Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 339 (2008) 774–784

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

The multisection method for triple products and identities of Rogers–Ramanujan type

Wenchang Chu *, Chenying Wang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China

Received 30 January 2007

Available online 21 July 2007

Submitted by M. Laurent

Abstract

By applying the bisection and trisection method to Jacobi's triple product identity, we establish several identities factorizing sum and difference of infinite products, which lead, in turn, to new and elementary proofs for twenty identities of Rogers–Ramanujan type.

Keywords: The bisection and trisection series; Basic hypergeometric series; Jacobi's triple product identity; The quintuple product identity; Rogers–Ramanujan identities

For two complex x and q, the shifted-factorial of x with base q is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right) \quad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}\left(1-q^k x\right) \quad \text{and} \quad (x;q)_n = (x;q)_\infty / \left(xq^n;q\right)_\infty.$$

For the sake of brevity, the product of shifted factorials is abbreviated to

$$[\alpha,\beta,\ldots,\gamma;q]_n = (\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n.$$

Following Gasper and Rahman [5], the basic hypergeometric series is defined by

$${}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix} \,\middle|\, q; z\right] = \sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r} \frac{[a_0,a_1,\ldots,a_r;q]_n}{[q,b_1,\ldots,b_s;q]_n} z^n$$

where the base q will be restricted to $|q| < 1$ for nonterminating q-series.

* Corresponding author. Current address: Dipartimento di Matematica, Università del Salento, Lecce-Arnesano PO Box 193, Lecce 73100, Italy.
E-mail addresses: chu.wenchang@unile.it (W. Chu), wang.chenying@163.com (C. Wang).

0022-247X/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2007.06.074

There are several important identities in the theta function theory, two of which are Jacobi's triple product identity [9]

$$\sum_{n=-\infty}^{+\infty}(-1)^n q^{\binom{n}{2}}x^n=[q,x,q/x;q]_\infty \tag{0.1}$$

and the celebrated quintuple product identity [4]

$$[q,z,q/z;q]_\infty\big[qz^2,q/z^2;q^2\big]_\infty=\sum_{n=-\infty}^{+\infty}\big\{1-z^{1+6n}\big\}q^{3\binom{n}{2}}\big(q^2/z^3\big)^n \tag{0.2a}$$

$$=\sum_{n=-\infty}^{+\infty}\big\{1-\big(q/z^2\big)^{1+3n}\big\}q^{3\binom{n}{2}}\big(qz^3\big)^n \tag{0.2b}$$

where the former has played fundamental role in the proof of Rogers–Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{[q,q^4;q^5]_\infty}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{[q^2,q^3;q^5]_\infty}.$$

The similar identities expressing infinite sums as infinite products are generally said to be Rogers–Ramanujan type. Their proofs require deep understanding of q-series and classical partition theory. Therefore direct and more accessible proofs have been desirable.

The purpose of this paper is to present a very elementary approach to a class of twenty Rogers–Ramanujan type identities. By employing bisection and trisection method to Jacobi's triple product identity, we shall derive several identities factorizing sum and difference of two infinite products. Then their combinations with Euler's q-exponential formulae, the q-binomial expansion, the q-Gauss theorem and the q-Kummer theorem will lead to new and elementary proofs for twenty identities of Rogers–Ramanujan type.

1. Bisection of the Jacobi triple products

For a natural number a and an integer c, define two infinite sums by

$$X(a,c):=\sum_k q^{4ak^2+2ck}=\big[q^{8a},-q^{4a+2c},-q^{4a-2c};q^{8a}\big]_\infty,$$

$$Y(a,c):=\sum_k q^{4ak^2-4ak+2ck}=\big[q^{8a},-q^{2c},-q^{8a-2c};q^{8a}\big]_\infty.$$

When $a\not\equiv c \pmod 2$, applying Jacobi's triple product identity (0.1), we have

$$X(a,c)+q^{a-c}Y(a,c)=\sum_k q^{ak^2+ck}=\big[q^{2a},-q^{a+c},-q^{a-c};q^{2a}\big]_\infty,$$

$$X(a,c)-q^{a-c}Y(a,c)=\sum_k(-1)^k q^{ak^2+ck}=\big[q^{2a},q^{a+c},q^{a-c};q^{2a}\big]_\infty,$$

which result in the following two equations:

$$\big[-q^{a+c},-q^{a-c};q^{2a}\big]_\infty+\big[q^{a+c},q^{a-c};q^{2a}\big]_\infty \tag{1.1a}$$

$$=\frac{2}{(q^{2a};q^{2a})_\infty}\big[q^{8a},-q^{4a+2c},-q^{4a-2c};q^{8a}\big]_\infty, \tag{1.1b}$$

$$\big[-q^{a+c},-q^{a-c};q^{2a}\big]_\infty-\big[q^{a+c},q^{a-c};q^{2a}\big]_\infty \tag{1.1c}$$

$$=\frac{2q^{a-c}}{(q^{2a};q^{2a})_\infty}\big[q^{8a},-q^{2c},-q^{8a-2c};q^{8a}\big]_\infty. \tag{1.1d}$$

They will be specialized to more concrete equations in order to derive identities of Rogers–Ramanujan type.

