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Further summation formulae related to generalized harmonic numbers

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Abstract

By employing the univariate series expansion of classical hypergeometric series formulae, Shen [L.-C. Shen, Remarks on some integrals and series involving the Stirling numbers and $\zeta(n)$, Trans. Amer. Math. Soc. 347 (1995) 1391–1399] and Choi and Srivastava [J. Choi, H.M. Srivastava, Certain classes of infinite series, Monatsh. Math. 127 (1999) 15–25; J. Choi, H.M. Srivastava, Explicit evaluation of Euler and related sums, Ramanujan J. 10 (2005) 51–70] investigated the evaluation of infinite series related to generalized harmonic numbers. More summation formulae have systematically been derived by Chu [W. Chu, Hypergeometric series and the Riemann Zeta function, Acta Arith. 82 (1997) 103–118], who developed fully this approach to the multivariate case. The present paper will explore the hypergeometric series method further and establish numerous summation formulae expressing infinite series related to generalized harmonic numbers in terms of the Riemann Zeta function $\zeta(m)$ with $m = 5, 6, 7$, including several known ones as examples.

Keywords: Riemann Zeta function; Hypergeometric series; Symmetric functions

1. Introduction

Following Bailey [2] and Slater [13], the generalized hypergeometric series reads as

$$ {}_{1+p}F_q\left[\begin{matrix} a_0, & a_1, & \ldots, & a_p \\ & b_1, & \ldots, & b_q \end{matrix} \,\middle|\, z\right] := \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} $$

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where the shifted factorial is defined, for a complex number w, by $(w)_0 := 1$ and $(w)_n := w(w+1)(w+2)\cdots(w+n-1)$ with $n = 1, 2, \ldots$. Most of the nonterminating hypergeometric summation theorems (cf. [2,13]) evaluate nonterminating series as Γ-function-fraction in the following form:

$$\Gamma\left[\begin{matrix} a, & b, & \ldots, & c \\ A, & B, & \ldots, & C \end{matrix}\right] := \frac{\Gamma(a)\Gamma(b)\cdots\Gamma(c)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

For the Γ-function, there hold the Weierstrass product expression (cf. [1])

$$\Gamma(z) = z^{-1}\prod_{n=1}^{\infty}\left\{(1+1/n)^z/(1+z/n)\right\}$$

and the logarithmic differentiation

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty}\frac{z-1}{(n+1)(n+z)}$$

with the Euler constant being given by

$$\gamma = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{1}{k} - \ln n\right\}.$$

We can further derive the following expansions:

$$\Gamma(1-z) = \exp\left\{\sum_{k=1}^{\infty}\frac{\sigma_k}{k}z^k\right\}, \tag{1.1a}$$

$$\Gamma\left(\frac{1}{2}-z\right) = \sqrt{\pi}\exp\left\{\sum_{k=1}^{\infty}\frac{\tau_k}{k}z^k\right\} \tag{1.1b}$$

where the Riemann Zeta sequences $\{\sigma_k, \tau_k\}$ are defined by

$$\begin{aligned} &\sigma_1 = \gamma, \qquad \sigma_m = \zeta(m), \quad m = 2, 3, \ldots; \\ &\tau_1 = \gamma + 2\ln 2, \qquad \tau_m = (2^m - 1)\zeta(m), \quad m = 2, 3, \ldots. \end{aligned}$$

For infinite series related to the Riemann Zeta function, De Doelder [8] established numerous interesting identities through evaluating improper integrals. Some of them are rederived in [3] by means of the Parseval identity on Fourier series. Based on hypergeometric summation formulae, Shen [12] and Choi and Srivastava [4,5] established several interesting infinite series identities.

The hypergeometric method has further been developed by Chu [6] systematically, who discovered numerous infinite series identities involving generalized harmonic numbers. For example, Chu [6] examined hypergeometric summation theorem due to Gauss

$${}_2F_1\left[\begin{matrix} x, & y \\ & 1-z \end{matrix}\,\middle|\, 1\right] = \Gamma\left[\begin{matrix} 1-z, & 1-x-y-z \\ 1-x-z, & 1-y-z \end{matrix}\right] \tag{1.2}$$

as well as Kummer and Dixon. The hypergeometric series on the left side just displayed may be expressed in terms of the partial sums of the Riemann Zeta function through the symmetric functions generated by the following finite products (cf. [10]):

$$\prod_{k=1}^{n}\left(1+\frac{x}{k}\right)=1+xH_n+\frac{x^2}{2}\left(H_n^2-H_n^{(2)}\right)+\frac{x^3}{6}\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)+\cdots, \tag{1.3a}$$

$$\prod_{k=1}^{n}\left(1-\frac{x}{k}\right)^{-1}=1+xH_n+\frac{x^2}{2}\left(H_n^2+H_n^{(2)}\right)+\frac{x^3}{6}\left(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}\right)+\cdots, \tag{1.3b}$$

$$\prod_{k=1}^{n}\left(1+\frac{y}{2k-1}\right)=1+yO_n+\frac{y^2}{2}\left(O_n^2-O_n^{(2)}\right)+\frac{y^3}{6}\left(O_n^3-3O_nO_n^{(2)}+2O_n^{(3)}\right)+\cdots, \tag{1.3c}$$

$$\prod_{k=1}^{n}\left(1-\frac{y}{2k-1}\right)^{-1}=1+yO_n+\frac{y^2}{2}\left(O_n^2+O_n^{(2)}\right)+\frac{y^3}{6}\left(O_n^3+3O_nO_n^{(2)}+2O_n^{(3)}\right)+\cdots, \tag{1.3d}$$

where the generalized harmonic numbers are defined by

$$H_n=\sum_{k=1}^{n}\frac{1}{k},\qquad H_n^{(r)}=\sum_{k=1}^{n}\frac{1}{k^r},\quad r=2,3,\ldots,$$
$$O_n=\sum_{k=1}^{n}\frac{1}{2k-1},\qquad O_n^{(r)}=\sum_{k=1}^{n}\frac{1}{(2k-1)^r},\quad r=2,3,\ldots.$$

Instead, the Γ-function-fraction on the right side may be expanded as multivariate formal power series by (1.1). Then term-by-term comparison of the coefficients from two power series may result in an infinite number of summation formulae.

