



# The weighted estimates for the operators $V^\alpha(-\Delta_G + V)^{-\beta}$ and $V^\alpha \nabla_G(-\Delta_G + V)^{-\beta}$ on the stratified Lie group $G$

Yu Liu<sup>1</sup>

Department of Mathematics and Mechanics, University of Science and Technology Beijing, Beijing 100083, China

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## ABSTRACT

In this paper we consider the Schrödinger operator  $-\Delta_G + V$  on the stratified Lie group  $G$  where the nonnegative potential  $V$  belongs to the reverse Hölder class  $B_{q_1}$  for some  $q_1 \geq \frac{Q}{2}$  and  $Q$  is the homogeneous dimension of  $G$ . The weighted  $L^p-L^q$  estimates for the operators  $V^\alpha(-\Delta_G + V)^{-\beta}$  and  $V^\alpha \nabla_G(-\Delta_G + V)^{-\beta}$  are obtained.

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## 1. Introduction

The investigation of Schrödinger operators on the Euclidean space  $\mathbb{R}^n$  with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [2,9,10]). Shen [9] studied the Schrödinger operator  $-\Delta + V$ , assuming the nonnegative potential  $V$  belongs to the reverse Hölder class  $B_q$  for  $q \geq \frac{n}{2}$  and he proved the  $L^p$  boundedness of the operators  $(-\Delta + V)^{i\gamma}$ ,  $\nabla^2(-\Delta + V)^{-1}$ ,  $\nabla(-\Delta + V)^{-\frac{1}{2}}$  and  $\nabla(-\Delta + V)^{-1}\nabla$ . Kurata and Sugano generalized Shen's results to uniformly elliptic operators in [5]. Sugano [8] also extended some results of Shen to the operator  $V^\alpha(-\Delta + V)^{-\beta}$ ,  $0 \leq \alpha \leq \beta \leq 1$ , and  $V^\alpha \nabla_G(-\Delta_G + V)^{-\beta}$ ,  $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$ ,  $\beta - \alpha \geq \frac{1}{2}$ . Later, Lu [7] and Li [6] investigated the Schrödinger operators in a more general setting.

The main purpose of this paper is to generalize Sugano's results in [8] to the stratified Lie group. More precisely, we investigate the weighted  $L^p-L^q$  boundedness of the operators

$$T_1 = V^\alpha(-\Delta_G + V)^{-\beta}, \quad 0 \leq \alpha \leq \beta \leq 1,$$

$$T_2 = V^\alpha \nabla_G(-\Delta_G + V)^{-\beta}, \quad 0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1, \quad \beta - \alpha \geq \frac{1}{2},$$

on the stratified Lie group  $G$ . Note that the operators  $V(-\Delta_G + V)^{-1}$  and  $V^{\frac{1}{2}}\nabla_G(-\Delta_G + V)^{-1}$  in [6] are the special case of  $T_1$  and  $T_2$ , respectively.

It is worth pointing out that we need to establish pointwise estimates for  $T_1$ ,  $T_2$  and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on nilpotent Lie group in [6]. And we prove the weighted estimates by using the weighted  $L^p-L^q$  boundedness of the fractional maximal operators and the fractional integral operators on spaces of homogeneous type in [1].

E-mail address: liuyu75@pku.org.cn.

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We first recall some basic knowledge of stratified Lie groups (cf. [3]). A Lie group  $G$  is called stratified if it is nilpotent, connected and simple connected, and its Lie algebra  $\mathfrak{g}$  admits a vector space decomposition  $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$  such that  $[V_1, V_k] = V_{k+1}$  for  $1 \leq k < m$  and  $[V_1, V_m] = 0$ . If  $G$  is stratified, its Lie algebra admits a family of dilations, namely,

$$\delta_r(X_1 + X_2 + \cdots + X_m) = rX_1 + r^2X_2 + \cdots + r^mX_m \quad (X_j \in V_j).$$

Assume that  $G$  is a Lie group with underlying manifold  $\mathbb{R}^n$  for some positive integer  $n$ .  $G$  inherits dilations from  $\mathfrak{g}$ : if  $x \in G$  and  $r > 0$ , we write

$$rx = (r^{d_1}x_1, \dots, r^{d_n}x_n), \quad (1)$$

where  $1 \leq d_1 \leq \cdots \leq d_n$ . The map  $x \rightarrow rx$  is an automorphism of  $G$ . The left (or right) Haar measure on  $G$  is simply  $dx_1 \cdots dx_n$ , which is the Lebesgue measure on  $\mathfrak{g}$ . The inverse of any  $x \in G$  is simply  $-x$ . The group law must have the form

$$xy = (p_1(x, y), \dots, p_n(x, y)) \quad (2)$$

for some polynomials  $p_1, \dots, p_n$  in  $x_1, \dots, x_n, y_1, \dots, y_n$ .