1.1. Case $a=2$ and $c=1$ in (1.1)

The corresponding equations read as:

$$\left(q;q^2\right)_\infty+\left(-q;q^2\right)_\infty=\frac{2}{(q^4;q^4)_\infty}\left[q^{16},-q^6,-q^{10};q^{16}\right]_\infty, \tag{1.2a}$$

$$\left(-q;q^2\right)_\infty-\left(q;q^2\right)_\infty=\frac{2q}{(q^4;q^4)_\infty}\left[q^{16},-q^2,-q^{14};q^{16}\right]_\infty. \tag{1.2b}$$

A. Recall q-binomial theorem [5, II-3]

$${}_1\phi_0\left[\begin{matrix}a\\-\end{matrix}\,\middle|\,q;x\right]=\frac{(ax;q)_\infty}{(x;q)_\infty}\quad\text{where } |x|<1, \tag{1.3}$$

which can be specified as

$${}_1\phi_0\left[\begin{matrix}-1\\-\end{matrix}\,\middle|\,q^2;q\right]=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

Then the linear combinations

$$1\pm{}_1\phi_0\left[\begin{matrix}-1\\-\end{matrix}\,\middle|\,q^2;q\right]$$

lead us to the following identities:

$$1+\sum_{m=1}^{\infty}\frac{(-q^2;q^2)_{m-1}}{(q^2;q^2)_m}q^m=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[q^{16},-q^6,-q^{10};q^{16}\right]_\infty, \tag{1.4a}$$

$$\sum_{m=0}^{\infty}(-1)^m\frac{(-q^2;q^2)_m}{(q^2;q^2)_{1+m}}q^m=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left[q^{16},-q^2,-q^{14};q^{16}\right]_\infty. \tag{1.4b}$$

B. Recall Euler's first q-exponential formula (cf. [5, p. 10])

$$\frac{1}{(\xi;q)_\infty}=\sum_{n=0}^{\infty}\frac{\xi^n}{(q;q)_n}\quad\text{where } |\xi|<1.$$

Then its combinations with

$$(\sqrt{q};q)_\infty^{-1}\pm(-\sqrt{q};q)_\infty^{-1}=\frac{(-\sqrt{q};q)_\infty\pm(\sqrt{q};q)_\infty}{(q;q^2)_\infty}$$

give rise to the following bisection series identities:

$$\sum_{m=0}^{\infty}\frac{q^m}{(q;q)_{2m}}=\frac{[q^8,-q^3,-q^5;q^8]_\infty}{(q;q)_\infty}, \tag{1.5a}$$

$$\sum_{m=0}^{\infty}\frac{q^m}{(q;q)_{1+2m}}=\frac{[q^8,-q,-q^7;q^8]_\infty}{(q;q)_\infty}. \tag{1.5b}$$

C. Euler's second q-exponential formula (cf. [5, p. 11]) reads as

$$(\xi;q)_\infty=\sum_{n=0}^{\infty}\frac{(-1)^n\xi^n}{(q;q)_n}q^{\binom{n}{2}}.$$

The corresponding bisection series with respect to the parity of summation index leads to the following identities:

$$\sum_{m=0}^{\infty}\frac{q^{m(2m-1)}}{(q;q)_{2m}}=\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(q;q)_{2m+1}}=(-q;q)_\infty.$$

Letting $\xi = \pm\sqrt{q}$ and then factorizing the two expressions

$$(\sqrt{q};q)_\infty \pm (-\sqrt{q};q)_\infty,$$

we recover the following identities of Rogers–Ramanujan type [11, Eqs. 39 and 38]:

$$\sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q)_{2m}} = \frac{[q^8,-q^3,-q^5;q^8]_\infty}{(q^2;q^2)_\infty}, \tag{1.6a}$$

$$\sum_{m=0}^{\infty} \frac{q^{2m(m+1)}}{(q;q)_{1+2m}} = \frac{[q^8,-q,-q^7;q^8]_\infty}{(q^2;q^2)_\infty}, \tag{1.6b}$$

where the first identity is actually due to Jackson [8, p. 170, the 5th equation]. It should be pointed out that the derivation of these two identities just presented has previously been given by Andrews and Santos [2].

D. Recall the q-Kummer summation theorem [5, II-9]

$${}_2\phi_1\left[\begin{matrix} a, & c \\ & qa/c \end{matrix} \middle| q; -q/c\right] = \frac{(q^2a/c^2;q^2)_\infty}{(q^2a;q^2)_\infty} \frac{[qa,-q;q]_\infty}{[qa/c,-q/c;q]_\infty} \quad \text{where } |q/c|<1.$$

Through its following specification

$$\sum_{m=0}^{\infty} \frac{(-1;q)_m}{(q;q)_m} q^{\binom{1+m}{2}} = \frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^4,q^2,q^2;q^4\big]_\infty = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$$

we may factorize the expressions

$$\pm 1 + \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$$

and recover two beautiful Rogers–Ramanujan type identities due to Gessel and Stanton [6, Eqs. 7.13 and 7.15]:

$$1 + \sum_{m=1}^{\infty} \frac{(-q;q)_{m-1}}{(q;q)_m} q^{\binom{1+m}{2}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^{16},-q^6,-q^{10};q^{16}\big]_\infty, \tag{1.7a}$$

$$\sum_{m=0}^{\infty} \frac{(-q;q)_m}{(q;q)_{m+1}} q^{m+\binom{1+m}{2}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^{16},-q^2,-q^{14};q^{16}\big]_\infty. \tag{1.7b}$$

