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Existence of global solutions to the Caginalp phase-field system with dynamic boundary conditions and singular potentials

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Abstract

Our aim in this article is to prove the global (in time) existence of solutions to a Caginalp phase-field system with dynamic boundary conditions and a singular potential. The main difficulty is to prove that the solutions are strictly separated from the singular values of the potential. This is achieved by studying an auxiliary elliptic problem.

Keywords: Caginalp system; Dynamic boundary conditions; Singular potentials; Existence of solutions

1. Introduction

We consider, in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$, the phase-field system

$$\begin{cases} \eta\partial_t w - \Delta w = -\partial_t u, & \text{in } \Omega,\ t > 0, \\ \partial_t u - \Delta u + f(u) = w, & \text{in } \Omega,\ t > 0, \\ \left.\dfrac{\partial w}{\partial \nu}\right|_\Gamma = 0, \quad u|_\Gamma = \psi, & t > 0, \\ \partial_t\psi - \Delta_\Gamma\psi + \lambda\psi + g(\psi) + \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma,\ t > 0, \\ u|_{t=0} = u_0, \quad w|_{t=0} = w_0, & \text{in } \Omega, \\ \psi|_{t=0} = \psi_0, & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where ν is the unit outer normal to the boundary, $\eta \in (0, 1)$, $\lambda > 0$ and Δ_Γ is the Laplace–Beltrami operator. Moreover, w represents the temperature, while u is the order parameter with trace ψ on Γ. The function g is a smooth function

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in $\mathbb{R}$, while f is smooth in $(-1,1)$, but tends to $\pm\infty$ approaching ± 1 (i.e., it is a singular potential). Thus, we say that $z_0 = (u_0, \psi_0, w_0)$ is an admissible initial datum if $\|u_0\|_{L^\infty(\Omega)} < 1$.

This system, proposed in [6] in order to model melting–solidification phenomena in certain classes of materials, has been extensively studied, for various types of boundary conditions and for regular potentials f, see, e.g., [3–6, 9,14,16,21,30] and the references therein. In particular, one has satisfactory results on the existence and uniqueness of solutions, the existence of finite-dimensional attractors and the convergence of solutions to steady states. We note however that, for regular potentials, it is not known whether the order parameter remains in the physically relevant interval $[-1,1]$ in general (see however [1] and [2]).

Now, singular potentials f are also important from a physical point of view; in particular, we have in mind the following thermodynamically relevant logarithmic potential:

$$f(s) = -\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s}, \quad s \in (-1,1),\ 0 < \kappa_0 < \kappa_1.$$

Such potentials, in the case of Dirichlet boundary conditions for both w and u, were considered in [17]; in particular, the existence and uniqueness of solutions and the existence of exponential attractors were proved in [17]. The convergence of solutions to steady states was proved in [18] for mixed Dirichlet (for the temperature) and Neumann (for the order parameter) boundary conditions. The case of Neumann boundary conditions, for both w and u, was treated in [7]. We can note that, contrary to regular potentials, such singular potentials allow to prove that the order parameter remains strictly between -1 and 1, as it is expected from the physical point of view.

In this article, we supplement the equations with the so-called dynamic boundary conditions for the order parameter (in the sense that the kinetics, i.e., the time derivative of the order parameter, appears explicitly in the boundary conditions). Such boundary conditions have been proposed by physicists (see [10,11] and [19]; see also [12]) in order to account for the interactions with the walls in confined systems. In particular, the Cahn–Hilliard equation, endowed with these boundary conditions, was studied in [9,12,15,22–25] and [28]. The Caginalp system, endowed with dynamic boundary conditions and with regular potentials, was considered in [9,13] and [14].

Our aim in this article is to prove the global (in time) existence of solutions for general nonlinear (smooth) functions g in the dynamic boundary conditions. The same issue, followed by an asymptotic analysis, was addressed in [8], under the assumption that g is positive/negative close to the singularities ± 1 of the potential f. In particular, we could not consider in [8] constant functions g which naturally appear in the physical derivation of the dynamic boundary conditions, see [10,11] and [19].

To accomplish our purpose, it is essential to prove that any solution originating from an admissible initial datum stays away from the singularities of f. In [8], this goal was achieved by considering constant sub- and super-solutions which, without the sign assumptions on g, are no longer adequate. We are able to overcome this difficulty by studying a suitable elliptic problem with nonhomogeneous Dirichlet boundary conditions.

Assumptions and notation. Our key hypotheses are those on the nonlinearities f and g, namely,

$$f \in C^2(-1,1), \quad \lim_{s\to\pm 1} f(s) = \pm\infty, \quad \lim_{s\to\pm 1} f'(s) = +\infty, \tag{1.2}$$

$$\begin{gathered} g \in C^2(\mathbb{R}), \quad \liminf_{|s|\to+\infty} g'(s) \geqslant 0 \quad \text{and either} \quad g(s)s \geqslant \mu s^2 - \mu', \quad \forall s \in \mathbb{R}, \\ \mu > 0, \quad \mu' \geqslant 0, \quad \text{or} \quad g \equiv \text{Const.}, \end{gathered} \tag{1.3}$$

while $\eta \in (0,1)$ and $\lambda > 0$. In particular, there exist $K_1 > 0$ and $c \geqslant 0$ such that

$$f'(s) \geqslant -K_1, \quad -c \leqslant F(s) \leqslant f(s)s + c, \quad \forall s \in (-1,1), \tag{1.4}$$

where $F(s) = \int_0^s f(\tau)\,d\tau$ (see [17]). Moreover, it is possible to find $K_2 > 0$ such that

$$g'(s) \geqslant -K_2, \qquad \langle G(v) - g(v)v, 1\rangle_\Gamma \leqslant K_2 \|v\|_\Gamma^2, \quad \forall v \in L^2(\Gamma), \tag{1.5}$$

where $G(s) = \int_0^s g(\tau)\,d\tau$ (see [29]).

