

# Extensions of the Nemytsky operator: Distributional solutions of nonlinear problems

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Received 19 January 2007

Available online 21 May 2007

Submitted by J. Mawhin

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## Abstract

The aim of this article is to develop a methodology for working with solutions of nonlinear differential problems that are signed measures.

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*Keywords:* Nonlinear differential problems; Signed measures; Nemytsky operator

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## 1. Introduction

Measures are distributional solutions of linear partial differential equations, and linear partial differential operators can be defined on distributions using duality and the formal adjoint operator.

As a very simple example, we can consider the first order boundary value problem

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \\ u(x, 0) = f(x), \end{cases} \quad (1)$$

where  $u = u(x, t)$ . Then,  $U(x, t) = \delta(x - t) = \delta_t$  is a fundamental solution to this problem. That is to say,  $U$  is a solution to the boundary value problem

$$\begin{cases} \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0, \\ U(x, 0) = \delta(x). \end{cases}$$

Therefore,  $u(x, t) = (\delta_t * f)(x) = f(x - t)$  is a solution to (1) under very general conditions on  $f$ .

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If we consider instead the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + g(u),$$

it is not known how to define the nonlinear term  $g(u)$ , when  $u$  is a general distribution.

The aim of this article is to develop a functional calculus for signed measures that allow us to give a meaning to the nonlinear term  $g(u)$  under fairly general conditions on  $g$ . Our main tool is the so-called Nemytsky operator  $N_g$  [9], which, as we explain later, is an extension of the composition operator  $u \rightarrow g(u)$ . There is an extensive literature on these operators, and their applications to nonlinear problems involving both partial differential equations and ordinary differential equations (see [1–7] and references therein for a survey on these topics).

The organization of this article is as follows: In Section 2 we present an account of the definitions and properties related to signed measures. We also define the Nemytsky operator and we state several basic properties. In Section 3 we construct an extension of the Nemytsky operator to a class of signed measures, assuming that the function  $g$  is nonnegative. In particular, we are able to define functions of infinite measures, such as the Lebesgue measure. In Section 4 we prove an explicit representation of the Nemytsky operator for finite signed measures, when  $g$  is, roughly speaking, a piecewise linear function with variable coefficients. In Section 5 we use our results to solve an initial value problem with a signed finite measure as initial condition for a class of nonlinear evolution equations.

## 2. Preliminary definitions and results

We begin this section with a review of definitions and properties related to the notion of signed measure. For more details, we refer to [8] and references therein.

Let  $\Sigma$  be a  $\sigma$ -algebra of subsets of a set  $X$  and let  $\mathbb{R}^*$  be the extended real line that consists of the real numbers and the symbols  $-\infty$  and  $\infty$ , with the usual operations. We adopt the convention  $0.(\pm\infty) = (\pm\infty).0 = 0$ . We leave  $\infty + (-\infty)$  and  $(-\infty) + \infty$  undefined.

We consider set functions  $\mu : \Sigma \rightarrow \mathbb{R}^*$  that take at most one of the two values  $\infty$  and  $-\infty$ .

**Definition 1.** The set function  $\mu : \Sigma \rightarrow \mathbb{R}^*$  is called a signed measure if

1.  $\mu(\emptyset) = 0$ .
2.  $\mu(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} \mu(A_i)$  whenever  $\{A_i\}_{i \geq 1} \subseteq \Sigma$  are pairwise disjoint.

As a consequence of 2., the series  $\sum_{i \geq 1} \mu(A_i)$  converges absolutely in  $\mathbb{R}^*$ . We also observe that if  $A$  and  $B \in \Sigma$  with  $B \subseteq A$  and  $\mu(B)$  is finite,

$$\mu(A - B) = \mu(A) - \mu(B).$$

**Definition 2.** A measure  $\mu$  is a signed measure that only takes nonnegative values in  $\mathbb{R}^*$ .

**Definition 3.** Given a measure space  $(X, \Sigma, \mu)$  and a  $\Sigma$ -measurable function  $f : X \rightarrow \mathbb{R}^*$ , we say that  $f$  has a  $\mu$ -integral if at least one of  $\int_X f^+ d\mu$  and  $\int_X f^- d\mu$  is finite. If this is the case, we write

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

If a function  $f$  has a  $\mu$ -integral, then the integral  $\int_E f d\mu$  exists in  $\mathbb{R}^*$  for every  $E \in \Sigma$ .

**Definition 4.** Given a signed measure  $\lambda : \Sigma \rightarrow \mathbb{R}^*$ , the set functions  $\lambda^+, \lambda^- : \Sigma \rightarrow \mathbb{R}^*$  are defined as

$$\begin{aligned} \lambda^+(E) &= \sup\{\lambda(A) : A \subseteq E, A \in \Sigma\}, \\ \lambda^-(E) &= -\inf\{\lambda(A) : A \subseteq E, A \in \Sigma\}. \end{aligned}$$

Since  $\lambda(\emptyset) = 0$ , this implies that  $\lambda^+$  and  $\lambda^-$  are nonnegative. They are in fact measures and they are called, respectively, the positive part or upper variation of  $\lambda$  and the negative part or lower variation of  $\lambda$ . We recall the following properties:

1.  $\lambda^+ \geq \lambda$  and  $\lambda^- \geq -\lambda$ .
2.  $\lambda^+$  and  $\lambda^-$  are increasing, and  $\lambda^- = (-\lambda)^+$ .
3. Given  $E \in \Sigma$ , if one of the numbers  $\lambda^+(E)$  and  $\lambda^-(E)$  is finite, then  $\lambda(E) = \lambda^+(E) - \lambda^-(E)$ .