It should be pointed out that the univariate case was extensively investigated in [4,5,12], while the multivariate approach has been shown more efficient in dealing with infinite sums concerning generalized harmonic numbers. By examining four typical hypergeometric summation formulae due to Gauss, Kummer, Dixon and Dougall, Chu [6] has derived several identities related to the first three values $\zeta(2), \zeta(3)$ and $\zeta(4)$ of Riemann Zeta function. In this paper, we explore this approach further and establish several infinite series identities for $\zeta(5)$, $\zeta(6)$ and $\zeta(7)$. Two typical examples may be displayed as follows:

$$\sum_{n=1}^{\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n^4}=-\frac{\pi^2}{6}\zeta(3)+3\zeta(5),$$
$$\sum_{n=1}^{\infty}\frac{(1+\frac{1}{2}+\cdots+\frac{1}{n})^2}{n^3}=-\frac{\pi^2}{6}\zeta(3)+\frac{7}{2}\zeta(5).$$

Because the hypergeometric method is quite mechanical and systematic, the tedious demonstration for summation formulae will not be presented in detail except for necessity. Instead, we will use a self-explained notation $[x^iy^jz^k]$ to identify the process of extracting the coefficients of monomial $x^iy^jz^k$ from multivariate power series expansions. Throughout the paper, the Euler summation formulae

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}=\frac{\pi}{4},$$
$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}=\frac{\pi^3}{32},$$
$$\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945},\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^5}=\frac{5\pi^5}{1536}$$

will be applied frequently without explanation.

2. The Gauss summation theorem

Recall the Gauss summation formula (1.2). We may reformulate it, by means of (1.1a), as a functional equation between two multivariate infinite series

$$\Gamma\begin{bmatrix}1-z,\ 1-x-y-z\\ 1-x-z,\ 1-y-z\end{bmatrix}$$
$$=1+xy\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{n-1}(1+\frac{x}{i})(1+\frac{y}{i})}{n^2\prod_{j=1}^{n}(1-\frac{z}{j})} \tag{2.1a}$$
$$=\exp\left\{\sum_{k=1}^{\infty}\frac{\sigma_k}{k}\big[z^k+(x+y+z)^k-(x+z)^k-(y+z)^k\big]\right\}$$
$$=\exp\Big\{\sigma_2xy+\sigma_3xy(x+y+2z)+\sigma_4xy\Big(x^2+y^2+\frac{3}{2}xy+3xz+3yz+3z^2\Big)$$
$$+\sigma_5xy\big(x^3+2x^2y+2xy^2+y^3+4x^2z+6xyz+4y^2z+6xz^2+6yz^2+4z^3\big)$$
$$+\cdots\Big\}, \tag{2.1b}$$

where the right-hand side of the first equality may be expanded, via (1.3a)–(1.3b), as a power series. Then term-by-term comparison of the coefficients from two power series result in an infinite number of summation formulae. If we identify by $[x^iy^jz^k]$ the extraction of the coefficients of monomial $x^iy^jz^k$ from both series, the first few terms may be displayed as infinite series identities.

Example 2.1 *(Summation formulae related to $\zeta(3)$ and $\zeta(4)$).*

$$[xyz]\qquad \sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\sigma_3=2\zeta(3)\qquad [6,(1.1\text{b})], \tag{2.2a}$$
$$\big[x^2yz\big]\qquad \sum_{n=1}^{\infty}\frac{H_{n-1}H_n}{n^2}=3\sigma_4=\frac{\pi^4}{30}\qquad [6,(1.2\text{a})], \tag{2.2b}$$
$$\big[x^2y^2\big]\qquad \sum_{n=1}^{\infty}\frac{H_{n-1}^2}{n^2}=\frac{1}{2}\sigma_2^2+\frac{3}{2}\sigma_4=\frac{11\pi^4}{360}\qquad [6,(1.2\text{b})]. \tag{2.2c}$$

In order to exemplify the method, we show only the first formula. It is trivial to see that the coefficient of xyz from the right member of (2.1a) results in $\sum_{n\geqslant 1} H_n/n^2$, thanks to (1.3b). Instead, the coefficient of xyz from the Maclaurin series of (2.1b) is contributed only by $[xyz]\exp\{\sigma_3 xy(x+3y+2z)\} = 2\sigma_3$. Equating both coefficients leads us immediately to (2.2a).

Chu [6] has exhausted the formulae which can be derived from the coefficients of monomials with degree $\leqslant 4$. We shall emphasize on the formulae by examining the coefficients of monomials whose degrees range from 5 to 7.