Before describing the phase space, we agree to denote the Lebesgue spaces of square summable functions in Ω and Γ by $(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|)$ and $(L^2(\Gamma), \langle\cdot,\cdot\rangle_\Gamma, \|\cdot\|_\Gamma)$, respectively. Then we often use the average

$$\langle w\rangle = \frac{1}{|\Omega|}\int_\Omega w\,dx, \quad \forall w \in L^1(\Omega),$$

to recover $\|w\|_{H^1(\Omega)}$, on account of the Friedrich's inequality

$$\|w - \langle w\rangle\| \leqslant c\|\nabla w\|, \quad \forall w \in H^1(\Omega),$$

where c is a positive constant which only depends on the domain Ω. In general, throughout the article, c stands for a positive constant which may also vary in the same line.

Finally, we recall that the admissible data (and solutions) stay away from the singularities of f: we thus introduce the notation

$$D[u] = \big(1 - \|u\|_{L^\infty(\Omega)}\big)^{-1}, \quad \forall u \in L^\infty(\Omega),\ \|u\|_{L^\infty(\Omega)} \neq 1.$$

Now, embodying the boundary condition for w in

$$H^2_N(\Omega) = \left\{ w \in H^2(\Omega)\colon \left.\frac{\partial w}{\partial \nu}\right|_\Gamma = 0 \right\},$$

we have at our disposal all the ingredients to define the phase space, namely,

$$\Phi = \big\{(u, \psi, w) \in H^2(\Omega) \times H^2(\Gamma) \times H^2_N(\Omega)\colon\ 0 < D[u] < +\infty,\ \psi = u|_\Gamma\big\},$$

endowed with the $H^2(\Omega) \times H^2(\Gamma) \times H^2_N(\Omega)$-norm. We denote by $\|\cdot\|_\Phi$ the norm on Φ.

2. The main result

Our main result is the following.

Theorem 2.1. *We assume that* (1.2) *and* (1.3) *hold. Then, for any initial datum $z_0 = (u_0, \psi_0, w_0) \in \Phi$, problem* (1.1) *possesses a unique solution $z(t) = (u(t), \psi(t), w(t)) \in \Phi$, for every $t \geqslant 0$.*

In order to prove this theorem, we first obtain several a priori estimates. To do so, we a priori assume that the first component $u(t)$ is separated from the singularities of f, i.e., that $\|u(t)\|_{L^\infty(\Omega)} < 1$, $\forall t \geqslant 0$. In particular, these estimates allow to prove that $u(t)$ is actually strictly separated from the singularities of f, i.e., that $\|u(t)\|_{L^\infty(\Omega)} \leqslant c < 1$, $\forall t \geqslant 0$.

First, repeating word by word the proof of [8, Theorem 3.1] (in that case, the sign hypotheses on g do not play any role), we deduce

Theorem 2.2. *Given any initial datum $z_0 = (u_0, \psi_0, w_0) \in \Phi$, every solution $z(t) = (u(t), \psi(t), w(t)) \in \Phi$ to* (1.1) *satisfies*

$$\begin{aligned}
&\big\|u(t)\big\|^2_{H^1(\Omega)} + \big\|\psi(t)\big\|^2_{H^1(\Gamma)} + \big\|w(t)\big\|^2_{H^2(\Omega)} + \big\|\partial_t u(t)\big\|^2 + \big\|\partial_t \psi(t)\big\|^2_\Gamma \\
&\quad + \int_0^t e^{-k(t-s)}\big[\big\|\partial_t u(s)\big\|^2_{H^1(\Omega)} + \big\|\partial_t \psi(s)\big\|^2_{H^1(\Gamma)} + \eta\big\|\partial_t w(s)\big\|^2_{H^1(\Omega)}\big]\,ds \\
&\leqslant Q_\eta\big(D[u_0], \|z_0\|_\Phi\big)e^{-kt} + C_\eta, \quad \forall t \geqslant 0,\ k > 0,
\end{aligned}$$

where the increasing function Q_η and the positive constant C_η depend on η.

The next task consists in obtaining estimates on u and ψ in $H^2(\Omega)$ and $H^2(\Gamma)$, respectively. These cannot be achieved directly, due to the singular values of the potential f, and we first need to derive L^∞-estimates on u and ψ.

However, the lack of sign assumptions on g prevents us from arguing as in [8, Theorem 3.2] to prove that, if the initial data are separated from ± 1, then u and ψ stay away from these values.

Theorem 2.3. *Given any $z_0 = (u_0, \psi_0, w_0) \in \Phi$, the first two components of any solution $z(t) = (u(t), \psi(t), w(t)) \in \Phi$ to* (1.1) *are strictly separated from the singularities of f, namely, there exists $\gamma \in (0, 1)$ such that*

$$\big\|u(t)\big\|_{L^\infty(\Omega)} \leqslant \gamma \quad \textit{and} \quad \big\|\psi(t)\big\|_{L^\infty(\Gamma)} \leqslant \gamma, \quad \forall t \geqslant 0.$$

Proof. Thanks to Theorem 2.2, there exists a constant $\beta > 0$ satisfying

$$\|w(t)\|_{L^\infty(\Omega)} \leqslant c\|w(t)\|_{H^2(\Omega)} \leqslant \beta, \quad \forall t \geqslant 0.$$