As a consequence, if  $\lambda : \Sigma \rightarrow \mathbb{R}$ ,

$$\lambda(E) = \lambda^+(E) - \lambda^-(E). \quad (2)$$

4. Let  $\lambda : \Sigma \rightarrow \mathbb{R}^*$  be a signed measure and let  $E \in \Sigma$ . Then,

$$\begin{aligned} \lambda^+(E) = \infty & \text{ implies } \lambda(E) = \infty, \\ \lambda^-(E) = \infty & \text{ implies } \lambda(E) = -\infty. \end{aligned} \quad (3)$$

Therefore, if  $\lambda(E)$  is finite, both  $\lambda^+(E)$  and  $\lambda^-(E)$  are finite. We can conclude that every signed measure  $\lambda : \Sigma \rightarrow \mathbb{R}$  is bounded.

As a consequence of (2) and (3), we obtain

**Theorem 5 (Jordan Decomposition).** Given a signed measure  $\lambda : \Sigma \rightarrow \mathbb{R}^*$ ,

$$\lambda = \lambda^+ - \lambda^-.$$

**Definition 6.** If  $\lambda : \Sigma \rightarrow \mathbb{R}^*$  is a signed measure, the total variation or variation of  $\lambda$  is defined as

$$|\lambda| = \lambda^+ + \lambda^-.$$

The properties of  $\lambda^+$  and  $\lambda^-$  imply that  $|\lambda|$  is a measure that satisfies the following properties:

1.  $|\lambda(A)| \leq |\lambda|(A)$ , for any  $A \in \Sigma$ .
2.  $|\lambda(A)| = \sup\{|\lambda(B)| + |\lambda(A \setminus B)| : B \subseteq A, B \in \Sigma\}$ .

$$3. \quad |\lambda|(A) = \sup \left\{ \sum_{i \in J} |\lambda(A_i)| \right\}$$

where  $\{A_i\}_{i \in J}$  are pairwise disjoint,  $A = \bigcup_{i \in J} A_i$  and  $J$  runs over all the finite subsets of  $\mathbb{N}$ .

Property 3. can be adopted as the definition of  $|\lambda|$ , avoiding any reference to the Jordan decomposition.

Thus, a signed measure  $\lambda : \Sigma \rightarrow \mathbb{R}^*$  is bounded if and only if it has finite total variation.

**Definition 7.** Two measures  $\lambda, \nu : \Sigma \rightarrow [0, \infty]$  are mutually singular, or singular, denoted  $\lambda \perp \nu$ , if there is a partition  $X = A \cup B$ , with  $A, B \in \Sigma$ , such that  $\lambda(A) = \nu(B) = 0$ .

If  $\lambda : \Sigma \rightarrow \mathbb{R}^*$  is a signed measure, then  $\lambda^+ \perp \lambda^-$ . Moreover, if there are measures  $\lambda_1$  and  $\lambda_2$  such that  $\lambda = \lambda_1 - \lambda_2$  and  $\lambda_1 \perp \lambda_2$ , then  $\lambda_1 = \lambda^+$  and  $\lambda_2 = \lambda^-$ . In particular, the Jordan decomposition of a signed measure is unique.

**Remark 8.** We extend Definition 7 to signed measures in the following way: Two signed measures  $\lambda, \nu : \Sigma \rightarrow \mathbb{R}^*$  are mutually singular, or singular, denoted  $\lambda \perp \nu$ , if there is a partition  $X = A \cup B$ , with  $A, B \in \Sigma$ , such that  $\lambda(A_1) = 0$  for every  $\Sigma$ -measurable subset  $A_1$  of  $A$  and  $\nu(B_1) = 0$  for every  $\Sigma$ -measurable subset  $B_1$  of  $B$ . Equivalently, two signed measures  $\lambda, \nu : \Sigma \rightarrow \mathbb{R}^*$  are mutually singular, or singular, if  $|\lambda| \perp |\nu|$ .

**Definition 9.** If  $\lambda$  and  $\nu$  are signed measures, we say that  $\nu$  is absolutely continuous with respect to  $\lambda$ , denoted  $\nu \ll \lambda$ , if  $E \in \Sigma$  and  $|\lambda|(E) = 0$  implies that  $|\nu|(E) = 0$ .