Example 2.2 *(Summation formulae related to $\zeta(3)$ and $\zeta(5)$).*

$$[x^4y] \qquad \sum_{n=1}^{\infty} \frac{H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}}{n^2} = 6\sigma_5 = 6\zeta(5), \tag{2.3a}$$

$$[x^3y^2] \qquad \sum_{n=1}^{\infty} \frac{H_{n-1}(H_{n-1}^2 - H_{n-1}^{(2)})}{n^2} = 2\sigma_2\sigma_3 + 4\sigma_5 = \frac{\pi^2}{3}\zeta(3) + 4\zeta(5), \tag{2.3b}$$

$$[x^3yz] \qquad \sum_{n=1}^{\infty} \frac{H_n(H_{n-1}^2 - H_{n-1}^{(2)})}{n^2} = 8\sigma_5 = 8\zeta(5), \tag{2.3c}$$

$$[x^2y^2z] \qquad \sum_{n=1}^{\infty} \frac{H_{n-1}^2 H_n}{n^2} = 2\sigma_2\sigma_3 + 6\sigma_5 = \frac{\pi^2}{3}\zeta(3) + 6\zeta(5), \tag{2.3d}$$

$$[x^2yz^2] \qquad \sum_{n=1}^{\infty} \frac{H_{n-1}(H_n^2 + H_n^{(2)})}{n^2} = 12\sigma_5 = 12\zeta(5), \tag{2.3e}$$

$$[xyz^3] \qquad \sum_{n=1}^{\infty} \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{n^2} = 24\sigma_5 = 24\zeta(5). \tag{2.3f}$$

For the formulae displayed in the last example, performing the replacements

$$H_{n-1} = H_n - \frac{1}{n}, \qquad H_{n-1}^{(2)} = H_n^{(2)} - \frac{1}{n^2}, \qquad H_{n-1}^{(3)} = H_n^{(3)} - \frac{1}{n^3}, \tag{2.4}$$

and then combining them properly, we can derive the following identities.

Example 2.3 *(More summation formulae related to $\zeta(3)$ and $\zeta(5)$).*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^4} = -\frac{\pi^2}{6}\zeta(3) + 3\zeta(5) \qquad \text{[9, p. 16] and [11, (4b)]}, \tag{2.5a}$$

$$\sum_{n=1}^{\infty} \left\{ \frac{H_n^3}{n^2} - \frac{2H_n^2}{n^3} \right\} = \frac{\pi^2}{2}\zeta(3) + 3\zeta(5), \tag{2.5b}$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)} - H_n^2}{n^3} = \frac{2\pi^2}{3}\zeta(3) - 8\zeta(5), \tag{2.5c}$$

$$\sum_{n=1}^{\infty} \frac{H_nH_n^{(2)}}{n^2} = \frac{\pi^2}{6}\zeta(3) + \zeta(5) \qquad \text{[5, (4.4)] and [11, (3a)]}, \tag{2.5d}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3+2H_n^{(3)}}{n^2}=-\frac{\pi^2}{2}\zeta(3)+21\zeta(5). \tag{2.5e}$$

3. The Dougall–Dixon summation theorem

3.1. The Dougall–Dixon theorem [13, p. 56]

$$\begin{aligned}
&{}_5F_4\left[\begin{matrix} w,\ 1+w/2, & x, & y, & z \\ w/2, & 1+w-x, & 1+w-y, & 1+w-z \end{matrix}\,\middle|\, 1\right] \\
&\quad=\Gamma\left[\begin{matrix} 1+w-x,\ 1+w-y,\ 1+w-z,\ 1+w-x-y-z \\ 1+w,\ 1+w-x-y,\ 1+w-x-z,\ 1+w-y-z \end{matrix}\right]
\end{aligned}$$

may be expressed, by means of (1.1a), as

$$\begin{aligned}
&\Gamma\left[\begin{matrix} 1+w-x,\ 1+w-y,\ 1+w-z,\ 1+w-x-y-z \\ 1+w,\ 1+w-x-y,\ 1+w-x-z,\ 1+w-y-z \end{matrix}\right] \\
&\quad=1+xyz\sum_{n=1}^{\infty}\frac{w+2n}{n^4}\frac{\prod_{i=1}^{n-1}(1+\frac{w}{i})(1+\frac{x}{i})(1+\frac{y}{i})(1+\frac{z}{i})}{\prod_{j=1}^{n}(1+\frac{w-x}{j})(1+\frac{w-y}{j})(1+\frac{w-z}{j})} \\
&\quad=\exp\left\{\sum_{k=1}^{\infty}(-1)^k\frac{\sigma_k}{k}\left[\begin{matrix}(w-x)^k+(w-y)^k+(w-z)^k+(w-x-y-z)^k \\ -w^k-(w-x-y)^k-(w-x-z)^k-(w-y-z)^k\end{matrix}\right]\right\} \\
&\quad=\exp\bigl\{2\sigma_3xyz+3\sigma_4xyz(x+y+z-2w)+2\sigma_5xyz\bigl(2x^2+2y^2+2z^2+6w^2 \\
&\qquad+3xy+3xz+3yz-6wx-6wy-6wz\bigr)+\cdots\bigr\}.
\end{aligned}$$

Similar to the process illustrated in the last section, this equation leads us to the following identities.

Example 3.1 *(Summation formulae related to $\zeta(5)$).*

$$[x^2y^2z]\qquad \sum_{n=1}^{\infty}\frac{(H_{n-1}+H_n)^2}{n^3}=3\sigma_5=3\zeta(5), \tag{3.1a}$$

$$[x^2yzw]\qquad \sum_{n=1}^{\infty}\left\{\frac{8H_n^2+2H_{n-1}^{(2)}}{n^3}-\frac{H_{n-1}+H_n}{n^4}\right\}=12\sigma_5=12\zeta(5). \tag{3.1b}$$

In view of (2.4), their combinations give other two identities.

Example 3.2 *(More summation formulae related to $\zeta(5)$).*

$$\sum_{n=1}^{\infty}\left\{\frac{H_n^2}{n^3}-\frac{H_n}{n^4}\right\}=\frac{1}{2}\zeta(5), \tag{3.2a}$$

$$\sum_{n=1}^{\infty}\left\{\frac{H_n^{(2)}}{n^3}+\frac{3H_n}{n^4}\right\}=\frac{9}{2}\zeta(5). \tag{3.2b}$$

Combining Example 2.3 with Example 3.2, we obtain further the following results.