Next, possibly enlarging β, we fix $\alpha \in (0, 1)$ and $\delta > 0$ such that $[\alpha, \alpha + \delta] \subset [\|u_0\|_{L^\infty(\Omega)}, 1)$,

$$f(\alpha) = \beta \quad \text{and} \quad f(\alpha + \delta) - \beta \geqslant 0. \tag{2.1}$$

Our aim is to prove that $u \leqslant \alpha + \delta$. To do so, we consider the solution u_ε to a suitable elliptic problem in a proper domain $\Omega - \overline{\Omega_\varepsilon}$. Indeed, provided that Ω is smooth enough, we can consider, for any $\varepsilon > 0$ small (without loss of generality, we may assume that $\varepsilon \in (0, 1]$), the set $\Omega_\varepsilon = \{x \in \Omega\colon \operatorname{dist}(x, \Gamma) > \varepsilon\}$ and

$$x \in \Gamma \quad \Longleftrightarrow \quad x - \varepsilon\nu(x) \in \Gamma_\varepsilon = \partial\Omega_\varepsilon,$$

where $\nu(x)$ is the unit outer normal to Γ at x.

We then introduce the nonlinear elliptic problem

$$\begin{cases} -\Delta u_\varepsilon + f(u_\varepsilon) = \beta, & \text{in } \Omega - \overline{\Omega_\varepsilon}, \\ u_\varepsilon = \alpha, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = \alpha + \delta, & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Our goals are twofold. First, we prove that, for every ε, problem (2.2) admits a classical solution $\alpha \leqslant u_\varepsilon \leqslant \alpha + \delta$ in $\Omega - \overline{\Omega_\varepsilon}$. Then we prove that the normal derivative of u_ε tends to $+\infty$ as $\varepsilon \to 0^+$, namely,

Theorem 2.4. *Problem* (2.2) *admits a unique classical solution $u_\varepsilon \in [\alpha, \alpha + \delta]$. Furthermore, the normal derivative of u_ε satisfies*

$$\lim_{\varepsilon \to 0^+} \frac{\partial u_\varepsilon}{\partial \nu}\bigg|_\Gamma = +\infty.$$

The proof of this theorem will be given in the next section. Actually, we will be able to relax (2.1), still getting the existence of a classical solution $u_\varepsilon \in [\alpha, \alpha + \delta]$ to (2.2), under the assumption that ε is small enough, with the same limit for the normal derivative (note however that (2.1) is needed in the proof of Theorem 2.3).

If u_ε solves problem (2.2) with the aforementioned constants, it can be extended to Ω_ε by taking $u_\varepsilon = \alpha$ in Ω_ε. It is straightforward to check that $U = u - u_\varepsilon$ satisfies

$$\partial_t U - \Delta U + f(u) - f(u_\varepsilon) \leqslant 0, \quad \text{in } \Omega - \overline{\Omega_\varepsilon} \text{ and in } \Omega_\varepsilon,\ t > 0, \tag{2.3}$$

since the right-hand side reads $w - \beta$ in $\Omega - \overline{\Omega_\varepsilon}$ and $w - f(\alpha)$ in Ω_ε. On the boundary Γ, $\Psi = \psi - u_\varepsilon|_\Gamma$ satisfies $\Psi = U|_\Gamma$ and solves

$$\partial_t \Psi - \Delta_\Gamma \Psi + \lambda\Psi + g(\psi) - g(u_\varepsilon) + \frac{\partial U}{\partial \nu} = G, \quad \text{on } \Gamma,\ t > 0, \tag{2.4}$$

where $G = -\lambda(\alpha + \delta) - g(\alpha + \delta) - \frac{\partial u_\varepsilon}{\partial \nu}$ (recall that $u_\varepsilon|_\Gamma = \alpha + \delta$ and note that $\Delta_\Gamma u_\varepsilon = 0$ on Γ). Due to Theorem 2.4, $G \leqslant 0$, provided that we fix ε small enough, since we can make the normal derivative of u_ε as large as we want (uniformly with respect to $x \in \Gamma$, see Remark 3.1 below).

Multiplying then (2.3) by $U_+ = \max\{U, 0\}$ and (2.4) by $\Psi_+ = \max\{\Psi, 0\}$, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|U_+\|^2 + \|\Psi_+\|_\Gamma^2\big) + \|\nabla U_+\|^2 + \|\nabla \Psi_+\|_\Gamma^2 + \lambda\|\Psi_+\|_\Gamma^2 \leqslant K_1\|U_+\|^2 + K_2\|\Psi_+\|_\Gamma^2,$$

on account of the inequality

$$-\big\langle f(u) - f(u_\varepsilon), U_+\big\rangle - \big\langle g(\psi) - g(u_\varepsilon|_\Gamma), \Psi_+\big\rangle_\Gamma \leqslant K_1\|U_+\|^2 + K_2\|\Psi_+\|_\Gamma^2$$

which follows from assumptions (1.2) and (1.3). Since $U_+(0) = 0$ in Ω and $\Psi_+(0) = 0$ on Γ, Gronwall's lemma yields $u \leqslant u_\varepsilon \leqslant \alpha + \delta$ in $\Omega \times [0, +\infty)$ and $\psi \leqslant u_\varepsilon|_\Gamma \leqslant \alpha + \delta$ on $\Gamma \times [0, +\infty)$. The lower estimates are proved analogously by considering the elliptic problem

$$\begin{cases} -\Delta v_\varepsilon + f(v_\varepsilon) = -\beta', & \text{in } \Omega - \overline{\Omega_\varepsilon}, \\ v_\varepsilon = -\alpha', & \text{on } \Gamma_\varepsilon, \\ v_\varepsilon = -\alpha' - \delta', & \text{on } \Gamma, \end{cases}$$

for proper positive constants α', β' and δ'. Mimicking the proof of Theorem 2.4, we show that this problem possesses a unique solution v_ε such that $-\alpha'-\delta' \leqslant v_\varepsilon \leqslant -\alpha'$ and

$$\lim_{\varepsilon\to 0^+} \frac{\partial v_\varepsilon}{\partial \nu}\bigg|_\Gamma = -\infty.$$

