g(M))$ implies that there exists a subsequence $\{gx_m\}$ of $\{gx_n\}$ such that $gx_m \to y$ as $m \to \infty$. Since $k_m \to 1$, $fx_m = (1 - k_m)q + k_m gx_m$ converges to y. Since g and f are continuous, $gfx_m \to gy$ and $fgx_m \to fy$ as $m \to \infty$. By the compatibility of f and g, we obtain $fy = gy$. Hence the pair $\{f, g\}$ is nontrivially compatible. Theorem 1.3 guarantees that, $M \cap F(f) \cap F(g) \neq \emptyset$. □

Corollary 2.4. *Let M be a nonempty q-starshaped subset of a normed space X and f and g be continuous self-maps of M. Suppose that g has d.o.d, f is affine with $q \in F(f)$ and $\operatorname{cl} g(M) \subset f(M)$. If $\operatorname{cl}(g(M))$ is compact, the pair $\{f, g\}$ is compatible and satisfies* (2.2), *for all $x, y \in M$, then $M \cap F(f) \cap F(g) \neq \emptyset$.*

Proof. Since g has d.o.d. g has proper orbits. As $\operatorname{cl}(g(M))$ is compact, g has relatively compact orbits. Therefore by Theorem 1.4, g satisfies condition (C). The result now follows by Theorem 2.3. □

Theorem 2.5. *Let M be a nonempty q-starshaped subset of a Banach space X and f and g be self-maps of M. Suppose that g satisfies condition* (C), *f is affine, $f(q) = q$, $\operatorname{wcl} g(M)$ is weakly compact and $\operatorname{wcl} g(M) \subset f(M)$. If the pair $\{f, g\}$ is continuous and compatible, then $M \cap F(f) \cap F(g) \neq \emptyset$ provided one of the following two conditions is satisfied*:

(a) *$f - g$ is demiclosed at 0 and the pair $\{f, g\}$ satisfies* (2.2), *for all $x, y \in M$*;
(b) *X satisfies Opial's condition and g is f-nonexpansive map.*

Proof. Let $\{k_n\}$ and $\{g_n\}$ be defined as in Theorem 2.3. The analysis in Theorem 2.3, and the completeness of $\operatorname{wcl}(g_n(M))$ guarantee that there exists $x_n \in M$ such that $fx_n = g_n x_n$. The weak compactness of $\operatorname{wcl}(g(M))$ implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ weakly as $m \to \infty$. Since $\{x_m\}$ is bounded, $k_m \to 1$, and $\|(f - g)(x_m)\| = \|((1 - k_m)q + k_m gx_m) - gx_m\| \leqslant (1 - k_m)(\|q\| + \|gx_m\|)$ converges to 0.

(a) Since $(f - g)$ is demiclosed at 0 so $(f - g)y = 0$ and hence $fy = gy$. Thus the pair $\{f, g\}$ is nontrivially compatible and the conclusion follows from Theorem 1.3.

(b) If $fy \neq gy$, then

$$\begin{aligned}\liminf_{m\to\infty} \|fx_m - fy\| &< \liminf_{m\to\infty} \|fx_m - gy\| \\ &\leqslant \liminf_{m\to\infty} \|fx_m - gx_m\| + \liminf_{m\to\infty} \|gx_m - gy\| \\ &= \liminf_{m\to\infty} \|gx_m - gy\| \leqslant \liminf_{m\to\infty} \|fx_m - fy\|,\end{aligned}$$

which is a contradiction. Thus $fy = gy$. Thus the pair $\{f, g\}$ is nontrivially compatible and the conclusion follows from Theorem 1.3. $\square$

Example 2.6. Let $X = R$ and $M = [0, 1]$. Let $f(x) = x^k$ for any positive integer k and $g(x) = 1 - (x-1)^2/2$, for $x \in M$. Let $q = 1$. To see that g satisfies condition (C), note that the nth composition $g^n(x) = 1 - (x-1)^{2^n}/2^{p(n)}$, where $p(n) = 1 + 2 + 2^2 + \cdots + 2^{n-1}$. Thus $g^n(x) \to 1$ as $n \to \infty$. In fact, the diameter of $O(g^n(x)) = (x-1)^{2^n}/2^{p(n)}$, which approaches 0 as $n \to \infty$; i.e., g has d.o.d. and hence g satisfies condition (C). Moreover, f and g commute at $x = 1$, the only coincidence point of f and g. Since f and g are continuous and M is compact, we know they are compatible (see [7]). To verify that f and g satisfy the inequality (2.2), note that since $q = 1$, if $x < y$, the maximum member of the five terms in the right member of (2.2) is $\|fx - gy\|$ and it is true that $\|gx - gy\| \leqslant \|fx - gy\|$. It is immediate that M is 1-starshaped and the remainder of the hypothesis of Theorem 2.3 is satisfied. And $x = 1$ is the promised common fixed point of f and g. Note that g is not f-nonexpansive for $k > 1$.