E. Recall the q-Gauss summation formula [5, II-8]

$${}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; c/ab\right] = \frac{[c/a,c/b;q]_\infty}{[c,c/ab;q]_\infty} \quad \text{where } |c/ab|<1. \tag{1.8}$$

Specializing it to

$${}_1\phi_1\left[\begin{matrix} -1 \\ q \end{matrix} \middle| q^2; -q\right] = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$$

and then computing the linear combinations

$$\pm 1 + {}_1\phi_1\left[\begin{matrix} -1 \\ q \end{matrix} \middle| q^2; -q\right]$$

we rederive two identities of Rogers–Ramanujan type [11, Eqs. 72 and 69]:

$$1 + \sum_{m=1}^{\infty} \frac{(-q^2;q^2)_{m-1}}{(q;q)_{2m}} q^{m^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^{16},-q^6,-q^{10};q^{16}\big]_\infty, \tag{1.9a}$$

$$\sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m}{(q;q)_{2m+2}} q^{m^2+2m} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^{16},-q^2,-q^{14};q^{16}\big]_\infty. \tag{1.9b}$$

1.2. Case $a=1$ and $c=0$ in (1.1)

The corresponding equations read as

$$\big(-q;q^2\big)_\infty^2+\big(q;q^2\big)_\infty^2=\frac{2}{(q^2;q^2)_\infty}\big[q^8,-q^4,-q^4;q^8\big]_\infty, \tag{1.10a}$$

$$\big(-q;q^2\big)_\infty^2-\big(q;q^2\big)_\infty^2=\frac{4q}{(q^2;q^2)_\infty}\big[q^8,-q^8,-q^8;q^8\big]_\infty. \tag{1.10b}$$

In view of the q-binomial theorem (1.3), we have

$${}_1\phi_0\left[\begin{matrix}-1\\ -\end{matrix}\,\Big|\,q;\xi\right]=\frac{(-\xi;q)_\infty}{(\xi;q)_\infty}=\frac{(-\xi;q)_\infty^2}{(\xi^2;q^2)_\infty}.$$

Letting $\xi=\pm\sqrt{q}$ and then considering the bisection series, we derive the following identities:

$$\sum_{m=0}^{\infty}\frac{(-1;q)_{2m}}{(q;q)_{2m}}q^m=\frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^4,-q^2,-q^2;q^4\big]_\infty, \tag{1.11a}$$

$$\sum_{m=0}^{\infty}\frac{(-q;q)_{2m}}{(q;q)_{1+2m}}q^m=\frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^4,-q^4,-q^4;q^4\big]_\infty. \tag{1.11b}$$

Instead, from the q-Gauss theorem (1.8), we have

$${}_2\phi_1\left[\begin{matrix}-1, & -1\\ & q\end{matrix}\,\Big|\,q^2;q\right]=\frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty^2}.$$

Then the linear combinations

$$\pm1+{}_2\phi_1\left[\begin{matrix}-1, & -1\\ & q\end{matrix}\,\Big|\,q^2;q\right]$$

lead us to the following two identities:

$$1+2\sum_{m=1}^{\infty}\frac{(-q^2;q^2)_{m-1}^2}{(q;q)_{2m}}q^m=\frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^8,-q^4,-q^4;q^8\big]_\infty, \tag{1.12a}$$

$$\sum_{m=0}^{\infty}\frac{(-q^2;q^2)_m^2}{(q;q)_{2+2m}}q^m=\frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^8,-q^8,-q^8;q^8\big]_\infty. \tag{1.12b}$$

1.3. Case $a=5$ and $c=4$ in (1.1)

The corresponding equations read as:

$$\big[q^{10},-q,-q^9;q^{10}\big]_\infty+\big[q^{10},q,q^9;q^{10}\big]_\infty=2\big[q^{40},-q^{12},-q^{28};q^{40}\big]_\infty, \tag{1.13a}$$

$$\big[q^{10},-q,-q^9;q^{10}\big]_\infty-\big[q^{10},q,q^9;q^{10}\big]_\infty=2q\big[q^{40},-q^8,-q^{32};q^{40}\big]_\infty. \tag{1.13b}$$

Applying these equations to the identity [11, Eq. 20]

$$\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^4;q^4)_m}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^5,q^2,q^3;q^5\big]_\infty=\frac{[q^{10},-q,-q^9;q^{10}]_\infty}{[q^{20},q^4,q^{16};q^{20}]_\infty}$$

we find that the two bisection series, under the base replacement $q\to q^{1/4}$, result in the following identities of Rogers–Ramanujan type [11, Eqs. 98 and 94]:

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_{2m}} = \frac{[q^{10},q^2,q^8;q^{10}]_\infty}{(q;q)_\infty}\big[q^{14},q^6;q^{20}\big]_\infty, \tag{1.14a}$$

$$\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q;q)_{1+2m}} = \frac{[q^{10},q^3,q^7;q^{10}]_\infty}{(q;q)_\infty}\big[q^{16},q^4;q^{20}\big]_\infty. \tag{1.14b}$$

Both identities can be traced back to an earlier work by Rogers [10, Eqs. 5 and 6], where the first identity can also be found in Bailey [3, Eq. 1.4].