We omit the end of the proof of Theorem 2.3, the arguments being the same as above. □

Lemma 2.5. *There exists $M_\gamma > 0$ depending on the constant γ introduced in Theorem* 2.3 (*and, through γ, on $D[u_0]$ and $\|z_0\|_\Phi$*) *such that the first two components of any solution $z(t) = (u(t), \psi(t), w(t)) \in \Phi$ to* (1.1) *satisfy*

$$\|u(t)\|_{H^2(\Omega)} + \|\psi(t)\|_{H^2(\Gamma)} \leqslant M_\gamma, \quad \forall t \geqslant 0.$$

To prove this lemma, it is sufficient to write the equations for u and ψ as a suitable elliptic system and then apply [22, Lemma A.1].

As far as the uniqueness and the continuous dependence of the solutions on the initial data are concerned, we have, arguing as in [7, Lemma 3.1] and [8, Lemma 3.3],

Theorem 2.6. *Under the assumptions of Theorem* 2.1, *if $z_i(t) = (u_i(t), \psi_i(t), w_i(t)) \in \Phi$ is a solution to* (1.1) *departing from $z_{0i} = (u_{0i}, \psi_{0i}, w_{0i}) \in \Phi$, $i = 1, 2$, there holds, $\forall t \geqslant 0$,*

$$\begin{aligned} &\|u_1(t) - u_2(t)\|^2 + \|\psi_1(t) - \psi_2(t)\|_\Gamma^2 + \|w_1(t) - w_2(t)\|^2 \\ &\quad \leqslant C_1\big(\|u_{01} - u_{02}\|^2 + \|\psi_{01} - \psi_{02}\|_\Gamma^2 + \|w_{01} - w_{02}\|^2\big)e^{C_2 t}, \end{aligned} \tag{2.5}$$

where the constants $C_1, C_2 > 0$ depend on η, but are independent of the initial data, and

$$\begin{aligned} &\|u_1(t) - u_2(t)\|_{H^1(\Omega)}^2 + \|\psi_1(t) - \psi_2(t)\|_{H^1(\Gamma)}^2 + \|w_1(t) - w_2(t)\|_{H^1(\Omega)}^2 \\ &\qquad + \big\langle \eta\big[w_1(t) - w_2(t)\big] + u_1(t) - u_2(t)\big\rangle^2 \\ &\quad \leqslant C_1\big(\|u_{01} - u_{02}\|_{H^2(\Omega)}^2 + \|\psi_{01} - \psi_{02}\|_{H^2(\Gamma)}^2 + \|w_{01} - w_{02}\|_{H^1(\Omega)}^2\big)e^{C_2 t}, \end{aligned} \tag{2.6}$$

where the constants $C_1, C_2 > 0$ depend on $D[u_{0i}]$, $\|z_{0i}\|_\Phi$, $i = 1, 2$, and on η.

Finally, the existence of a solution can be proved by arguing as in [8, Theorem 4.1]. To do so, we consider the same problem, in which the singular potential is replaced by the continuous function

$$h(s) = \begin{cases} s + \gamma + f(-\gamma), & s \in (-\infty, -\gamma], \\ f(s), & s \in [-\gamma, \gamma], \\ s - \gamma + f(\gamma), & s \in [\gamma, +\infty), \end{cases}$$

where γ is the positive constant which appears in Theorem 2.3.

This function meets all the requirements of [14] to have the existence of a regular solution $z_h(t) = (u_h(t), \psi_h(t), w_h(t))$, namely, $z_h(t)$ belongs to $W^{(1,2),p}(Q_T) \times W^{(1,2),p}(\partial Q_T) \times W^{(1,2),p}(Q_T)$, $Q_T = (0, T) \times \Omega$, $\forall T \geqslant 0$, $p \in (3, \frac{10}{3})$ (here, $W^{(1,2),p}(Q_T)$ consists of the functions belonging to $L^p(Q_T)$ whose first-order time and first- and second-order space derivatives belong to $L^p(Q_T)$), except that $h \notin C^1(\mathbb{R})$; one can however make a further regularization. To be more precise, one can consider a regularized function h_ξ, $\xi > 0$ small, which, e.g., coincides with h on $(-\infty, -\gamma - \xi]$, $[-\gamma + \xi, \gamma - \xi]$ and $[\gamma + \xi, +\infty)$ and which satisfies (1.4) and then pass to the limit.