If  $\lambda$  is a measure, we have that  $\nu \ll \lambda$  if and only if  $\nu^+ \ll \lambda$  and  $\nu^- \ll \lambda$ . Moreover, we can say that  $\nu \ll \lambda$  if and only if  $E \in \Sigma$  and  $\lambda(E) = 0$  implies that  $\nu(E_1) = 0$  for every  $\Sigma$ -measurable subset  $E_1$  of  $E$ .

Given a measure space  $(X, \Sigma, \mu)$  and a function  $f : X \rightarrow \mathbb{R}^*$  that has a  $\mu$ -integral, the set function  $\lambda$  defined for  $E \in \Sigma$  as  $\lambda(E) = \int_E f d\mu$ , is a signed measure and  $\lambda \ll \mu$ . We will frequently denote this signed measure  $\lambda$  as  $f d\mu$ .

**Theorem 10 (Radon–Nikodym Theorem).** *Let  $(X, \Sigma, \mu)$  be a  $\sigma$ -finite measure space and let  $\lambda : \Sigma \rightarrow \mathbb{R}^*$  be a signed measure such that  $\lambda \ll \mu$ . Then, there exists a  $\Sigma$ -measurable function  $f : X \rightarrow \mathbb{R}^*$  such that  $\lambda = f d\mu$ . The function  $f$  is unique up to  $\mu$ -a.e.*

We remark that this result does not require the signed measure  $\lambda$  to be  $\sigma$ -finite. Moreover, there are known conditions characterizing those measures  $\mu$  for which every signed measure  $\lambda \ll \mu$  can be represented as  $f d\mu$ .

**Theorem 11 (Lebesgue Decomposition).** *Let  $(X, \Sigma, \mu)$  be a measure space and let  $\lambda : \Sigma \rightarrow \mathbb{R}^*$  be a  $\sigma$ -finite signed measure. Then, there exist unique signed measures  $\lambda_a$  and  $\lambda_s$  defined on  $\Sigma$  such that  $\lambda = \lambda_a + \lambda_s$ ,  $\lambda_a \ll \mu$  and  $\lambda_s \perp \mu$ .*

This concludes our review of signed measures. Additional results will be presented at the appropriate time.

We will now review the definition and some of the properties of the Nemytsky operator. We will follow for the most part of the presentation in [9, Chapter VI].

We assume that we have fixed a complete measure space  $(X, \Sigma, \mu)$ .

**Definition 12.** A function  $g : X \times \mathbb{R}^* \rightarrow \mathbb{R}^*$  is called an  $N$ -function if it satisfies the following properties:

1. The function  $u \rightarrow g(x, u)$  is continuous for  $\mu$ -a.e.  $x \in X$ .
2. The function  $x \rightarrow g(x, u)$  is  $\Sigma$ -measurable for each  $u \in \mathbb{R}^*$ .

We denote

$$L^0 = \{f : X \rightarrow \mathbb{R}^* : f \text{ is } \Sigma\text{-measurable}\}.$$

**Lemma 13.** *Given an  $N$ -function  $g$  and given  $f \in L^0$ , the composite function  $g(x, f(x))$  belongs to  $L^0$ .*

**Proof.** We first assume that  $f$  is a simple function,

$$f = \sum_{i=1}^k c_i \chi_{E_i}$$

with  $c_i \in \mathbb{R}$  and  $\{E_i\} \subset \Sigma$ , pairwise disjoint. Given  $a \in \mathbb{R}$ ,

$$\{x \in X : g(x, f(x)) < a\} = \left\{ x \in X \setminus \bigcup_{i=1}^k E_i : g(x, 0) < a \right\} \cup \bigcup_{i=1}^k \{x \in E_i : g(x, c_i) < a\},$$

which is  $\Sigma$ -measurable according to Definition 12. If  $f \in L^0$ , there exists a sequence  $\{\varphi_n\}$  of simple functions,  $\varphi_n : X \rightarrow \mathbb{R}$ , such that  $\varphi_n(x) \rightarrow f(x)$  in  $\mathbb{R}^*$  for each  $x \in X$  [8, p. 78]. Then, Definition 12 implies that  $g(x, \varphi_n(x)) \rightarrow g(x, f(x))$  for  $\mu$ -a.e.  $x \in X$ . Thus, the function  $g(x, f(x))$  is  $\Sigma$ -measurable. This completes the proof of the lemma.  $\square$

**Definition 14.** Given an  $N$ -function  $g$ , the Nemytsky operator  $N_g$  is defined for  $f \in L^0$  as

$$N_g(f)(x) = g(x, f(x)). \quad (4)$$

Lemma 13 shows that the Nemytsky operator maps  $L^0$  into  $L^0$ . Given  $1 \leq p, q < \infty$ , (4) defines a continuous and bounded operator from  $L^p$  into  $L^q$  if and only if [9, p. 155] there exist a function  $a \in L^q$  and a constant  $b \geq 0$  so

$$|g(x, u)| \leq a(x) + b|u|^{p/q}.$$

In the following sections we will describe extensions of the Nemytsky operator to various signed measures, under particular conditions on the  $N$ -function  $g$ .