1.4. *Case $a = 5$ and $c = 2$ in (1.1)*

The corresponding equations read as:

$$\big[q^{10},-q^3,-q^7;q^{10}\big]_\infty + \big[q^{10},q^3,q^7;q^{10}\big]_\infty = 2\big[q^{40},-q^{16},-q^{24};q^{40}\big]_\infty, \tag{1.15a}$$

$$\big[q^{10},-q^3,-q^7;q^{10}\big]_\infty - \big[q^{10},q^3,q^7;q^{10}\big]_\infty = 2q^3\big[q^{40},-q^4,-q^{36};q^{40}\big]_\infty. \tag{1.15b}$$

As observed by Andrews and recorded by Alladi [1, p. 222] that the bisection series of another identity [11, Eq. 16]

$$\sum_{m=0}^{\infty} \frac{q^{m(2+m)}}{(q^4;q^4)_m} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^5,q,q^4;q^5\big]_\infty = \frac{[q^{10},-q^3,-q^7;q^{10}]_\infty}{[q^{20},q^8,q^{12};q^{20}]_\infty}$$

leads, under the base replacement $q \to q^{1/4}$, to the following two identities of Rogers–Ramanujan type [11, Eqs. 99 and 96]:

$$\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q;q)_{2m}} = \frac{[q^{10},q,q^9;q^{10}]_\infty}{(q;q)_\infty}\big[q^{12},q^8;q^{20}\big]_\infty, \tag{1.16a}$$

$$\sum_{m=0}^{\infty} \frac{q^{m(m+2)}}{(q;q)_{1+2m}} = \frac{[q^{10},q^4,q^6;q^{10}]_\infty}{(q;q)_\infty}\big[q^{18},q^2;q^{20}\big]_\infty. \tag{1.16b}$$

These two identities have been found earlier by Rogers [10, Eqs. 13 and 7], where the last one appeared also in Bailey [3, Eq. 1.5].

1.5. *Case $a = 3$ and $c = 1$ in (1.1)*

The corresponding equations read as:

$$\big[-q,-q^2;q^3\big]_\infty + \big[q,q^2;q^3\big]_\infty = \frac{2}{(q^3;q^3)_\infty}\big[q^{12},-q^5,-q^7;q^{12}\big]_\infty,$$

$$\big[-q,-q^2;q^3\big]_\infty - \big[q,q^2;q^3\big]_\infty = \frac{2q}{(q^3;q^3)_\infty}\big[q^{12},-q,-q^{11};q^{12}\big]_\infty.$$

Similarly, combining "± 1" with the identity [11, Eq. 6]

$$\sum_{m=0}^{\infty} \frac{(-1;q)_m}{(q;q)_m(q;q^2)_m} q^{m^2} = \frac{[q^3,-q,-q^2;q^3]_\infty}{(q;q)_\infty} = \frac{[-q,-q^2;q^3]_\infty}{[q,q^2;q^3]_\infty} \tag{1.17}$$

we obtain the following two further identities [11, Eqs. 58 and 56]

$$1 + \sum_{m=1}^{\infty} \frac{(-q;q)_{m-1}}{(q;q)_m(q;q^2)_m} q^{m^2} = \frac{[q^{12},-q^5,-q^7;q^{12}]_\infty}{(q;q)_\infty}, \tag{1.18a}$$

$$\sum_{m=0}^{\infty} \frac{(-q;q)_m}{(q;q)_{m+1}(q;q^2)_{m+1}} q^{m^2+2m} = \frac{[q^{12},-q,-q^{11};q^{12}]_\infty}{(q;q)_\infty}. \tag{1.18b}$$

It should be pointed out that the errors appeared in the original sums of Slater [11, Eqs. 6 and 56] have been corrected respectively by (1.17) and (1.18b).

1.6.

Finally, define two infinite products by

$$\mathcal{X} := \big[q^{12}, q^5, q^7; q^{12}\big]_\infty = \sum_k (-1)^k q^{12\binom{k}{2}+5k},$$
$$\mathcal{Y} := \big[q^{12}, q, q^{11}; q^{12}\big]_\infty = \sum_k (-1)^k q^{12\binom{k}{2}+k}.$$

Then for $\varepsilon = \pm 1$, considering the Jacobi triple product

$$\mathcal{X} + q\varepsilon\mathcal{Y} = \big[-q^3, -q\varepsilon, \varepsilon q^2; -q^3\big]_\infty = \sum_k (-1)^{\binom{k}{2}} q^{3\binom{k}{2}+k}\varepsilon^k$$

we get the following two equations:

$$\big[-q^3, -q, q^2; -q^3\big]_\infty + \big[-q^3, q, -q^2; -q^3\big]_\infty = 2\big[q^{12}, q^5, q^7; q^{12}\big]_\infty, \tag{1.19a}$$
$$\big[-q^3, -q, q^2; -q^3\big]_\infty - \big[-q^3, q, -q^2; -q^3\big]_\infty = 2q\big[q^{12}, q, q^{11}; q^{12}\big]_\infty. \tag{1.19b}$$