Let $D_M^{R,f}(u) = P_M(u) \cap G_M^{R,f}(u)$, where $G_M^{R,f}(u) = \{x \in M\colon \|fx - u\| \leqslant (2R+1)\operatorname{dist}(u, M)\}$.
The following result extends Theorem 2.5 of [12] and Theorem 2.5 in [16] to R-weakly commuting maps satisfying a more general inequality where f need not be linear.

Theorem 2.7. *Let M be subset of a normed space X and $f, g : X \to X$ be mappings such that $u \in F(f) \cap F(g)$ for some $u \in X$ and $g(\partial M \cap M) \subset M$. Suppose that f and g are continuous on $D_M^{R,f}(u)$, $D_M^{R,f}(u)$ is q-starshaped, closed and $f(D_M^{R,f}(u)) = D_M^{R,f}(u)$. If* $\operatorname{cl}(g(D_M^{R,f}(u)))$ *is compact, g satisfies condition* (C) *and the pair $\{f, g\}$ is R -weakly commuting and satisfies for all $x \in D_M^{R,f}(u) \cup \{u\}$,*

$$\|gx - gy\| \leqslant \begin{cases} \|fx - fu\| & \text{if } y = u, \\ \max\{\|fx - fy\|, \operatorname{dist}(fx, [q, gx]), \operatorname{dist}(fy, [q, gy]), \\ \qquad \operatorname{dist}(fx, [q, gy]), \operatorname{dist}(fy, [q, gx])\} & \text{if } y \in D_M^{R,f}(u), \end{cases} \tag{2.3}$$

then $P_M(u) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Let $x \in D_M^{R,f}(u)$. Then $x \in P_M(u)$ and hence $\|x - u\| = \operatorname{dist}(u, M)$. Note that for any $k \in (0, 1)$,

$$\big\|ku + (1-k)x - u\big\| = (1-k)\|x - u\| < \operatorname{dist}(u, M).$$

It follows that the line segment $\{ku + (1-k)x\colon 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $g(\partial M \cap M) \subset M$, gx must be in M. Also since $fx \in P_M(u)$, $u \in F(f) \cap F(g)$ and f and g satisfy (2.3) we have

$$\|gx - u\| = \|gx - gu\| \leqslant \|fx - fu\| = \|fx - u\| = \operatorname{dist}(u, M).$$

Thus $gx \in P_M(u)$. From the R-weak commutativity of the pair $\{f, g\}$ and (2.3), it follows that,

$$\begin{aligned} \|fgx - u\| &= \|fgx - gfx + gfx - gu\| \leqslant R\|gx - fx\| + \big\|f^2x - fu\big\| \\ &= R\|gx - u + u - fx\| + \big\|f^2x - u\big\| \\ &\leqslant R\big(\|gx - gu\| + \|fx - u\|\big) + \big\|f^2x - u\big\| \end{aligned}$$

$$\leqslant R\big(\|fx-u\|+\|fx-u\|\big)+\|f^2x-u\|$$
$$\leqslant (2R+1)\,\mathrm{dist}(u,M).$$

Thus $gx\in G_M^{R,f}(u)$. Consequently, $g(D_M^{R,f}(u))\subset D_M^{R,f}(u)=f(D_M^{R,f}(u))$. Now by Theorem 2.3 we obtain, $P_M(u)\cap F(f)\cap F(g)\neq\emptyset$. $\square$

Remark 2.8. Let $C_M^f(u)=\{x\in M\colon\ fx\in P_M(u)\}$. Then $f(P_M(u))\subset P_M(u)$ implies $P_M(u)\subset C_M^f(u)\subset G_M^{R,f}(u)$ and hence $D_M^{R,f}(u)=P_M(u)$. Consequently, Theorem 2.7 remains valid when $D_M^{R,f}(u)=P_M(u)$ and the pair $\{f,g\}$ is compatible on $P_M(u)$, which in turn extends the results in [1,4,5,12–17].

As an application of Theorem 2.3, we obtain the following generalization of the corresponding results in [1,14–16,18,19].