### 3. Extension of the Nemytsky operator for nonnegative $N$ -functions

We assume that  $(X, \Sigma, \mu)$  is a complete  $\sigma$ -finite measure space. We denote

$$\mathcal{L}^0 = \{f \in L^0 : f \text{ has a } \mu\text{-integral}\}$$

and

$$\mathcal{M}_a = \{\lambda : \Sigma \rightarrow \mathbb{R}^* \text{ signed measure: } \lambda \ll \mu\}$$

The Radon–Nikodym Theorem (Theorem 10) implies that there is a map  $\Lambda : \mathcal{L}^0 \rightarrow \mathcal{M}_a$ , defined as  $\Lambda(f) = f d\mu$ .

**Proposition 15.** *Let  $g$  be a nonnegative  $N$ -function. Then, there exists a unique operator  $\bar{N}_g : \mathcal{M}_a \rightarrow \mathcal{M}_a$  such that*

$$\Lambda \circ N_g(f) = \bar{N}_g \circ \Lambda(f) \quad (5)$$

for all  $f \in \mathcal{L}^0$ .

**Proof.** We first observe that  $N_g$  maps  $\mathcal{L}^0$  into  $\mathcal{L}^0$ . In fact,  $N_g$  maps  $L^0$  into  $L^0$  according to Lemma 13, and then the nonnegative measurable function  $g(x, f(x))$  has a  $\mu$ -integral for every  $f \in \mathcal{L}^0$ . Given  $\lambda \in \mathcal{M}_a$ ,  $\lambda = f d\mu$ , we propose the definition

$$\bar{N}_g(\lambda) = g(x, f(x)) d\mu. \quad (6)$$

Since the Radon–Nikodym Theorem assures the uniqueness of  $f$  up to  $\mu$ -a.e., we need to show that  $\bar{N}_g$  is well defined. Indeed, if  $f = h$  outside of a null set  $O \in \Sigma$ , and  $E \in \Sigma$ ,

$$\int_E g(x, h(x)) d\mu = \int_{E \cap (X \setminus O)} g(x, f(x)) d\mu = \int_E g(x, f(x)) d\mu.$$

So,  $\bar{N}_g(\lambda) = \bar{N}_g(h d\mu)$ . The definition given by (6) implies that (5) holds. Suppose that  $T : \mathcal{M}_a \rightarrow \mathcal{M}_a$  is another operator satisfying

$$\Lambda \circ N_g(f) = T \circ \Lambda(f).$$

Then,

$$T(f d\mu) = T \circ \Lambda(f) = \Lambda \circ N_g(f) = \bar{N}_g \circ \Lambda(f) = \bar{N}_g(f d\mu).$$

This completes the proof of the proposition.  $\square$

**Remark 16.** The operator given by (6) satisfies several properties that are natural to expect from a functional calculus:

1. From Proposition 15 we have

$$\bar{N}_g(\mu) = g(x, 1) d\mu,$$

since  $\Lambda$  maps the measure  $\mu$  to the identically one function. So, we can define, in particular, the Nemytsky operator of complete  $\sigma$ -finite measures, finite or not, for a fairly general class of  $N$ -functions. In particular, this observation applies to the Lebesgue measure on  $\mathbb{R}^n$ .

2. If  $g_1$  and  $g_2$  are two nonnegative  $N$ -functions, the sum  $g_1 + g_2$  is also an  $N$ -function. Moreover, the operator  $\bar{N}_g$  is additive in  $g$ . That is,

$$\bar{N}_{g_1+g_2} = \bar{N}_{g_1} + \bar{N}_{g_2}.$$

3. Given a  $\Sigma$ -measurable function  $\alpha : X \rightarrow [0, \infty]$  and a nonnegative  $N$ -function  $g$ , the multiplicative product  $\alpha g$  is also a nonnegative  $N$ -function and

$$\overline{N}_{\alpha g} = \alpha \overline{N}_g. \quad (7)$$

Observe that the product  $\alpha \overline{N}_g(f d\mu)$  is the measure defined on  $E \in \Sigma$  as  $\int_E \alpha(x) g(x, f(x)) d\mu$ . In particular, if  $g$  is independent of  $u$ , we can use (7) and 1. to obtain

$$\overline{N}_g(\lambda) = g\mu,$$

for every  $\lambda \in \mathcal{M}_a$ .

4. The product of two  $N$ -functions is also an  $N$ -function. Given two nonnegative  $N$ -functions  $g_1$  and  $g_2$ ,

$$\overline{N}_{g_1 g_2}(f d\mu) = g_1(x, f) g_2(x, f) d\mu.$$

5. Given two  $N$ -functions  $g_1, g_2 : X \times \mathbb{R}^* \rightarrow \mathbb{R}^*$ , the function  $g_2(x, g_1(x, u))$  is also an  $N$ -function. Furthermore,

$$\overline{N}_{g_2(x, u)} \circ \overline{N}_{g_1(x, u)} = \overline{N}_{g_2(x, g_1(x, u))}.$$

We observe that if  $g(x, u) = |u|$ , then  $\overline{N}_g(\lambda) = |\lambda|$ , the total variation of  $\lambda$ . In the next section we obtain an explicit representation of the operator  $\overline{N}_g$  for  $N$ -functions that generalize the function  $|u|$ .