Example 3.3 *(More summation formulae related to $\zeta(3)$ and $\zeta(5)$).*

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}=-\frac{\pi^2}{6}\zeta(3)+\frac{7}{2}\zeta(5) \qquad [11, (3c)], \tag{3.3a}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=\frac{\pi^2}{2}\zeta(3)-\frac{9}{2}\zeta(5), \tag{3.3b}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}=\frac{\pi^2}{6}\zeta(3)+10\zeta(5) \qquad [11, (3b)], \tag{3.3c}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}=-\frac{\pi^2}{3}\zeta(3)+\frac{11}{2}\zeta(5). \tag{3.3d}$$

If we consider the coefficients of monomials $x^i y^j z^k$ of degrees 6 and 7, then we shall obtain further summation formulae related to $\zeta(6)$ and $\zeta(7)$, whose proper combinations result in more compact formulae. Some of them are displayed in the following two examples.

Example 3.4 *(Summation formulae related to $\zeta(6)$).*

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5}=\frac{\pi^6}{540}-\frac{1}{2}\zeta^2(3) \qquad [7, (B.7b)], \tag{3.4a}$$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}=\frac{97\pi^6}{22680}-2\zeta^2(3) \qquad [7, (B.8a)] \text{ and } [9, p. 24], \tag{3.4b}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^3}=\frac{31\pi^6}{5040}-\frac{5}{2}\zeta^2(3) \qquad [7, (B.8b)], \tag{3.4c}$$

$$\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}=\frac{979\pi^6}{22680}+3\zeta^2(3) \qquad [5, (4.48)], \tag{3.4d}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=-\frac{\pi^6}{2835}+\zeta^2(3) \qquad [5, (1.15)] \text{ and } [7, (B.9a)], \tag{3.4e}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}=\frac{\pi^6}{1890}+\frac{\zeta(3)^2}{2} \qquad [7, (B.9a)] \text{ and } [9, p. 23], \tag{3.4f}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^2}=\frac{37\pi^6}{11340}-\zeta^2(3). \tag{3.4g}$$

Example 3.5 *(Summation formulae related to $\zeta(7)$).*

$$\sum_{n=1}^{\infty}\frac{H_n}{n^6}=-\frac{\pi^4}{90}\zeta(3)-\frac{\pi^2}{6}\zeta(5)+4\zeta(7), \tag{3.5a}$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = -\frac{\pi^4}{36}\zeta(3) - \frac{\pi^2}{6}\zeta(5) + 6\zeta(7) \qquad [9, \text{p. } 16], \tag{3.5b}$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} = \frac{\pi^4}{45}\zeta(3) + \frac{5\pi^2}{6}\zeta(5) - 10\zeta(7) \qquad [5, (1.16)], \tag{3.5c}$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} = \frac{\pi^4}{90}\zeta(3) + \frac{5\pi^2}{3}\zeta(5) - 17\zeta(7), \tag{3.5d}$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(5)}}{n^2} = -\frac{\pi^4}{45}\zeta(3) - \frac{2\pi^2}{3}\zeta(5) + 11\zeta(7). \tag{3.5e}$$

3.2. With replacement of z by $z + 1/2$ in the Dougall–Dixon theorem, the resulting identity reads as

$$\begin{aligned} &{}_5F_4\left[\begin{matrix} w, & 1+\frac{w}{2}, & x, & y, & \frac{1}{2}+z \\ & \frac{w}{2}, & 1+w-x, & 1+w-y, & \frac{1}{2}+w-z \end{matrix}\,\middle|\, 1\right] \\ &\quad = \Gamma\left[\begin{matrix} 1+w-x, \ 1+w-y, \ \frac{1}{2}+w-z, \ \frac{1}{2}+w-x-y-z \\ 1+w, \ 1+w-x-y, \ \frac{1}{2}+w-x-z, \ \frac{1}{2}+w-y-z \end{matrix}\right] \end{aligned}$$

which may be restated as

$$\begin{aligned} &\Gamma\left[\begin{matrix} 1+w-x, \ 1+w-y, \ \frac{1}{2}+w-z, \ \frac{1}{2}+w-x-y-z \\ 1+w, \ 1+w-x-y, \ \frac{1}{2}+w-x-z, \ \frac{1}{2}+w-y-z \end{matrix}\right] \\ &\quad = 1 + xy\sum_{n=1}^{\infty} \frac{w+2n}{n^3} \frac{\prod_{i=1}^{n-1}(1+\frac{w}{i})(1+\frac{x}{i})(1+\frac{y}{i})}{\prod_{j=1}^{n}(1+\frac{w-x}{j})(1+\frac{w-y}{j})} \prod_{k=1}^{n} \frac{1+2z/(2k-1)}{1+2(w-z)/(2k-1)} \\ &\quad = \exp\left\{ \sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k}\big[(w-x)^k + (w-y)^k - w^k - (w-x-y)^k\big] \right. \\ &\qquad \left. + (-1)^k \frac{\tau_k}{k}\big[(w-z)^k + (w-x-y-z)^k - (w-x-z)^k - (w-y-z)^k\big] \right\} \\ &\quad = \exp\big\{2\sigma_2 xy - 2\sigma_3 xy(6w-3x-3y-7z) + \sigma_4 xy\big(42w^2 - 42wx + 14x^2 \\ &\qquad - 42wy + 21xy + 14y^2 - 90wz + 45xz + 45yz + 45z^2\big) - 2\sigma_5 xy\big(60w^3 \\ &\qquad - 90w^2x + 60wx^2 - 15x^3 - 90w^2y + 90wxy - 30x^2y + 60wy^2 - 30xy^2 \\ &\qquad - 15y^3 - 186w^2z + 186wxz - 62x^2z + 186wyz - 93xyz - 62y^2z \\ &\qquad + 186wz^2 - 93xz^2 - 93yz^2 - 62z^3\big) + \cdots\big\}. \end{aligned}$$

The summation formulae derived from the coefficients of monomials with degree 5 are as follows.