Observe that the two identities [11, Eqs. 4 and 25]:

$$\sum_{m=0}^{\infty} \frac{(q;q^2)_m}{(q^4;q^4)_m} q^{m^2} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \big[q^2, -q, -q; q^2\big]_\infty, \tag{1.20a}$$
$$\sum_{m=0}^{\infty} \frac{(-q;q^2)_m}{(q^4;q^4)_m} q^{m^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \big[q^6, q^3, q^3; q^6\big]_\infty \tag{1.20b}$$

may be unified as

$$\sum_{m=0}^{\infty} \varepsilon^m \frac{(q;q^2)_m}{(q^4;q^4)_m} q^{m^2} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \begin{cases} [q^2, -q, -q; q^2]_\infty, & \varepsilon = +1, \\ [q^6, -q^3, -q^3; q^6]_\infty, & \varepsilon = -1, \end{cases} \tag{1.21}$$

where (1.20a) follows easily from the special case of the q-Gauss theorem (1.8):

$${}_2\phi_1\left[\begin{matrix} q, & -M \\ & -q^2 \end{matrix} \,\Big|\, q^2; q/M\right] = \left[\begin{matrix} -q, & q^2/M \\ -q^2, & q/M \end{matrix} \,\Big|\, q^2\right]_\infty.$$

Then two equations (1.19a) and (1.19b) together with the equivalent products

$$\varepsilon = +1 \quad \Rightarrow \quad \big[-q^3, -q, q^2; -q^3\big]_\infty = \big(q;q^2\big)_\infty \big(-q^2;q^2\big)_\infty \big[q^2, -q, -q; q^2\big]_\infty,$$
$$\varepsilon = -1 \quad \Rightarrow \quad \big[-q^3, q, -q^2; -q^3\big]_\infty = \big(q;q^2\big)_\infty \big(-q^2;q^2\big)_\infty \big[q^6, -q^3, -q^3; q^6\big]_\infty;$$

give rise to the following identities of Rogers–Ramanujan type [11, Eqs. 53 and 55]:

$$\sum_{m=0}^{\infty} \frac{(q;q^2)_{2m}}{(q^4;q^4)_{2m}} q^{4m^2} = \frac{[q^{12}, q^5, q^7; q^{12}]_\infty}{(q^4;q^4)_\infty}, \tag{1.22a}$$
$$\sum_{m=0}^{\infty} \frac{(q;q^2)_{1+2m}}{(q^4;q^4)_{1+2m}} q^{4m(m+1)} = \frac{[q^{12}, q, q^{11}; q^{12}]_\infty}{(q^4;q^4)_\infty}. \tag{1.22b}$$

2. Trisection of Jacobi-triple products

For a natural number a and an integer c with $|c| \leqslant a$, the Jacobi triple product identity asserts that

$$\sum_k q^{ak^2+ck} = \big[q^{2a}, -q^{a+c}, -q^{a-c}; q^{2a}\big]_\infty = \Theta_0 + q^{a+c}\Theta_1 + q^{a-c}\Theta_{-1}$$

where the Θ-functions are defined through the trisection of the series:

$$\Theta_0[a,c] := \sum_k q^{9ak^2+3ck} = \left[q^{18a}, -q^{9a+3c}, -q^{9a-3c}; q^{18a}\right]_\infty,$$
$$\Theta_1[a,c] := \sum_k q^{9ak^2+6ak+3ck} = \left[q^{18a}, -q^{3a-3c}, -q^{15a+3c}; q^{18a}\right]_\infty,$$
$$\Theta_{-1}[a,c] := \sum_k q^{9ak^2-6ak+3ck} = \left[q^{18a}, -q^{3a+3c}, -q^{15a-3c}; q^{18a}\right]_\infty.$$

2.1. Case $c = a$

The corresponding Θ-functions become

$$U(a) := \Theta_0[a,a] = \left[q^{18a}, -q^{6a}, -q^{12a}; q^{18a}\right]_\infty,$$
$$V(a) := \Theta_1[a,a]/2 = \left[q^{18a}, -q^{18a}, -q^{18a}; q^{18a}\right]_\infty.$$