#### 4. Extension of the Nemytsky operator for piecewise linear $N$ -functions

We fix again a complete  $\sigma$ -finite measure space  $(X, \Sigma, \mu)$ . We begin with the following known lemma:

**Lemma 17.** *If  $\lambda_1, \lambda_2 : \Sigma \rightarrow \mathbb{R}^*$  are mutually singular signed measures, then*

$$|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2|.$$

**Proof.** For any signed measures  $\lambda_1$  and  $\lambda_2$ , we have that

$$|\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2|.$$

So, it is enough to prove that if  $\lambda_1 \perp \lambda_2$ ,

$$|\lambda_1 + \lambda_2| \geq |\lambda_1| + |\lambda_2|.$$

According to (8), we can find a partition  $X = A \cup B$ , with  $A, B \in \Sigma$ , such that  $|\lambda_1|(B) = 0$  and  $|\lambda_2|(A) = 0$ . It follows that given  $E \in \Sigma$ ,

$$|\lambda_1|(E) = |\lambda_1|(A \cap E)$$

and

$$|\lambda_2|(E) = |\lambda_2|(B \cap E).$$

We now fix  $\varepsilon > 0$  and consider a partition  $(C_i)_{i \in J}$  of  $A \cap E$  such that

$$\sum_{i \in J} |\lambda_1(C_i)| \geq |\lambda_1|(A \cap E) - \frac{\varepsilon}{2}$$

and a partition  $(D_j)_{j \in L}$  of  $B \cap E$  such that

$$\sum_{j \in L} |\lambda_2(D_j)| \geq |\lambda_2|(B \cap E) - \frac{\varepsilon}{2}.$$

Since

$$E = \left( \bigcup_{i \in J} C_i \right) \cup \left( \bigcup_{j \in L} D_j \right)$$

is a partition of  $E$ ,

$$\begin{aligned} |\lambda_1 + \lambda_2|(E) &\geq \sum_{i \in J} |\lambda_1(C_i)| + \sum_{j \in L} |\lambda_2(D_j)| \\ &\geq |\lambda_1|(A \cap E) + |\lambda_2|(B \cap E) - \varepsilon \\ &= |\lambda_1|(E) + |\lambda_2|(E) - \varepsilon. \end{aligned}$$

So we can conclude that

$$|\lambda_1 + \lambda_2|(E) \geq |\lambda_1|(E) + |\lambda_2|(E).$$

This completes the proof of the lemma.  $\square$

**Definition 18.** An  $N$ -function  $g : X \times \mathbb{R}^* \rightarrow \mathbb{R}^*$  is piecewise linear if

$$g(x, u) = \sum_{i=1}^n a_i(x) |u - b_i(x)| + c(x)u$$

for  $\mu$ -a.e.  $x \in X$ , with  $a_i, c \in L^\infty$  and  $b_i \in L^1 \cap L^\infty$ .

If  $g$  is a piecewise linear  $N$ -function, the Nemytsky operator  $N_g$  is well defined from  $L^1$  into itself. We denote

$$\mathcal{M} = \{\lambda : \Sigma \rightarrow \mathbb{R} \text{ signed measure}\}.$$

Given  $\lambda \in \mathcal{M}$ , the properties observed in Remark 16 imply that we can define

$$\bar{N}_g(\lambda) = \sum_{i=1}^n a_i |\lambda - b_i \mu| + c \lambda, \quad (8)$$

where  $|\lambda - b_i \mu|$  is the total variation of the measure  $\lambda - b_i \mu$ . If we consider the operator  $\Lambda$  defined from  $L^1$  into  $\mathcal{M}$  as  $\Lambda(f) = f d\mu$ , then

$$\Lambda \circ N_g(f) = \bar{N}_g \circ \Lambda(f)$$

for every  $f \in L^1$ . In fact, given  $E \in \Sigma$ ,

$$\begin{aligned} \bar{N}_g(f d\mu)(E) &= \sum_{i=1}^n a_i |f d\mu - b_i \mu|(E) + c(f d\mu)(E) \\ &= \int_E \left( \sum_{i=1}^n a_i |f - b_i| + cf \right) d\mu = (N_g(f) d\mu)(E). \end{aligned}$$

Given  $\lambda \in \mathcal{M}$ , we want to obtain an explicit expression for  $\bar{N}_g(\lambda)$ . According to the Lebesgue Decomposition Theorem, we can write

$$\lambda = f d\mu + \lambda_s,$$

with  $f \in L^1$  and  $\lambda_s \perp \mu$ .

We write

$$l^+(g)(x) = \lim_{u \rightarrow +\infty} \frac{g(x, u)}{u} = \sum_{i=1}^n a_i(x) + c(x) \quad (9)$$

and

$$l^-(g)(x) = \lim_{u \rightarrow -\infty} \frac{g(x, u)}{u} = - \sum_{i=1}^n a_i(x) + c(x). \quad (10)$$

We observe that  $l^+(g)(x)$  and  $l^-(g)(x)$  exist for  $\mu$ -a.e.  $x \in X$ .