Theorem 2.9. *Let f and g be self-mappings of a normed space X with $u\in F(f)\cap F(g)$ and $M\in\Im_0$ such that $g(M_u)\subset f(M)=M$. Suppose that $\|fx-u\|=\|x-u\|$ for all $x\in M$, $\|gx-u\|\leqslant\|fx-u\|$ for all $x\in M_u$, the pair $\{f,g\}$ is continuous on M_u and one of the following two conditions is satisfied*:

(a) $\mathrm{cl}\,f(M_u)$ *is compact,*
(b) $\mathrm{cl}\,g(M_u)$ *is compact.*

Then

(i) $P_M(u)$ *is nonempty, closed and convex,*
(ii) $g(P_M(u))\subset f(P_M(u))=P_M(u)$,
(iii) $P_M(u)\cap F(f)\cap F(g)\neq\emptyset$ *provided g satisfies condition* (C)*, the pair $\{f,g\}$ is compatible on $P_M(u)$ and satisfies for all $q\in F(g)$,*

$$\|gx-gy\|\leqslant\max\big\{\|fx-fy\|,\mathrm{dist}\big(fx,[q,gx]\big),\mathrm{dist}\big(fy,[q,gy]\big),$$
$$\mathrm{dist}\big(fx,[q,gy]\big),\mathrm{dist}\big(fy,[q,gx]\big)\big\},$$

for all $x,y\in P_M(u)$.

Proof. (i) We follow the arguments used in [6,12]. We may assume that $u\notin M$. If $x\in M\setminus M_u$, then $\|x\|>2\|u\|$. Note that

$$\|x-u\|\geqslant\|x\|-\|u\|>\|u\|\geqslant\mathrm{dist}(u,M_u).$$

Thus, $\mathrm{dist}(u,M_u)=\mathrm{dist}(u,M)\leqslant\|u\|$. Also $\|z-u\|=\mathrm{dist}(u,\mathrm{cl}\,f(M_u))$ for some $z\in\mathrm{cl}\,f(M_u)$. This implies that

$$\mathrm{dist}(u,M_u)\leqslant\mathrm{dist}\big(u,\mathrm{cl}\,f(M_u)\big)\leqslant\mathrm{dist}\big(u,f(M_u)\big)\leqslant\|fx-u\|\leqslant\|x-u\|,$$

for all $x\in M_u$. Hence $\|z-u\|=\mathrm{dist}(u,M)$ and so $P_M(u)$ is nonempty. Moreover it is closed and convex. The same conclusion holds whenever $\mathrm{cl}\,g(M_u)$ is compact where we replace f by g and utilize inequalities $\|gx-u\|\leqslant\|fx-u\|$ and $\|fx-u\|=\|x-u\|$ to obtain that $P_M(u)$ is nonempty.

(ii) Let $z \in P_M(u)$. Then $\|fz-u\| = \|fz-fu\| \leqslant \|z-u\| = \mathrm{dist}(u, M)$. This implies that $fz \in P_M(u)$ and so $f(P_M(u)) \subset P_M(u)$. For the converse assume that $y \in P_M(u)$, then $y \in M = f(M)$. Thus there is some $x \in M$ such that $y = fx$. Now

$$\|x-u\| = \|fx-u\| = \|y-u\| = \mathrm{dist}(u, M).$$

This implies that $x \in P_M(u)$ and so $f(P_M(u)) = P_M(u)$.

Let $y \in g(P_M(u))$. Since $g(M_u) \subset f(M)$ and $P_M(u) \subset M_u$, there exist $z \in P_M(u)$ and $x_0 \in M$ such that $y = gz = fx_0$. Further, we have

$$\|fx_0-u\| = \|gz-gu\| \leqslant \|fz-fu\| = \|fz-u\| \leqslant \|z-u\| = \mathrm{dist}(u, M).$$

Thus, $x_0 \in C_M^f(u) = P_M(u)$ and so (ii) holds.

$P_M(u)$ is closed and g-invariant, so by condition (C), $P_M(u) \cap F(g) \neq \emptyset$; it follows that there exists $q \in P_M(u)$ such that $q \in F(g)$. In both of the cases (a) and (b), $\mathrm{cl}\, g(P_M(u))$ is compact. Hence (iii) follows from Theorem 2.3. □

The following result extends and improves [1, Theorem 4.2], [4, Theorem 8], [12, Theorem 2.9], [14, Theorem 2.4], [15, Theorem 2.1] and [16, Theorem 2.9].

Theorem 2.10. *Let f and g be self-mappings of a normed space X with $u \in F(f) \cap F(g)$ and $M \in \Im_0$ such that $g(M_u) \subset f(M) \subset M$. Suppose that $\|fx-u\| \leqslant \|x-u\|$ for all $x \in M_u$, $\|gx-u\| \leqslant \|fx-u\|$ for all $x \in M_u$, the pair $\{f, g\}$ is continuous on M_u and one of the following two conditions is satisfied*:

(a) $\mathrm{cl}\, f(M_u)$ *is compact,*
(b) $\mathrm{cl}\, g(M_u)$ *is compact.*