Example 3.6 *(Summation formulae related to $\zeta(3)$ and $\zeta(5)$).*

$$[x^3yz] \quad \sum_{n=1}^{\infty} \frac{\{(H_{n-1}+H_n)^2 - H_{n-1}^{(2)} + H_n^{(2)}\}O_n}{n^2} = 31\sigma_5 = 31\zeta(5), \tag{3.6a}$$

$$[x^2y^2z] \quad \sum_{n=1}^{\infty} \frac{(H_{n-1}+H_n)^2 O_n}{n^2} = \frac{7}{2}\sigma_2\sigma_3 + \frac{93}{4}\sigma_5 = \frac{7\pi^2}{12}\zeta(3) + \frac{93}{4}\zeta(5), \tag{3.6b}$$

$$[x^2yz^2] \quad \sum_{n=1}^{\infty} \frac{(H_{n-1}+H_n)O_n^2}{n^2} = \frac{93}{8}\sigma_5 = \frac{93}{8}\zeta(5), \tag{3.6c}$$

$$[xyz^3] \quad \sum_{n=1}^{\infty} \frac{2O_n^3 + O_n^{(3)}}{n^2} = \frac{93}{8}\sigma_5 = \frac{93}{8}\zeta(5). \tag{3.6d}$$

Example 3.7 *(More summation formulae related to $\zeta(3)$ and $\zeta(5)$).*

$$\sum_{n=1}^{\infty} \frac{O_n}{n^4} = -\frac{7\pi^2}{12}\zeta(3) + \frac{31}{4}\zeta(5), \tag{3.7a}$$

$$\sum_{n=1}^{\infty} \frac{(H_{n-1}^2 + H_n^2)O_n}{n^2} = \frac{31}{2}\zeta(5), \tag{3.7b}$$

$$\sum_{n=1}^{\infty} \frac{H_{n-1}H_nO_n}{n^2} = \frac{7\pi^2}{24}\zeta(3) + \frac{31}{8}\zeta(5). \tag{3.7c}$$

Remark. According to (2.4), we observe that (3.7a) can be derived from the difference between (3.6a) and (3.6b), the identity (3.7b) from (3.6a) and (3.7c) from the difference between (3.6b) and (3.7b).

3.3. Replacing w, x, y, z respectively by $1+w, x+1/2, y+1/2, z+1/2$ in the Dougall–Dixon theorem, we can express the resulting equation as

$$\begin{aligned}
&{}_5F_4\left[\begin{matrix} 1+w, & \frac{3+w}{2}, & x+\frac{1}{2}, & y+\frac{1}{2}, & z+\frac{1}{2} \\ & \frac{1+w}{2}, & \frac{3}{2}+w-x, & \frac{3}{2}+w-y, & \frac{3}{2}+w-z \end{matrix} \,\middle|\, 1\right] \\
&= \Gamma\left[\begin{matrix} \frac{3}{2}+w-x, & \frac{3}{2}+w-y, & \frac{3}{2}+w-z, & \frac{1}{2}+w-x-y-z \\ 2+w, & 1+w-x-y, & 1+w-x-z, & 1+w-y-z \end{matrix}\right].
\end{aligned}$$

The last relation may in turn be reformulated as

$$\begin{aligned}
&\Gamma\left[\begin{matrix} \frac{1}{2}+w-x, & \frac{1}{2}+w-y, & \frac{1}{2}+w-z, & \frac{1}{2}+w-x-y-z \\ 1+w, & 1+w-x-y, & 1+w-x-z, & 1+w-y-z \end{matrix}\right] \\
&= \pi^2 \exp\left\{ \sum_{k=1}^{\infty} (-1)^k \frac{\tau_k}{k}\left[(w-x)^k + (w-y)^k + (w-z)^k + (w-x-y-z)^k\right] \right\} \\
&\quad \times \exp\left\{ -\sum_{k=1}^{\infty} (-1)^k \frac{\sigma_k}{k}\left[w^k + (w-x-y)^k + (w-x-z)^k + (w-y-z)^k\right] \right\}
\end{aligned}$$

whose expansion yields the following equality:

$$\sum_{n=1}^{\infty} \frac{w+2n-1}{(n-\frac{1}{2})^3} \frac{\prod_{i=1}^{n-1}(1+\frac{w}{i})(1+\frac{2x}{2i-1})(1+\frac{2y}{2i-1})(1+\frac{2z}{2i-1})}{\prod_{j=1}^{n}(1+\frac{2w-2x}{2j-1})(1+\frac{2w-2y}{2j-1})(1+\frac{2w-2z}{2j-1})}$$
$$= \exp\bigl\{4(x+y+z-2w)\ln 2 + 2\sigma_2\bigl(2w^2-2wx+x^2-2wy+xy+y^2$$
$$-2wz+xz+yz+z^2\bigr) - 2\sigma_3\bigl(4w^3-6w^2x+6wx^2-2x^3-6w^2y$$
$$+6wxy-3x^2y+6wy^2-3xy^2-2y^3-6w^2z+6wxz-3x^2z+6wyz$$
$$-7xyz-3y^2z+6wz^2-3xz^2-3yz^2-2z^3\bigr) + \sigma_4\bigl(14w^4-28w^3x$$
$$+42w^2x^2-28wx^3+7x^4-28w^3y+42w^2xy-42wx^2y+14x^3y$$
$$+42w^2y^2-42wxy^2+21x^2y^2-28wy^3+14xy^3+7y^4-28w^3z$$
$$+42w^2xz-42wx^2z+14x^3z+42w^2yz-90wxyz+45x^2yz-42wy^2z$$
$$+45xy^2z+14y^3z+42w^2z^2-42wxz^2+21x^2z^2-42wyz^2+45xyz^2$$
$$+21y^2z^2-28wz^3+14xz^3+14yz^3+7z^4\bigr)+\cdots\bigr\}.$$

The first few terms of its power series expansion yield the following summation formulae.