**Proposition 19.** Given  $\lambda \in \mathcal{M}$ ,  $\lambda = f d\mu + \lambda_s$ , we can write for each  $E \in \Sigma$ ,

$$\bar{N}_g(\lambda)(E) = \int_E N_g(f) d\mu + (l^+(g)\lambda_s^+)(E) - (l^-(g)\lambda_s^-)(E).$$

**Proof.** According to (8),

$$\bar{N}_g(\lambda) = \sum_{i=1}^n a_i |f d\mu + \lambda_s - b_i \mu| + c(f d\mu + \lambda_s).$$

We observe that  $f d\mu - b_i \mu \perp \lambda_s$ . Therefore, using Lemma 17,

$$\bar{N}_g(\lambda) = \sum_{i=1}^n a_i |f d\mu - b_i \mu| + c f d\mu + \sum_{i=1}^n a_i |\lambda_s| + c \lambda_s = \bar{N}_g(f d\mu) + \sum_{i=1}^n a_i |\lambda_s| + c \lambda_s.$$

Using (9) and (10), we obtain, for each  $E \in \Sigma$ ,

$$\bar{N}_g(\lambda)(E) = \int_E N_g(f) d\mu + \left( \frac{l^+(g) - l^-(g)}{2} |\lambda_s| \right)(E) + \left( \frac{l^+(g) + l^-(g)}{2} \lambda_s \right)(E).$$

Since

$$\lambda_s = \lambda_s^+ - \lambda_s^-$$

and

$$|\lambda_s| = \lambda_s^+ + \lambda_s^-,$$

we can write

$$\bar{N}_g(\lambda)(E) = \int_E N_g(f) d\mu + (l^+(g)\lambda_s^+)(E) - (l^-(g)\lambda_s^-)(E).$$

This completes the proof of the proposition.  $\square$

## 5. A class of nonlinear evolution equations

As before, we fix a complete  $\sigma$ -finite measure space  $(X, \Sigma, \mu)$ . We denote

$$\mathcal{M}_b = \{\lambda : \Sigma \rightarrow \mathbb{R} \text{ signed measure: } \lambda \ll \mu\}.$$

The space  $\mathcal{M}_b$  is a Banach space with the norm

$$\|\lambda\|_{\mathcal{M}_b} = |\lambda|(X).$$

For  $0 < T < \infty$  fixed, the space  $C[0, T; \mathcal{M}_b]$  of continuous functions  $\lambda : [0, T] \rightarrow \mathcal{M}_b$  becomes a Banach space with the norm

$$\|\lambda\| = \sup_{0 \leq t \leq T} \|\lambda(t)\|_{\mathcal{M}_b}.$$

Likewise, the space  $C^1[0, T; \mathcal{M}_b]$  of continuously differentiable functions  $\lambda : [0, T] \rightarrow \mathcal{M}_b$  becomes a Banach space with the norm  $\|\lambda\| + \|\lambda'\|$ .

**Remark 20.** We observe that  $C[0, T; \mathcal{M}_b]$  is isometrically isomorphic to  $C[0, T; L^1]$  endowed with the norm

$$\|f\| = \sup_{0 \leq t \leq T} \|f(t)\|_{L^1}.$$

Indeed,



$$\|(f d\mu)(t)\|_{\mathcal{M}_b} = |(f d\mu)(t)| = \|f(t)\|_{L^1},$$

for each  $0 \leq t \leq T$  [8, p. 94].

Given  $\lambda_0 \in \mathcal{M}_b$ , we consider the initial value problem

$$\begin{cases} \frac{d\lambda}{dt} + \mathcal{A}(\lambda)(t) = 0 & \text{for } 0 < t < T, \\ \lambda(0) = \lambda_0. \end{cases} \quad (11)$$

We want to formulate conditions on the operator  $\mathcal{A}$  so that (11) has a unique solution  $\lambda \in C^1[0, T; \mathcal{M}_b]$ . We begin with a lemma.

**Lemma 21.** *Let  $G : [0, T] \times X \times \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying the conditions:*

1.  $|G(t, x, u)| \leq a(x) + b|u|$ , for some  $a \in L^1$  and  $b \geq 0$ , for  $\mu$ -a.e.  $x \in X$ ,  $0 \leq t \leq T$  and  $u \in \mathbb{R}$ .
2. The function  $x \rightarrow G(t, x, u)$  is  $\Sigma$ -measurable for each  $0 \leq t \leq T$  and  $u \in \mathbb{R}$ .
3. There exists  $C > 0$  such that  $|G(t, x, u_1) - G(t, x, u_2)| \leq C|u_1 - u_2|$ , for  $0 \leq t \leq T$ ,  $u_1, u_2 \in \mathbb{R}$  and for  $\mu$ -a.e.  $x \in X$ .
4. There exists  $C > 0$  such that  $|G(t_1, x, u) - G(t_2, x, u)| \leq C|u||t_1 - t_2|$ , for  $0 \leq t_1, t_2 \leq T$ ,  $u \in \mathbb{R}$  and for  $\mu$ -a.e.  $x \in X$ .