Then

(i) $P_M(u)$ *is nonempty, closed and convex,*
(ii) $g(P_M(u)) \subset f(P_M(u)) \subset P_M(u)$, *provided that* $\|fx-u\| = \|x-u\|$ *for all* $x \in C_M^f(u)$, *and*
(iii) $P_M(u) \cap F(f) \cap F(g) \neq \emptyset$ *provided that* $\|fx-u\| = \|x-u\|$ *for all* $x \in C_M^f(u)$, f *and* g *satisfy condition* (C), $f(P_M(u))$ *is closed, the pair* $\{f, g\}$ *is compatible on* $P_M(u)$ *and satisfies for all* $q \in F(f)$,

$$\|gx-gy\| \leqslant \max\bigl\{\|fx-fy\|, \mathrm{dist}\bigl(fx, [q, gx]\bigr), \mathrm{dist}\bigl(fy, [q, gy]\bigr), \mathrm{dist}\bigl(fx, [q, gy]\bigr), \mathrm{dist}\bigl(fy, [q, gx]\bigr)\bigr\},$$

for all $x, y \in P_M(u)$.

Proof. (i) and (ii) follows as in Theorem 2.9.

(iii)(a) By (i) $P_M(u)$ is closed and by (ii) $P_M(u)$ is f-invariant, so by condition (C) $P_M(u) \cap F(f) \neq \emptyset$. It follows that there exists $q \in P_M(u)$ such that $q \in F(f)$. By (ii), the compactness of $\mathrm{cl}\, f(M_u)$ implies that $\mathrm{cl}\, g(P_M(u))$ is compact. The conclusion now follows from Theorem 2.3 applied to $P_M(u)$.

(iii)(b) By (i) $P_M(u)$ is closed and by (ii) $P_M(u)$ is f-invariant, so by condition (C) $P_M(u) \cap F(f) \neq \emptyset$, it follows that there exists $q \in P_M(u)$ such that $q \in F(f)$. Theorem 2.3 further guarantees that $P_M(u) \cap F(g) \cap F(f) \neq \emptyset$. □

Remark 2.11. The results similar to [2, Theorems 4.3, 4.4] for the compatible pair satisfying a more general contractive condition can be obtained as an application of Theorem 2.5(a), (b).

The following result extends [1, Theorem 4.1], [4, Theorem 8], [6, Theorem 2.14], [14, Theorems 2.3, 2.4] and [16, Theorem 2.9].

Theorem 2.12. *Let f, g and T be self-mappings of a normed space X with $u \in F(T) \cap F(f) \cap F(g)$ and $M \in \Im_0$ such that $T(M_u) \subset g(M) \subset M = f(M)$. Suppose that $\|gx - u\| \leqslant \|x - u\|$, $\|fx - u\| = \|x - u\|$ and $\|Tx - u\| \leqslant \|gx - fu\|$ for all $x \in M$, $\operatorname{cl} g(M_u)$ is compact, then*

(i) *$P_M(u)$ is nonempty, closed and convex,*
(ii) *$T(P_M(u)) \subset g(P_M(u)) \subset P_M(u) = f(P_M(u))$,*
(iii) *$P_M(u) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$ provided f and g are compatible affine, continuous, and satisfy for some $q \in P_M(u)$*

$$\|gx - gy\| \leqslant \max\bigl\{\|fx - fy\|, \operatorname{dist}\bigl(fx, [q, gx]\bigr), \operatorname{dist}\bigl(fy, [q, gy]\bigr), \operatorname{dist}\bigl(fx, [q, gy]\bigr), \operatorname{dist}\bigl(fy, [q, gx]\bigr)\bigr\},$$

for all $x, y \in P_M(u)$, g satisfies condition (C)*, T is continuous, the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting on $P_M(u)$ and satisfy for all $q \in F(f) \cap F(g)$,*

$$\|Tx - Ty\| \leqslant \max\left\{\|fx - gy\|, \operatorname{dist}\bigl(fx, [q, Tx]\bigr), \operatorname{dist}\bigl(gy, [q, Ty]\bigr), \frac{1}{2}\bigl[\operatorname{dist}\bigl(fx, [q, Ty]\bigr) + \operatorname{dist}\bigl(gy, [q, Tx]\bigr)\bigr]\right\}.$$

Proof. (i) and (ii) follows from [6, Theorem 2.14].

Since by Theorem 2.3, $P_M(u) \cap F(f) \cap F(g) \neq \emptyset$, it follows that there exists $q \in P_M(u)$ such that $q \in F(f) \cap F(g)$. Hence (iii) follows from [6, Theorem 2.2(i)] which holds for C_q-commuting maps (cf. Note added in proof of [2]). □

Compatible maps are different from those of C_q-commuting maps as is obvious from the following example, so our results cannot be implied by those of [2].

Example 2.13. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $f(x) = 2x - 1$ and $g(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $fq = q$ and $C_q(f, g) = [1, \infty)$. Note that f and g are compatible maps and g satisfies condition (C) but f and g are not C_q-commuting maps.

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