Example 3.8 *(Summation formulae related to $\zeta(3)$ and $\zeta(4)$).*

$$\sum_{n=1}^{\infty} \frac{(O_{n-1}+O_n)^3}{(2n-1)^2} = \frac{\pi^4}{16}\ln 2 + \pi^2 \ln^3 2 + \frac{7\pi^2}{32}\zeta(3), \tag{3.8a}$$

$$\sum_{n=1}^{\infty} \frac{O_{n-1}+O_n}{(2n-1)^4} = \frac{\pi^4}{48}\ln 2 - \frac{\pi^2}{32}\zeta(3), \tag{3.8b}$$

$$\sum_{n=1}^{\infty} \frac{O_{n-1}^{(3)}+O_n^{(3)}}{(2n-1)^2} = \frac{\pi^2}{8}\zeta(3). \tag{3.8c}$$

Sketch of proof. The coefficients of xyz from both formal power series expansions lead us directly to the first formula in Example 3.8. By combining (3.8a) with the following two identities:

$$[x^2y] \quad \sum_{n=1}^{\infty} \frac{(O_{n-1}+O_n)\{(O_{n-1}+O_n)^2 - O_{n-1}^{(2)} + O_n^{(2)}\}}{(2n-1)^2}$$
$$= \frac{\pi^4}{12}\ln 2 + \pi^2 \ln^3 2 + \frac{3\pi^2}{16}\zeta(3), \tag{3.9a}$$

$$[x^3] \quad \sum_{n=1}^{\infty} \frac{(O_{n-1}+O_n)\{(O_{n-1}+O_n)^2 - 3(O_{n-1}^{(2)} - O_n^{(2)})\} + 2(O_{n-1}^{(3)} + O_n^{(3)})}{(2n-1)^2}$$
$$= \frac{\pi^4}{8}\ln 2 + \pi^2 \ln^3 2 + \frac{3\pi^2}{8}\zeta(3), \tag{3.9b}$$

we can easily derive (3.8b) and (3.8c). □

Example 3.9 *(Summation formulae related to $\zeta(3)$ and $\zeta(4)$).*

$$\sum_{n=1}^{\infty}\frac{(O_{n-1}+O_n)^2}{(2n-1)^4}=\frac{\pi^6}{5760}+\frac{\pi^4}{24}\ln^2 2-\frac{\pi^2}{8}\ln 2\zeta(3), \tag{3.10a}$$

$$\sum_{n=1}^{\infty}\frac{(O_{n-1}+O_n)^4}{(2n-1)^2}=\frac{\pi^6}{90}+\frac{\pi^4}{4}\ln^2 2+2\pi^2\ln^4 2+\frac{7\pi^2}{4}\ln 2\zeta(3), \tag{3.10b}$$

$$\sum_{n=1}^{\infty}\frac{(O_{n-1}^2+O_n^2)^2}{(2n-1)^2}=\frac{\pi^6}{320}+\frac{\pi^4}{12}\ln^2 2+\frac{\pi^2}{2}\ln^4 2+\frac{3\pi^2}{8}\ln 2\zeta(3). \tag{3.10c}$$

Sketch of proof. The comparison of the coefficients gives the following:

$$[x^2yz]\quad \sum_{n=1}^{\infty}\frac{(O_{n-1}+O_n)^2\{(O_{n-1}+O_n)^2-O_{n-1}^{(2)}+O_n^{(2)}\}}{(2n-1)^2}$$
$$=\frac{13\pi^6}{1152}+\frac{7\pi^4}{24}\ln^2 2+2\pi^2\ln^4 2+\frac{13\pi^2}{8}\ln 2\zeta(3), \tag{3.11a}$$

$$[xyzw]\quad \sum_{n=1}^{\infty}\frac{\{(O_{n-1}+O_n)^2-O_{n-1}^{(2)}+O_n^{(2)}\}^2}{(2n-1)^2}$$
$$=\frac{\pi^6}{80}+\frac{\pi^4}{3}\ln^2 2+2\pi^2\ln^4 2+\frac{3\pi^2}{2}\ln 2\zeta(3). \tag{3.11b}$$

The linear combinations of the above two equations result in the formulae (3.10a) and (3.10b) on account of two relations:

$$O_n^{(2)}-O_{n-1}^{(2)}=\frac{1}{(2n-1)^2}\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}=\frac{\pi^6}{960}.$$

The identity (3.10c) is derived by reformulating the formula (3.11b). □

4. The Dixon–Kummer summation theorem

4.1. When $z\to\infty$, the Dougall–Dixon theorem in last section reduces to the Dixon–Kummer theorem [13]

$${}_4F_3\left[\begin{matrix}w,\ 1+w/2, & x, & y\\ w/2, & 1+w-x, & 1+w-y\end{matrix}\,\middle|\,-1\right]=\Gamma\left[\begin{matrix}1+w-x,\ 1+w-y\\ 1+w,\ 1+w-x-y\end{matrix}\right]$$

which may be expressed, through (1.1a), as

$$\Gamma\left[\begin{matrix}1+w-x,\ 1+w-y\\ 1+w,\ 1+w-x-y\end{matrix}\right]$$
$$=1+xy\sum_{n=1}^{\infty}(-1)^n\frac{w+2n}{n^3}\frac{\prod_{i=1}^{n-1}(1+\frac{w}{i})(1+\frac{x}{i})(1+\frac{y}{i})}{\prod_{j=1}^{n}(1+\frac{w-x}{j})(1+\frac{w-y}{j})}$$
$$=\exp\left\{\sum_{k=1}^{\infty}(-1)^k\frac{\sigma_k}{k}\left[(w-x)^k+(w-y)^k-w^k-(w-x-y)^k\right]\right\}$$

$$= \exp\left\{-\sigma_2 xy + \sigma_3 xy(2w - x - y) - \sigma_4 xy\left(3w^2 + x^2 + y^2 - 3wx - 3wy + \frac{3}{2}xy\right) + \sigma_5 xy(2w - x - y)\left(2w^2 + x^2 + y^2 - 2wx - 2wy + xy\right) + \cdots\right\}.$$

Its power series expansion via (1.3a)–(1.3b) leads us to infinite series identities.