Noting that $W^{(1,2),p}(Q_T)$ is continuously embedded into $C(\overline{Q_T})$ and that $\|u_0\|_{L^\infty(\Omega)} < 1$, it follows that $\|u_h(t)\|_{L^\infty(\Omega)} < 1$, $\forall t \in [0, t_\star)$, for some $t_\star > 0$ possibly small. Then, noting that f and h satisfy (1.4), for the same constants K_1, c (see [8, Lemma 4.1]), we can derive the same estimates as above on $z_h(t)$, with the very same constants, for $t \in [0, t_\star)$. Indeed, we can note that the bound on w in the proofs of Theorems 2.2 and 2.3 (and, thus, β) only depends on these constants. Strictly speaking, we should obtain a bound on w_{h_ξ} (i.e., perform estimates on the solution to the regularized problem) and then pass to the limit $\xi \to 0^+$; this procedure actually yields the desired bound on w_h, since we can choose the constants $K_{1,\xi}$ and c_ξ for h_ξ in (1.4) such that they converge to K_1 and c, respectively. We thus deduce that $\|u_h(t)\|_{L^\infty(\Omega)} < \gamma$, $\forall t \in [0, t_\star)$ (we also note that $\alpha + \delta \leqslant \gamma$ and $-\gamma \leqslant -\alpha' - \delta'$, so that (2.1) (respectively, the analogous property for the lower bound) still holds for h, for the same values of α and δ (respectively, of α' and δ')).

Thus, $z_h(t),\ t \in [0, t_\star)$, is a local solution to (1.1). Finally, it follows from standard arguments, based on the above a priori estimates and Gronwall's lemma, that this solution is actually global, which finishes the proof of existence.

Remark 2.7. Note that Lemma 2.5 does not prevent M_γ from blowing up as $\|u_0\|_{L^\infty(\Omega)} \to 1$. Thus, necessary conditions for an asymptotic analysis are proper dissipative estimates which are independent of the L^∞-norm of the initial data. Unfortunately, without the sign assumptions on g, we can no longer argue as in [8, Section 5], where suitable super- and sub-solutions were obtained by joining a decreasing linear function taking the value 1 at time zero (this certainly dominates the L^∞-norm of any admissible initial datum) to a proper constant; in our case, it seems much more difficult to link a super(sub)-solution starting at 1 to some other super(sub)-solution.

Remark 2.8. Arguing as in [7,8,17,22], Theorem 2.6 ensures (by continuity) the existence, as well as the uniqueness, of solutions with initial data belonging to the closure L of Φ in $L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$, namely,

$$L = \big\{(u, \psi, w) \in L^\infty(\Omega) \times L^2(\Gamma) \times L^2(\Omega)\colon\ \|u\|_{L^\infty(\Omega)} \leqslant 1\big\}.$$

In particular, this allows to consider initial data which contain the pure states (i.e., u_0 can take the values ± 1). However, we have not been able to prove that the solutions mix instantaneously (i.e., $\|u(t)\|_{L^\infty(\Omega)} < 1$ as soon as $t > 0$), as it is the case for classical boundary conditions (see [8] and [17]).

3. Proof of Theorem 2.4

We replace f by some potential $\phi = \phi_\delta \in C^1(\mathbb{R})$ which coincides with f on $[\alpha, \alpha+\delta]$ in (2.2), that is, we consider the problem

$$\begin{cases} -\Delta u_\varepsilon + \phi(u_\varepsilon) = \beta, & \text{in } \Omega - \overline{\Omega_\varepsilon}, \\ u_\varepsilon = \alpha, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = \alpha + \delta, & \text{on } \Gamma. \end{cases} \tag{3.1}$$

We still denote by u_ε the solution (belonging to $[\alpha, \alpha+\delta]$) to the corresponding system, since, eventually, it will be proved to be the solution to (2.2).

Thanks to (2.1), the functions $\overline{u}_\varepsilon = \alpha + \delta$ and $\underline{u}_\varepsilon = \alpha$ are super- and sub-solutions to (3.1), respectively. Thus, problem (3.1) admits a classical solution u_ε which takes values in $[\alpha, \alpha+\delta]$ (see, e.g., [26]): this amounts to saying that u_ε actually solves (2.2).

Since, provided that β is large enough, we can assume that $\phi' = f' \geqslant 0$ on $[\alpha, \alpha+\delta]$, problem (3.1) (and, analogously, problem (2.2)) has a unique solution taking values in $[\alpha, \alpha+\delta]$.

Actually, (2.1) can be relaxed, asking only $f' \geqslant 0$ on $[\alpha, \alpha+\delta]$, provided that ε is small enough. Indeed, the regularity of the domain ensures that the distance $d_\varepsilon(x) = \operatorname{dist}(x, \Gamma)$ is smooth in $\Omega - \overline{\Omega_\varepsilon}$. Then, having set

$$\underline{\theta}_\varepsilon(s) = \frac{\delta}{2\varepsilon^2}s^2 - \frac{3\delta}{2\varepsilon}s + \alpha + \delta \quad \text{and} \quad \overline{\theta}_\varepsilon(s) = -\frac{\delta}{2\varepsilon^2}s^2 - \frac{\delta}{2\varepsilon}s + \alpha + \delta,$$

the functions $\underline{\psi}_\varepsilon(x) = \underline{\theta}_\varepsilon(d_\varepsilon(x))$ and $\overline{\psi}_\varepsilon(x) = \overline{\theta}_\varepsilon(d_\varepsilon(x))$ satisfy $\underline{\psi}_\varepsilon, \overline{\psi}_\varepsilon \in C^2(\overline{\Omega - \overline{\Omega_\varepsilon}})$. Furthermore,

$$\underline{\psi}_\varepsilon|_{\Gamma_\varepsilon} = \overline{\psi}_\varepsilon|_{\Gamma_\varepsilon} = \alpha \quad \text{and} \quad \underline{\psi}_\varepsilon|_{\Gamma} = \overline{\psi}_\varepsilon|_{\Gamma} = \alpha + \delta,$$

with $\underline{\psi}_\varepsilon, \overline{\psi}_\varepsilon \in [\alpha, \alpha+\delta]$. It is straightforward to check that

$$\underline{\theta}'_\varepsilon(s) = \frac{\delta}{\varepsilon^2}s - \frac{3\delta}{2\varepsilon} \quad \text{and} \quad \overline{\theta}'_\varepsilon(s) = -\frac{\delta}{\varepsilon^2}s - \frac{\delta}{2\varepsilon}$$

both remain in $[-\frac{3\delta}{2\varepsilon}, -\frac{\delta}{2\varepsilon}]$ and

$$\underline{\theta}''_\varepsilon(s) = \frac{\delta}{\varepsilon^2} > 0 \quad \text{and} \quad \overline{\theta}''_\varepsilon(s) = -\frac{\delta}{\varepsilon^2} < 0.$$