Then, the following properties hold:

- (a) For each  $0 \leq t \leq T$ , the function  $G_t : X \times \mathbb{R} \rightarrow \mathbb{R}$  defined as  $G_t(x, u) = G(t, x, u)$  is an  $N$ -function.
- (b) For each  $0 \leq t \leq T$ , the Nemytsky operator  $N_{G_t}$  maps  $L^1$  to itself.
- (c) The function  $f(t, x) \rightarrow N_{G_t}(f(t, \cdot))(x)$  maps  $C[0, T; L^1]$  continuously into itself.

**Proof.** The proof of (a) is a direct application of conditions 2. and 3., while the proof of (b) follows from 1. To prove (c) we begin by observing that given  $f \in C[0, T; L^1]$ , the function  $N_{G_t}(f(t, \cdot))(x)$  belongs to  $L^1$  for each  $0 \leq t \leq T$ , as a consequence of (b). Moreover,  $N_{G_t}(f(t, \cdot))(x)$  belongs to  $C[0, T; L^1]$  because of condition 4. Finally, if  $f_n$  converges to  $f$  in  $C[0, T; L^1]$ , we use condition 3. to write

$$\begin{aligned} \|N_{G_t}(f_n(t, \cdot)) - N_{G_t}(f(t, \cdot))\|_{L^1} &\leq C \|f_n(t, \cdot) - f(t, \cdot)\|_{L^1} \\ &\leq C \|f_n - f\|. \end{aligned}$$

Thus,  $\sup_{0 \leq t \leq T} \|N_{G_t}(f_n(t, \cdot)) - N_{G_t}(f(t, \cdot))\|_{L^1} \rightarrow 0$  as  $n \rightarrow \infty$ . This completes the proof of the lemma.  $\square$

Given a function  $G$  satisfying the hypotheses of Lemma 21, we define for  $\lambda \in C[0, T; \mathcal{M}_b]$ ,

$$\mathcal{A}(\lambda)(t) = \overline{N}_{G_t}(\lambda(t)). \quad (12)$$

Remark 20 and Lemma 21 imply that the operator  $\mathcal{A}$  is continuous from  $C[0, T; \mathcal{M}_b]$  into itself.

We recall the following known extension of the Banach Fixed Point Theorem:

**Proposition 22.** *Let  $(S, d)$  be a complete metric space and consider a map  $f : S \rightarrow S$ . If there exists  $k \in \{1, 2, \dots\}$  such that the composite map  $f^{(k)}$  is a contraction, then the map  $f$  has a unique fixed point.*

**Theorem 23.** *The initial value problem (11) has one and only one solution in  $C^1[0, T; \mathcal{M}_b]$  if we assume that the operator  $\mathcal{A}$  is given by (12) and the function  $G$  satisfies the conditions stated in Lemma 21.*

**Proof.** We begin by observing that the initial value problem (11) has the same solutions in  $C^1[0, T; \mathcal{M}_b]$  as the integral equation

$$\lambda(t) = \lambda_0 + \int_0^t \mathcal{A}(\lambda)(s) ds. \quad (13)$$

We will show that (13) has one and only one solution in  $C^1[0, T; \mathcal{M}_b]$  by proving that the operator  $\mathcal{T}$  defined on  $C[0, T; \mathcal{M}_b]$  as  $\mathcal{T}(\lambda) = \lambda_0 + \int_0^t \mathcal{A}(\lambda)(s) ds$  has a unique fixed point. According to Proposition 22, it suffices to show that  $\mathcal{T}^{(k)}$  is a contraction in  $C[0, T; \mathcal{M}_b]$  for some  $k \in \{1, 2, \dots\}$ , for which it is enough to prove that for  $\lambda_1, \lambda_2 \in C[0, T; \mathcal{M}_b]$  and  $k \in \{1, 2, \dots\}$ ,

$$\|\mathcal{T}^{(k)}(\lambda_1) - \mathcal{T}^{(k)}(\lambda_2)\| \leq \frac{C^k T^k}{k!} \|\lambda_1 - \lambda_2\|. \quad (14)$$

When  $k = 1$ , if  $\lambda_i = f_i d\mu$ ,

$$\begin{aligned} \|\mathcal{T}(\lambda_1)(t) - \mathcal{T}(\lambda_2)(t)\|_{\mathcal{M}_b} &\leq \int_0^t \|\mathcal{A}(\lambda_1)(s) - \mathcal{A}(\lambda_2)(s)\|_{\mathcal{M}_b} ds \\ &= \int_0^t \|G(s, \cdot, f_1(s, \cdot)) - G(s, \cdot, f_2(s, \cdot))\|_{L^1} ds \\ &\leq Ct \sup_{0 \leq s \leq T} \|f_1(s, \cdot) - f_2(s, \cdot)\|_{L^1} \\ &= Ct \sup_{0 \leq s \leq T} \|\lambda_1(s) - \lambda_2(s)\|_{\mathcal{M}_b} = Ct \|\lambda_1 - \lambda_2\|. \end{aligned}$$