According to the Jacobi triple and quintuple product identities, we get the relations

$$U(a) + q^{2a}V(a) = \frac{(q^{2a};q^{2a})_\infty}{(-q^{2a};q^{2a})_\infty}\left[-q^{2a};q^{2a}\right]_\infty^3,$$
$$U(a) - 2q^{2a}V(a) = \frac{(q^{2a};q^{2a})_\infty}{(-q^{2a};q^{2a})_\infty}\left[-q^{6a};q^{6a}\right]_\infty,$$

which lead us to the following two equations

$$2\left[-q^{2a};q^{2a}\right]_\infty^3 + \left[-q^{6a};q^{6a}\right]_\infty = 3\frac{(-q^{2a};q^{2a})_\infty}{(q^{2a};q^{2a})_\infty}\left[q^{18a}, -q^{6a}, -q^{12a}; q^{18a}\right]_\infty,$$
$$\left[-q^{2a};q^{2a}\right]_\infty^3 - \left[-q^{6a};q^{6a}\right]_\infty = 3q^{2a}\frac{(-q^{2a};q^{2a})_\infty}{(q^{2a};q^{2a})_\infty}\left[q^{18a}, -q^{18a}, -q^{18a}; q^{18a}\right]_\infty.$$

They are essentially equivalent to the equations specified with $a = 1/2$:

$$2(-q;q)_\infty^3 + \left(-q^3;q^3\right)_\infty = 3\frac{(-q;q)_\infty}{(q;q)_\infty}\left[q^9, -q^3, -q^6; q^9\right]_\infty, \tag{2.1a}$$
$$(-q;q)_\infty^3 - \left(-q^3;q^3\right)_\infty = 3q\frac{(-q;q)_\infty}{(q;q)_\infty}\left[q^9, -q^9, -q^9; q^9\right]_\infty. \tag{2.1b}$$

With $\omega \neq 1$ being a cubic root of unity, we can specify the q-Gauss summation theorem (1.8) as

$${}_2\phi_1\left[\begin{matrix}\omega, & \omega^2\\ & -q\end{matrix}\,\middle|\, q; -q\right] = \frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}.$$

Then the last two equations allow us to compute

$$2 + {}_2\phi_1\left[\begin{matrix}\omega, & \omega^2\\ & -q\end{matrix}\,\middle|\, q; -q\right] \quad \text{and} \quad 1 - {}_2\phi_1\left[\begin{matrix}\omega, & \omega^2\\ & -q\end{matrix}\,\middle|\, q; -q\right].$$

The resulting equations can be stated as the following q-series identities:

$$1 + \sum_{m=1}^{\infty}(-1)^m\frac{(q^3;q^3)_{m-1}q^m}{(q;q)_{m-1}(q^2;q^2)_m} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left[q^9, -q^3, -q^6; q^9\right]_\infty, \tag{2.2a}$$
$$\sum_{m=0}^{\infty}(-1)^m\frac{(q^3;q^3)_m q^m}{(q;q)_m(q^2;q^2)_{1+m}} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left[q^9, -q^9, -q^9; q^9\right]_\infty. \tag{2.2b}$$

2.2. *Case* $c = 0$

The corresponding Θ-functions read as

$$\mathcal{U}(a) := \Theta_0[a, 0] = \left[q^{18a}, -q^{9a}, -q^{9a}; q^{18a}\right]_\infty,$$
$$\mathcal{V}(a) := \Theta_1[a, 0] = \left[q^{18a}, -q^{3a}, -q^{15a}; q^{18a}\right]_\infty.$$

Applying again the Jacobi triple and quintuple product identities, we can factorize the following linear combinations:

$$\mathcal{U}(a) + 2q^a\mathcal{V}(a) = \frac{(q^{2a}; q^{2a})_\infty}{(-q^a; q^{2a})_\infty}\left[-q^a; q^{2a}\right]^3_\infty,$$
$$\mathcal{U}(a) - q^a\mathcal{V}(a) = \frac{(q^{2a}; q^{2a})_\infty}{(-q^a; q^{2a})_\infty}\left[-q^{3a}; q^{6a}\right]_\infty,$$

which lead us to the two factorization equations:

$$\left[-q^a; q^{2a}\right]^3_\infty + 2\left[-q^{3a}; q^{6a}\right]_\infty = 3\frac{(-q^a; q^{2a})_\infty}{(q^{2a}; q^{2a})_\infty}\left[q^{18a}, -q^{9a}, -q^{9a}; q^{18a}\right]_\infty,$$
$$\left[-q^a; q^{2a}\right]^3_\infty - \left[-q^{3a}; q^{6a}\right]_\infty = 3q^a\frac{(-q^a; q^{2a})_\infty}{(q^{2a}; q^{2a})_\infty}\left[q^{18a}, -q^{3a}, -q^{15a}; q^{18a}\right]_\infty.$$

They are essentially equivalent to the equations specified with $a = 1$:

$$\left(q; q^2\right)^3_\infty + 2\left(q^3; q^6\right)_\infty = 3\frac{(q; q^2)_\infty}{(q^2; q^2)_\infty}\left[q^{18}, q^9, q^9; q^{18}\right]_\infty, \tag{2.3a}$$
$$\left(q^3; q^6\right)_\infty - \left(q; q^2\right)^3_\infty = 3q\frac{(q; q^2)_\infty}{(q^2; q^2)_\infty}\left[q^{18}, q^3, q^{15}; q^{18}\right]_\infty. \tag{2.3b}$$

For the same cubic root ω of unity, specify the q-Gauss theorem (1.8) as

$${}_2\phi_1\left[\begin{matrix}\omega, & \omega^2 \\ & q\end{matrix}\,\middle|\, q^2; q\right] = \frac{(q^3; q^6)_\infty}{(q; q^2)^3_\infty}.$$