Moreover, extending an argument from [20] (see also [27]) to the 3D case, we can see that

$$|\nabla d_\varepsilon| = 1, \qquad \nabla d_\varepsilon|_\Gamma = -\nu, \qquad |\Delta d_\varepsilon| \leqslant c, \tag{3.2}$$

where c is some constant which is independent of ε (for the reader's convenience, the detailed proof is written in Appendix A). Collecting the previous computations, we end up with

$$\Delta\underline{\psi}_\varepsilon(x) = \underline{\theta}'_\varepsilon\big(d_\varepsilon(x)\big)\Delta d_\varepsilon(x) + \frac{\delta}{\varepsilon^2} \quad \text{and} \quad \Delta\overline{\psi}_\varepsilon(x) = \bar{\theta}'_\varepsilon\big(d_\varepsilon(x)\big)\Delta d_\varepsilon(x) - \frac{\delta}{\varepsilon^2},$$

where, thanks to (3.2),

$$\left|\underline{\theta}'_\varepsilon\big(d_\varepsilon(x)\big)\Delta d_\varepsilon(x)\right| + \left|\bar{\theta}'_\varepsilon\big(d_\varepsilon(x)\big)\Delta d_\varepsilon(x)\right| \leqslant \frac{c}{\varepsilon},$$

for some constant c which is independent of ε. Thus, if ε is small enough, we obtain

$$\Delta\underline{\psi}_\varepsilon(x) \geqslant \frac{\delta}{2\varepsilon^2} \quad \text{and} \quad \Delta\overline{\psi}_\varepsilon(x) \leqslant -\frac{\delta}{2\varepsilon^2}, \tag{3.3}$$

and, since both $\phi(\underline{\psi}_\varepsilon) - \beta$ and $\phi(\overline{\psi}_\varepsilon) - \beta$ are bounded independently of ε, (3.3) then yields that, if ε is small enough, $\overline{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ are super- and sub-solutions to (3.1), respectively. Thus, for ε small, there exists $u_\varepsilon \in [\alpha, \alpha+\delta]$ classical solution to (3.1) (note that $\overline{\psi}_\varepsilon - \underline{\psi}_\varepsilon \geqslant 0$ in $\Omega - \overline{\Omega_\varepsilon}$), which actually also solves (2.2). Furthermore, if we assume that $f' \geqslant 0$ in $[\alpha, \alpha+\delta]$, this solution is unique.

To accomplish our second purpose, note that, for $x_0 \in \Gamma$ and $x \in \Omega - \overline{\Omega_\varepsilon}$, the inequalities

$$\overline{\psi}_\varepsilon(x_0) - \overline{\psi}_\varepsilon(x) \leqslant u_\varepsilon(x_0) - u_\varepsilon(x) \leqslant \underline{\psi}_\varepsilon(x_0) - \underline{\psi}_\varepsilon(x)$$

imply

$$\left.\frac{\partial\overline{\psi}_\varepsilon}{\partial\nu}\right|_\Gamma \leqslant \left.\frac{\partial u_\varepsilon}{\partial\nu}\right|_\Gamma \leqslant \left.\frac{\partial\underline{\psi}_\varepsilon}{\partial\nu}\right|_\Gamma.$$

Then, by definition of $\underline{\psi}_\varepsilon, \overline{\psi}_\varepsilon$ and (3.2), we obtain

$$\frac{\delta}{2\varepsilon} \leqslant -\bar{\theta}'_\varepsilon(d_\varepsilon)\Big|_\Gamma \leqslant \left.\frac{\partial u_\varepsilon}{\partial\nu}\right|_\Gamma \leqslant -\underline{\theta}'_\varepsilon(d_\varepsilon)\Big|_\Gamma \leqslant \frac{3\delta}{2\varepsilon}$$

and, passing to the limit $\varepsilon \to 0^+$, we can conclude.

Remark 3.1. Note that the growth of the normal derivative $\frac{\partial u_\varepsilon}{\partial\nu}|_\Gamma$ is uniform, in the sense that it only depends on δ.

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Appendix A. Proof of property (3.2)

Arguing exactly as in [20], it suffices, by localization and rotation, to consider the case where

$$\Gamma = \big\{(x, y, z) \in \mathbb{R}^3\colon\ (x, y) \in I,\ z = \varphi(x, y)\big\},$$

for some $I = (a, b) \times (c, d)$ and $\varphi \in C^2(I)$ bounded in I, together with its first and second derivatives (for simplicity, we write, e.g., φ_x instead of $\frac{\partial\varphi}{\partial x}$). Then, given $P = (x, y, z) \in \Omega$ with $(x, y) \in I$, there exists a unique $\bar{P} = (\bar{x}, \bar{y}, \bar{z}) \in \Gamma$ such that $d_\varepsilon(x, y, z) = \operatorname{dist}(P, \Gamma) = \operatorname{dist}(P, \bar{P})$, hence

$$d_\varepsilon(x, y, z) = \sqrt{(x-\bar{x})^2 + (y-\bar{y})^2 + \big[z - \varphi(\bar{x}, \bar{y})\big]^2}.$$