Or

$$\|\mathcal{T}(\lambda_1) - \mathcal{T}(\lambda_2)\| \leq CT \|\lambda_1 - \lambda_2\|.$$

We prove now that (14) holds for  $k = n + 1$ , assuming that it holds for  $k = n$ ,

$$\begin{aligned} \|\mathcal{T}^{(n+1)}(\lambda_1)(t) - \mathcal{T}^{(n+1)}(\lambda_2)(t)\|_{\mathcal{M}_b} &= \left\| \int_0^t (\mathcal{A}(\mathcal{T}^{(n)}(\lambda_1))(s) - \mathcal{A}(\mathcal{T}^{(n)}(\lambda_2))(s)) ds \right\|_{\mathcal{M}_b} \\ &\leq \int_0^t \|G(s, \cdot, \mathcal{T}^{(n)}(\lambda_1)(s)) - G(s, \cdot, \mathcal{T}^{(n)}(\lambda_2)(s))\|_{\mathcal{M}_b} ds \\ &\leq C \int_0^t \|\mathcal{T}^{(n)}(\lambda_1)(s) - \mathcal{T}^{(n)}(\lambda_2)(s)\|_{\mathcal{M}_b} ds \\ &\leq C \int_0^t \frac{C^n s^n}{n!} \|\lambda_1 - \lambda_2\| ds = \frac{C^{n+1} t^{n+1}}{(n+1)!} \|\lambda_1 - \lambda_2\|. \end{aligned}$$

This completes the proof of the theorem.  $\square$

**Remark 24.** As an illustration, we present now an example of an operator  $\mathcal{A}$  as considered in Theorem 23. With this purpose, we construct first a function  $G$  that satisfies the hypothesis of Lemma 21. We fix a function  $H : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the two conditions:

1. There exists  $C_1 > 0$  such that  $|H(r)| \leq C_1 |r|$  for all  $r \in \mathbb{R}$ .
2.  $H$  is a Lipschitz function; that is to say, there exists  $C_2 > 0$  such that  $|H(r_1) - H(r_2)| \leq C_2 |r_1 - r_2|$  for all  $r_1, r_2 \in \mathbb{R}$ .

Then, given  $a \in L^1$  we define  $G : [0, T] \times X \times \mathbb{R} \rightarrow \mathbb{R}$  as  $G(t, x, u) = H(a(x) + tu)$ .

We claim that  $G$  satisfies conditions 1.–4. in Lemma 21.

In fact,  $|G(t, x, u)| = |H(a(x) + tu)| \leq C_1 |a(x) + tu| \leq C_1 (|a(x)| + T|u|)$ , so condition 1. is satisfied.

If we fix  $0 \leq t \leq T$ ,  $u \in \mathbb{R}$ , the function  $x \rightarrow H(a(x) + tu)$  is  $\Sigma$ -measurable, because it is the composition, in the necessary order, of the  $\Sigma$ -measurable function  $x \rightarrow a(x)$  and the continuous function  $r \rightarrow H(r + tu)$ . So condition 2. holds.

We can write

$$\begin{aligned} |G(t, x, u_1) - G(t, x, u_2)| &= |H(a(x) + tu_1) - H(a(x) + tu_2)| \\ &\leq C_2 t |u_1 - u_2| \leq C_2 T |u_1 - u_2|. \end{aligned}$$

Therefore, condition 3. is satisfied.

Finally, if we fix  $0 \leq t_1, t_2 \leq T$ ,  $x \in X$ ,  $u \in \mathbb{R}$ , we have that

$$\begin{aligned} |G(t_1, x, u) - G(t_2, x, u)| &= |H(a(x) + t_1 u) - H(a(x) + t_2 u)| \\ &\leq C_2 |u| |t_1 - t_2|. \end{aligned}$$

Thus, condition 4. is satisfied as well.

According to Lemma 21, for each  $0 \leq t \leq T$ , the function  $G_t : X \times \mathbb{R} \rightarrow \mathbb{R}$  defined as  $G_t(x, u) = H(a(x) + tu)$  is an  $N$ -function.

If  $\lambda \in C^0[0, T; \mathcal{M}_b]$ , Remark 20 implies that  $\lambda(t) = f(t, \cdot) d\mu$  for a unique  $f \in C^0[0, T; L^1]$ . So, we can define the operator

$$\mathcal{A} : C^0[0, T; \mathcal{M}_b] \rightarrow C^0[0, T; \mathcal{M}_b]$$

as

$$\mathcal{A}(\lambda)(t) = \overline{N_{G_t}}(\lambda(t)) = H(a(\cdot) + tf(t, \cdot)) d\mu.$$

## Acknowledgment

The authors thank the anonymous referee for the careful reading of the manuscript and the helpful remarks.

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