Then from the linear combinations

$$1 + 2\,{}_2\phi_1\left[\begin{matrix}\omega, & \omega^2 \\ & q\end{matrix}\,\middle|\, q^2; q\right] \quad \text{and} \quad -1 + {}_2\phi_1\left[\begin{matrix}\omega, & \omega^2 \\ & q\end{matrix}\,\middle|\, q^2; q\right]$$

we can establish the following q-series identities:

$$1 + 2\sum_{m=1}^{\infty}\frac{(q^6; q^6)_{m-1}q^m}{(q; q)_{2m}(q^2; q^2)_{m-1}} = \frac{(-q; q)_\infty}{(q; q)_\infty}\left[q^{18}, q^9, q^9; q^{18}\right]_\infty, \tag{2.4a}$$
$$\sum_{m=0}^{\infty}\frac{(q^6; q^6)_m q^m}{(q; q)_{2+2m}(q^2; q^2)_m} = \frac{(-q; q)_\infty}{(q; q)_\infty}\left[q^{18}, q^3, q^{15}; q^{18}\right]_\infty. \tag{2.4b}$$

3. Bisection of identities due to Gessel–Stanton

Applying the bisection method to the identities of Gessel–Stanton type, this section will further rederive six Rogers–Ramanujan type identities.

3.1.

Gessel and Stanton [7, Eq. 3.10] found the following strange identity:

$$\sum_{k=0}^{\infty}\frac{(-q^{4+4k};q^8)_\infty}{(q^4;q^4)_k}q^{3k^2-2k}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^5,q^2,q^3;q^5\big]_\infty \tag{3.1a}$$

$$=\frac{[q^{12},q^8;q^{20}]_\infty}{(q^4;q^4)_\infty}\big[q^{10},-q,-q^9;q^{10}\big]_\infty. \tag{3.1b}$$

In view of (1.13a) and (1.13b), applying the bisection method to (3.1a)–(3.1b) and then performing the base replacements $q\to q^{1/8}$ and $q\to q^{1/4}$ respectively for the series consisting of even and odd terms, we obtain the following identities of Rogers–Ramanujan type [11, Eqs. 46 and 97]

$$\sum_{m=0}^{\infty}\frac{(-q;q)_m}{(q;q)_{2m}}q^{m^2+\binom{m}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^5,q,q^4;q^5\big]_\infty\times\big[q^7,q^3;q^{10}\big]_\infty, \tag{3.2a}$$

$$\sum_{m=0}^{\infty}\frac{(-q;q^2)_{1+m}}{(q^2;q^2)_{1+2m}}q^{3m^2+2m}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^{10},q^3,q^7;q^{10}\big]_\infty\times\big[q^{16},q^4;q^{20}\big]_\infty. \tag{3.2b}$$

We remark that the right member of the last equation is simpler than the original expression of two-terms difference due to Slater [11, Eq. 97].

3.2.

Gessel and Stanton [7, Eq. 3.11] found also another strange summation formula:

$$\sum_{k=0}^{\infty}\frac{(-q^{8+4k};q^8)_\infty}{(q^4;q^4)_k}q^{3k^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^5,q,q^4;q^5\big]_\infty \tag{3.3a}$$

$$=\frac{[q^{16},q^4;q^{20}]_\infty}{(q^4;q^4)_\infty}\big[q^{10},-q^3,-q^7;q^{10}\big]_\infty. \tag{3.3b}$$

By means of (1.15a) and (1.15b), we can similarly derive from the bisection series of the last equation the following two identities of Rogers–Ramanujan type [11, Eqs. 44 and 100]

$$\sum_{m=0}^{\infty}\frac{(-q;q)_m}{(q;q)_{1+2m}}q^{3\binom{1+m}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}\big[q^5,q^2,q^3;q^5\big]_\infty\times\big[q^9,q;q^{10}\big]_\infty, \tag{3.4a}$$

$$\sum_{m=0}^{\infty}\frac{(-q;q^2)_m}{(q^2;q^2)_{2m}}q^{3m^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big[q^{10},q,q^9;q^{10}\big]_\infty\times\big[q^{12},q^8;q^{20}\big]_\infty, \tag{3.4b}$$

where the last formula corrects a misprint appeared in Slater [11, Eq. 100].

3.3.

For two sequences defined by

$$\lambda_k:=\frac{q^{3k^2-k}}{(q^2;q^2)_{k-1}(q^2;q^4)_k}\quad\text{and}\quad\mu_k:=\frac{q^{3k^2-2k}}{(q;q^2)_k(q^4;q^4)_{k-1}}$$

consider the following two equations obtained respectively from (3.2a) with $q\to q^2$ and (3.2b):

$$\sum_{k\geqslant 0}\frac{q^{3k^2-k}}{(q^2;q^2)_k(q^2;q^4)_k}=\sum_{k\geqslant 0}\frac{q^{3k^2+k}}{(q^2;q^2)_k(q^2;q^4)_{k+1}}+\sum_{k\geqslant 0}\frac{q^{3k^2-k}(1-q^{2+4k}-q^{2k})}{(q^2;q^2)_k(q^2;q^4)_{k+1}}$$