Since the vector $(x-\bar{x}, y-\bar{y}, z-\varphi(\bar{x}, \bar{y}))$ lies on the normal line to $\mathrm{Gr}(\varphi)$ at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$, we obtain

$$x - \bar{x} + \varphi_x(\bar{x}, \bar{y})\big[z - \varphi(\bar{x}, \bar{y})\big] = 0, \tag{A.1}$$

$$y - \bar{y} + \varphi_y(\bar{x}, \bar{y})\big[z - \varphi(\bar{x}, \bar{y})\big] = 0. \tag{A.2}$$

Computing the partial derivatives of d_ε from (A.1) and (A.2), it is straightforward to check that

$$\nabla d_\varepsilon|_\Gamma = -\nu,$$

since $\nabla d_\varepsilon = \frac{1}{\sqrt{1+|\nabla\varphi(\bar{x},\bar{y})|^2}}(\varphi_x(\bar{x},\bar{y}), \varphi_y(\bar{x},\bar{y}), -1)$.

Henceforth, abusing the notation, we neglect the argument $(\bar{x},\bar{y})$. Then, in order to determine the Laplacian of d_ε, we perform further differentiations, which leads us to

$$\begin{aligned}
\frac{\partial^2 d_\varepsilon}{\partial x^2} &= \frac{\partial}{\partial \bar{x}}\left(\frac{\varphi_x}{\sqrt{1+|\nabla\varphi|^2}}\right)\frac{\partial \bar{x}}{\partial x} + \frac{\partial}{\partial \bar{y}}\left(\frac{\varphi_x}{\sqrt{1+|\nabla\varphi|^2}}\right)\frac{\partial \bar{y}}{\partial x} \\
&= \frac{\varphi_{xx}(1+{\varphi_y}^2) - \varphi_x\varphi_y\varphi_{xy}}{(1+|\nabla\varphi|^2)^{3/2}}\frac{\partial \bar{x}}{\partial x} + \frac{\varphi_{xy}(1+{\varphi_y}^2) - \varphi_x\varphi_y\varphi_{yy}}{(1+|\nabla\varphi|^2)^{3/2}}\frac{\partial \bar{y}}{\partial x}, \\
\frac{\partial^2 d_\varepsilon}{\partial y^2} &= \frac{\partial}{\partial \bar{x}}\left(\frac{\varphi_y}{\sqrt{1+|\nabla\varphi|^2}}\right)\frac{\partial \bar{x}}{\partial y} + \frac{\partial}{\partial \bar{y}}\left(\frac{\varphi_y}{\sqrt{1+|\nabla\varphi|^2}}\right)\frac{\partial \bar{y}}{\partial y} \\
&= \frac{\varphi_{xy}(1+{\varphi_x}^2) - \varphi_x\varphi_y\varphi_{xx}}{(1+|\nabla\varphi|^2)^{3/2}}\frac{\partial \bar{x}}{\partial y} + \frac{\varphi_{yy}(1+{\varphi_x}^2) - \varphi_x\varphi_y\varphi_{xy}}{(1+|\nabla\varphi|^2)^{3/2}}\frac{\partial \bar{y}}{\partial y}, \\
\frac{\partial^2 d_\varepsilon}{\partial z^2} &= \frac{\partial}{\partial \bar{x}}\left(\frac{-1}{\sqrt{1+|\nabla\varphi|^2}}\right)\frac{\partial \bar{x}}{\partial z} + \frac{\partial}{\partial \bar{y}}\left(\frac{-1}{\sqrt{1+|\nabla\varphi|^2}}\right)\frac{\partial \bar{y}}{\partial z} \\
&= \frac{\varphi_x\varphi_{xx} + \varphi_y\varphi_{xy}}{(1+|\nabla\varphi|^2)^{3/2}}\frac{\partial \bar{x}}{\partial z} + \frac{\varphi_x\varphi_{xy} + \varphi_y\varphi_{yy}}{(1+|\nabla\varphi|^2)^{3/2}}\frac{\partial \bar{y}}{\partial z}.
\end{aligned}$$

Thanks to these formulae, we only need to compute $\nabla\bar{x}$ and $\nabla\bar{y}$, which can be obtained from (A.1) and (A.2). Indeed, we deduce the following systems:

$$\begin{cases}
\frac{\partial \bar{x}}{\partial x}\left[1+{\varphi_x}^2 - (z-\varphi)\varphi_{xx}\right] + \frac{\partial \bar{y}}{\partial x}\left[\varphi_x\varphi_y - (z-\varphi)\varphi_{xy}\right] = 1, \\
\frac{\partial \bar{x}}{\partial x}\left[\varphi_x\varphi_y - (z-\varphi)\varphi_{xy}\right] + \frac{\partial \bar{y}}{\partial x}\left[1+{\varphi_y}^2 - (z-\varphi)\varphi_{yy}\right] = 0,
\end{cases} \tag{A.3}$$

$$\begin{cases}
\frac{\partial \bar{x}}{\partial y}\left[1+{\varphi_x}^2 - (z-\varphi)\varphi_{xx}\right] + \frac{\partial \bar{y}}{\partial y}\left[\varphi_x\varphi_y - (z-\varphi)\varphi_{xy}\right] = 0, \\
\frac{\partial \bar{x}}{\partial y}\left[\varphi_x\varphi_y - (z-\varphi)\varphi_{xy}\right] + \frac{\partial \bar{y}}{\partial y}\left[1+{\varphi_y}^2 - (z-\varphi)\varphi_{yy}\right] = 1,
\end{cases} \tag{A.4}$$

$$\begin{cases}
\frac{\partial \bar{x}}{\partial z}\left[1+{\varphi_x}^2 - (z-\varphi)\varphi_{xx}\right] + \frac{\partial \bar{y}}{\partial z}\left[\varphi_x\varphi_y - (z-\varphi)\varphi_{xy}\right] = \varphi_x, \\
\frac{\partial \bar{x}}{\partial z}\left[\varphi_x\varphi_y - (z-\varphi)\varphi_{xy}\right] + \frac{\partial \bar{y}}{\partial z}\left[1+{\varphi_y}^2 - (z-\varphi)\varphi_{yy}\right] = \varphi_y.
\end{cases} \tag{A.5}$$