Example 4.1 *(Alternating series related to the generalized harmonic numbers).*

$$[x^4 y] \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{-\frac{1}{2n^5} + \frac{H_{n-1}^3 + H_n^3 + H_n^{(3)}}{n^2}\right\} = \frac{3}{4}\sigma_5 = \frac{3}{4}\zeta(5), \tag{4.1a}$$

$$[x^3 y^2] \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{-\frac{H_{n-1} + H_n}{n^4} + \frac{2H_{n-1}^3 + 2H_n^3}{n^2}\right\} = -\frac{\pi^2}{12}\zeta(3) + \zeta(5), \tag{4.1b}$$

$$[wx^3 y] \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{1}{n^5} + \frac{2(H_{n-1} + H_n)(H_n^2 + H_n^{(2)})}{n^2} + \frac{2H_n^{(3)}}{n^2}\right\} = 4\zeta(5), \tag{4.1c}$$

$$[wx^2 y^2] \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{-\frac{H_{n-1} + H_n}{n^4} + \frac{4(H_{n-1} + H_n)(H_n^2 + H_n^{(2)})}{n^2}\right\} = -\frac{\pi^2}{3}\zeta(3) + 6\zeta(5), \tag{4.1d}$$

$$[w^2 x^2 y] \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{H_{n-1}}{n^4} - \frac{(H_{n-1} + H_n)H_n^2}{n^2} + \frac{4H_n(H_n^2 + H_n^{(2)})}{n^2} + \frac{2H_n^{(3)}}{n^2}\right\} = 6\zeta(5), \tag{4.1e}$$

$$[w^3 xy] \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{6}{n^5} + \frac{6H_n}{n^4} + \frac{3(H_n^2 + H_n^{(2)})}{n^3} + \frac{2(H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)})}{n^2}\right\} = 24\zeta(5). \tag{4.1f}$$

Simplifying these identities by means of (2.4), we further obtain the following summation formulae.

Example 4.2 *(Alternating series related to the generalized harmonic numbers).*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5} = \frac{15}{16}\zeta(5), \tag{4.2a}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} = -\frac{\pi^2}{12}\zeta(3) + \frac{59}{32}\zeta(5), \tag{4.2b}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3} = \frac{5\pi^2}{48}\zeta(3) - \frac{11}{32}\zeta(5), \tag{4.2c}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2} = \frac{\pi^2}{8}\zeta(3) - \frac{21}{32}\zeta(5), \tag{4.2d}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{2H_n^3}{n^2}-\frac{3H_n^2}{n^3}\right\}=\frac{\pi^2}{8}\zeta(3)-\frac{87}{32}\zeta(5), \tag{4.2e}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{H_nH_n^{(2)}}{n^2}+\frac{H_n^2}{n^3}\right\}=-\frac{7\pi^2}{96}\zeta(3)+\frac{73}{32}\zeta(5). \tag{4.2f}$$

4.2. With replacement y by $y+1/2$ in the Dixon–Kummer theorem, the resulting equation reads as

$${}_4F_3\left[\begin{matrix}w,\ 1+\frac{w}{2}, & x, & y+\frac{1}{2}\\ \frac{w}{2}, & 1+w-x, & \frac{1}{2}+w-y\end{matrix}\,\Big|\,-1\right]=\Gamma\left[\begin{matrix}1+w-x,\ \frac{1}{2}+w-y\\ 1+w,\ \frac{1}{2}+w-x-y\end{matrix}\right]$$

which may be restated as follows:

$$\begin{aligned}
&\Gamma\left[\begin{matrix}1+w-x,\ \frac{1}{2}+w-y\\ 1+w,\ \frac{1}{2}+w-x-y\end{matrix}\right]\\
&=1+x\sum_{n=1}^{\infty}(-1)^n\frac{w+2n}{n^2}\prod_{i=1}^{n-1}\left(1+\frac{w}{i}\right)\left(1+\frac{x}{i}\right)\prod_{j=1}^{n}\frac{1+2y/(2j-1)}{(1+\frac{w-x}{j})(1+\frac{2w-2y}{2j-1})}\\
&=\exp\left\{\sum_{k=1}^{\infty}(-1)^k\left[\frac{\sigma_k}{k}\left\{(w-x)^k-w^k\right\}+\frac{\tau_k}{k}\left\{(w-y)^k-(w-x-y)^k\right\}\right]\right\}\\
&=\exp\Big\{-2x\ln 2+\sigma_2x(2w-x-3y)-\sigma_3x\big(6w^2-6wx+2x^2-14wy+7xy+7y^2\big)\\
&\quad+\frac{1}{2}\sigma_4x\big(28w^3-42w^2x+28wx^2-7x^3-90w^2y+90wxy\\
&\quad-30x^2y+90wy^2-45xy^2-30y^3\big)+\cdots\Big\}.
\end{aligned}$$