$$=\sum_{k\geqslant 0}\frac{q^{3k^2+k}}{(q^2;q^2)_k(q^2;q^4)_{k+1}}+\sum_{k\geqslant 0}\{\lambda_k-\lambda_{k+1}\},$$

$$\sum_{k\geqslant 0}\frac{q^{3k^2+2k}}{(q;q^2)_{k+1}(q^4;q^4)_k}=\sum_{k\geqslant 0}\frac{q^{3k^2-2k}}{(q;q^2)_k(q^4;q^4)_k}+\sum_{k\geqslant 0}\frac{q^{3k^2-2k}(q^{4k}+q^{1+2k}-1)}{(q;q^2)_{k+1}(q^4;q^4)_k}$$
$$=\sum_{k\geqslant 0}\frac{q^{3k^2-2k}}{(q;q^2)_k(q^4;q^4)_k}+\sum_{k\geqslant 0}\{\mu_{k+1}-\mu_k\}.$$

Then the telescoping method shows that the two sums on λ_k and μ_k just displayed result in zero, in view of

$$\lambda_0=\lim_{k\to\infty}\lambda_k=0 \quad\text{and}\quad \mu_0=\lim_{k\to\infty}\mu_k=0.$$

We therefore derive from (3.2a) and (3.2b) the following two further identities of Rogers–Ramanujan type [11, Eqs. 62 and 95]:

$$\sum_{m=0}^{\infty}\frac{(-q;q)_m}{(q;q)_{1+2m}}q^{m^2+\binom{1+m}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}\left[q^5,q,q^4;q^5\right]_\infty\times\left[q^7,q^3;q^{10}\right]_\infty, \tag{3.5a}$$

$$\sum_{m=0}^{\infty}\frac{(-q;q^2)_m}{(q^2;q^2)_{2m}}q^{3m^2-2m}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[q^{10},q^3,q^7;q^{10}\right]_\infty\times\left[q^{16},q^4;q^{20}\right]_\infty. \tag{3.5b}$$

Interestingly, if we consider the following combination:

$$\left(-q^4;q^8\right)_\infty\times\text{Eq. (3.5a)}|_{q\to q^8}+q\left(-q^8;q^8\right)_\infty\times\text{Eq. (3.5b)}|_{q\to q^4}, \tag{3.6}$$

then we find a third strange identity similar to Gessel and Stanton [7, Eqs. 3.10 and 3.11]:

$$\sum_{k=0}^{\infty}\frac{(-q^{8+4k};q^8)_\infty}{(q^4;q^4)_k}q^{3k^2-4k+1}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[q^5,q^2,q^3;q^5\right]_\infty$$

where the right member corresponding to (3.6) has been simplified, according to the quintuple product identity (0.2a)–(0.2b), as follows:

$$q\left[q^{40},q^{12},q^{28};q^{40}\right]_\infty\left[q^{64},q^{16};q^{80}\right]_\infty+\left[q^{40},q^8,q^{32};q^{40}\right]_\infty\left[q^{56},q^{24};q^{80}\right]_\infty$$
$$=\left[q^{10},-q,-q^9;q^{10}\right]_\infty\left[q^{12},q^8;q^{20}\right]_\infty=(-q;q)_\infty\times\left[q^5,q^2,q^3;q^5\right]_\infty.$$

Acknowledgment

The authors are sincerely grateful to the anonymous referee for the careful reading and useful suggestions, which have improved the manuscript to the present version.

References

[1] K. Alladi, On the modified convergence of some continued fractions of Rogers–Ramanujan type, J. Combin. Theory Ser. A 65 (2) (1994) 214–245.
[2] G.E. Andrews, J.P.O. Santos, Rogers–Ramanujan type identities for partitions with attached odd parts, Ramanujan J. 1 (1997) 91–99.
[3] W.N. Bailey, On the simplification of some identities of the Rogers–Ramanujan type, Proc. London Math. Soc. (3) 1 (1951) 217–221.
[4] W.Y.C. Chen, W. Chu, N.S.S. Gu, Finite form of the quintuple product identity, J. Combin. Theory Ser. A 113 (1) (2006) 185–187.
[5] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge University Press, Cambridge, 2004.
[6] I. Gessel, D. Stanton, Applications of q-Lagrange inversion to basic hypergeometric series, Trans. Amer. Math. Soc. 277 (1) (1983) 173–201.
[7] I. Gessel, D. Stanton, Another family of q-Lagrange inversion formulas, Rocky Mountain J. Math. 16 (1986) 373–384.
[8] F.H. Jackson, Examples of a generalization of Euler's transformation for power series, Messenger of Math. 57 (1928) 169–187.
[9] C.G.J. Jacobi, Fundamenta Nova Theoriae Functionum Ellipticarum, Fratrum Bornträger Regiomonti, 1829; Gesammelte werke, Erster Band, G. Reimer, Berlin, 1881.
[10] L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25 (1894) 318–343.
[11] L.J. Slater, Further Identities of the Rogers–Ramanujan type, Proc. London Math. Soc. (2) 54 (1952) 147–167.