All systems of equations (A.3)–(A.5) share the same determinant

$$\begin{aligned}
\mathcal{D} &= \left[1+{\varphi_x}^2 - (z-\varphi)\varphi_{xx}\right]\left[1+{\varphi_y}^2 - (z-\varphi)\varphi_{yy}\right] - \left[(z-\varphi)\varphi_{xy} - \varphi_x\varphi_y\right]^2 \\
&= 1+{\varphi_x}^2+{\varphi_y}^2+{\varphi_x}^2{\varphi_y}^2 - (z-\varphi)\left[\left(1+{\varphi_x}^2\right)\varphi_{yy} + \left(1+{\varphi_y}^2\right)\varphi_{xx}\right] + (z-\varphi)^2\varphi_{xx}\varphi_{yy} \\
&\quad - (z-\varphi)^2{\varphi_{xy}}^2 - {\varphi_x}^2{\varphi_y}^2 + 2(z-\varphi)\varphi_x\varphi_y\varphi_{xy} \\
&= 1+|\nabla\varphi|^2 - (z-\varphi)\left[\left(1+{\varphi_x}^2\right)\varphi_{yy} + \left(1+{\varphi_y}^2\right)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy}\right] + (z-\varphi)^2\left(\varphi_{xx}\varphi_{yy} - {\varphi_{xy}}^2\right)
\end{aligned}$$

which, provided that $|z-\varphi|$ is small enough (with $|z-\varphi| < \varepsilon$), does not vanish. Besides, recalling that the Gaussian and the mean curvatures read

$$\begin{aligned}
K &= \frac{\varphi_{xx}\varphi_{yy} - {\varphi_{xy}}^2}{[1+|\nabla\varphi|^2]^2}, \\
H &= \frac{1}{2}\frac{[(1+{\varphi_x}^2)\varphi_{yy} + (1+{\varphi_y}^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy}]}{[1+|\nabla\varphi|^2]^{3/2}},
\end{aligned}$$

we can express $\mathcal{D}$ in terms of H and K, namely,

$$\mathcal{D} = \left(1 + |\nabla\varphi|^2\right)\left[1 - 2(z-\varphi)\sqrt{1+|\nabla\varphi|^2}H + (z-\varphi)^2\left(1+|\nabla\varphi|^2\right)K\right].$$

Then, solving the Cramer systems (A.3)–(A.5), we obtain

$$\begin{aligned}
\frac{\partial \bar{x}}{\partial x} &= \frac{1+{\varphi_y}^2 - (z-\varphi)\varphi_{yy}}{\mathcal{D}},\\
\frac{\partial \bar{y}}{\partial x} &= \frac{\partial \bar{x}}{\partial y} = \frac{(z-\varphi)\varphi_{xy} - \varphi_x\varphi_y}{\mathcal{D}},\\
\frac{\partial \bar{y}}{\partial y} &= \frac{1+{\varphi_x}^2 - (z-\varphi)\varphi_{xx}}{\mathcal{D}},\\
\frac{\partial \bar{x}}{\partial z} &= \frac{\varphi_x[1+{\varphi_y}^2 - (z-\varphi)\varphi_{yy}] + \varphi_y[(z-\varphi)\varphi_{xy} - \varphi_x\varphi_y]}{\mathcal{D}},\\
\frac{\partial \bar{y}}{\partial z} &= \frac{\varphi_x[(z-\varphi)\varphi_{xy} - \varphi_x\varphi_y] + \varphi_y[1+{\varphi_x}^2 - (z-\varphi)\varphi_{xx}]}{\mathcal{D}}.
\end{aligned}$$

We now have at our disposal all the ingredients to compute

$$\begin{aligned}
\Delta d_\varepsilon &= \frac{[(1+{\varphi_x}^2)\varphi_{yy} + (1+{\varphi_y}^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy}] - 2(z-\varphi)[\varphi_{xx}\varphi_{yy} - {\varphi_{xy}}^2]}{\mathcal{D}\sqrt{1+|\nabla\varphi|^2}}\\
&= \frac{[(1+{\varphi_x}^2)\varphi_{yy} + (1+{\varphi_y}^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy}] - 2(z-\varphi)[\varphi_{xx}\varphi_{yy} - {\varphi_{xy}}^2]}{(1+|\nabla\varphi|^2)^{3/2}\{1 - 2(z-\varphi)\sqrt{1+|\nabla\varphi|^2}H + (z-\varphi)^2(1+|\nabla\varphi|^2)K\}}\\
&= 2\frac{H - (z-\varphi)\sqrt{1+|\nabla\varphi|^2}K}{1 - 2(z-\varphi)\sqrt{1+|\nabla\varphi|^2}H + (z-\varphi)^2(1+|\nabla\varphi|^2)K}
\end{aligned}$$

and the desired bound easily follows.

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