Example 4.3 *(Alternating series related to the generalized harmonic numbers).*

$$\begin{aligned}
[x^4]\qquad &\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2(H_{n-1}^3+H_n^3)+H_{n-1}^{(3)}+H_n^{(3)}}{n}\\
&=-\ln^4 2+3\sigma_2\ln^2 2-\frac{3}{4}\sigma_2^2-6\sigma_3\ln 2+\frac{21}{4}\sigma_4\\
&=\frac{3\pi^4}{80}+\frac{\pi^2}{2}\ln^2 2-\ln^4 2-6\ln 2\,\zeta(3),
\end{aligned} \tag{4.3a}$$

$$\begin{aligned}
[x^3y]\qquad &\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_{n-1}^2+H_n^2)O_n}{n}=\frac{3}{4}\sigma_2\ln^2 2-\frac{3}{8}\sigma_2^2-\frac{7}{4}\sigma_3\ln 2+\frac{15}{8}\sigma_4\\
&=\frac{\pi^4}{96}+\frac{\pi^2}{8}\ln^2 2-\frac{7}{4}\ln 2\,\zeta(3),
\end{aligned} \tag{4.3b}$$

$$[x^2y^2] \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_{n-1}+H_n)O_n^2}{n} = -\frac{9}{32}\sigma_2^2 - \frac{7}{8}\sigma_3\ln 2 + \frac{45}{32}\sigma_4$$

$$= \frac{\pi^4}{128} - \frac{7}{8}\ln 2\zeta(3), \tag{4.3c}$$

$$[xy^3] \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2O_n^3 + O_n^{(3)}}{n} = \frac{45}{32}\sigma_4 = \frac{\pi^4}{64}, \tag{4.3d}$$

$$[wxy^2] \quad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{O_n^2}{n^2} + \frac{4O_n^3 + 4O_nO_n^{(2)} + 2O_n^{(3)}}{n}\right\} = \frac{45}{8}\sigma_4 = \frac{\pi^4}{16}, \tag{4.3e}$$

$$[w^2xy] \quad \sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{O_n}{n^3} + \frac{2O_n^2 + O_n^{(2)}}{n^2} + \frac{4O_n^3 + 8O_nO_n^{(2)} + 4O_n^{(3)}}{n}\right\}$$

$$= \frac{45}{4}\sigma_4 = \frac{\pi^4}{8}. \tag{4.3f}$$

Example 4.4 *(Alternating series related to the generalized harmonic numbers).*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{O_n^2}{n^2} + \frac{4O_nO_n^{(2)}}{n}\right\} = \frac{45}{16}\sigma_4 = \frac{\pi^4}{32}, \tag{4.4a}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left\{\frac{O_n}{n^3} + \frac{O_n^{(2)}}{n^2} + \frac{2O_n^{(3)}}{n}\right\} = \frac{45}{16}\sigma_4 = \frac{\pi^4}{32}. \tag{4.4b}$$

Remark. By means of linear combination (4.3e) – (4.3d) × 2, we find (4.4a). Similarly, we get (4.4b) from (4.3f) – (4.4a) × 2 – (4.3d) × 2.

4.3. Replacing w, x, y respectively by $1+w, x+1/2, y+1/2$ in the Dixon–Kummer theorem, we can state the result as

$${}_4F_3\left[\begin{matrix}1+w, & \frac{3+w}{2}, & x+\frac{1}{2}, & y+\frac{1}{2} \\ & \frac{1+w}{2}, & \frac{3}{2}+w-x, & \frac{3}{2}+w-y\end{matrix}\,\middle|\, -1\right] = \Gamma\left[\begin{matrix}\frac{3}{2}+w-x, & \frac{3}{2}+w-y \\ 2+w, & 1+w-x-y\end{matrix}\right]$$

which may be reformulated, through (1.1a) and (1.1b), as

$$\Gamma\left[\begin{matrix}\frac{1}{2}+w-x, & \frac{1}{2}+w-y \\ 1+w, & 1+w-x-y\end{matrix}\right]$$
$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{w+2n-1}{(n-\frac{1}{2})^2}\frac{\prod_{i=1}^{n-1}(1+\frac{w}{i})(1+\frac{2x}{2i-1})(1+\frac{2y}{2i-1})}{\prod_{j=1}^{n}(1+\frac{2w-2x}{2j-1})(1+\frac{2w-2y}{2j-1})}$$
$$= \pi\exp\left\{\sum_{k=1}^{\infty}(-1)^k\left[\frac{\tau_k}{k}\left\{(w-x)^k+(w-y)^k\right\} - \frac{\sigma_k}{k}\left\{w^k+(w-x-y)^k\right\}\right]\right\}$$
$$= \pi\exp\left\{2(x+y-2w)\ln 2 + \sigma_2\left(2w^2+x^2+y^2-xy-2wx-2wy\right)\right.$$
$$\left. + \sigma_3(x+y-2w)\left(2x^2+2y^2+2w^2-3xy-2wx-2wy\right) + \cdots\right\}.$$

Its power series expansion via (1.3a)–(1.3d) gives rise to the following two infinite series identities.

Example 4.5 *(Alternating series related to the generalized harmonic numbers).*

$$[x^3] \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2O_{n-1}^3+2O_n^3+O_{n-1}^{(3)}+O_n^{(3)}}{2n-1}$$
$$=\frac{\pi}{8}\ln^3 2+\frac{3\pi}{16}\sigma_2\ln 2+\frac{3\pi}{16}\sigma_3=\frac{\pi^3}{32}\ln 2+\frac{\pi}{8}\ln^3 2+\frac{3\pi}{16}\zeta(3), \tag{4.5a}$$
$$[x^2y] \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(O_{n-1}+O_n)(O_{n-1}^2+O_n^2)}{2n-1}$$
$$=\frac{\pi}{8}\ln^3 2-\frac{\pi}{32}\sigma_3=\frac{\pi}{8}\ln^3 2-\frac{\pi}{32}\zeta(3). \tag{4.5b}$$

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