The number  $Q = \sum_{j=1}^m j(\dim V_j)$  is called the homogeneous dimension of  $G$ . We fix a homogeneous norm function  $|\cdot|$  on  $G$  which is smooth away from 0. Thus,  $|rx| = r|x|$  for all  $x \in G$ ,  $r > 0$ ,  $|x^{-1}| = |x|$  for all  $x \in G$ , and  $|x| > 0$  if  $x \neq 0$ . The homogeneous norm induces a quasi-distance  $d$  which is defined by  $d(x, y) := |x^{-1}y|$ . Then the ball of radius  $r$  centered at  $x$  is given by  $B(x, r) = \{y \in G: d(x, y) < r\}$ .

Let  $X_1, \dots, X_l$  be a basis for  $V_1$  (viewed as left-invariant vector fields on  $G$ ). It follows from [3] that  $X_j$ ,  $j = 1, 2, \dots, l$ , are skew adjoint, that is,  $X_j^* = -X_j$ . Let  $\Delta_G = \sum_{i=1}^l X_i^2$  be the sub-Laplacian on  $G$ . This operator (which is hypoelliptic by Hörmander's theorem [4]) is well known to play the same fundamental role on  $G$  as the ordinary does on  $\mathbb{R}^n$ . Also, the gradient operator  $\nabla_G$  is denoted by  $\nabla_G = (X_1, \dots, X_l)$ .

### Definition 1.

- (1) A nonnegative locally  $L^q$  integrable function  $V$  on  $G$  is said to belong to the reverse Hölder class  $B_q$  ( $1 < q < \infty$ ) if there exists  $C > 0$  such that the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V(x)^q dx \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V(x) dx \right) \quad (3)$$

holds for every ball  $B$  in  $G$ .

- (2) Let  $V \geq 0$ . We say  $V \in B_\infty$ , if there exists a constant  $C > 0$  such that

$$\|V\|_{L^\infty(B)} \leq \frac{C}{|B|} \int_B V(x) dx \quad (4)$$

holds for every ball  $B$  in  $G$ .

Clearly,  $B_\infty \subseteq B_q$  for  $1 < q < \infty$ . But it is important that the  $B_q$  class has a property of “self-improvement”; that is, if  $V \in B_q$ , then  $V \in B_{q+\varepsilon}$  for some  $\varepsilon > 0$  (see [6]).

Now we recall the definitions of fractional maximal operator  $M_\gamma$  and  $A_{p,q}$ -weight on  $G$ .

**Definition 2.** Let  $f \in L^1_{\text{loc}}(G)$ . For  $0 \leq \gamma < Q$ , the fractional maximal operator is defined by

$$M_\gamma f(x) = \sup_{x \in B} \frac{1}{|B|^{1-\frac{\gamma}{Q}}} \int_B |f(y)| dy, \quad x \in G,$$

where the supremum on the right side is taken over all balls  $B$  such that  $x \in B$ .

**Definition 3.** Let  $1 < p < \infty$  and  $1 < q < \infty$ . For a nonnegative function  $w(x)$ , we say  $w \in A_{p,q}$  if

$$\left( \frac{1}{|B|} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|B|} \int_B w(x)^{-p/(p-1)} dx \right)^{\frac{p-1}{p}} \leq C$$

holds for every ball  $B$  in  $G$ , where  $C$  is a positive constant independent of  $B$ .

The following two pointwise estimates for  $T_1$  and  $T_2$  which generalize the results in [10, Lemma 3.2] to the stratified Lie group with the potential  $V \in B_\infty$ .

**Theorem 1.** Suppose  $V \in B_\infty$  and  $0 < \alpha \leq \beta \leq 1$ . Then there exists a constant  $C > 0$  such that

$$|T_1 f(x)| \leq CM_\gamma(|f|)(x), \quad f \in C_0^\infty(G),$$

where  $\gamma = 2(\beta - \alpha)$ .

**Theorem 2.** Suppose  $V \in B_\infty$ ,  $0 < \alpha \leq \frac{1}{2} < \beta \leq 1$  and  $\beta - \alpha \geq \frac{1}{2}$ . Then there exists a constant  $C > 0$  such that

$$|T_2 f(x)| \leq CM_\gamma(|f|)(x), \quad f \in C_0^\infty(G),$$

where  $\gamma = 2(\beta - \alpha) - 1$ .

We also obtain the similar estimates for the adjoint operators  $T_1^*$  and  $T_2^*$  with the potential  $V \in B_{q_1}$  for some  $q_1 > \frac{Q}{2}$ .

**Theorem 3.** Suppose  $V \in B_{q_1}$  for some  $q_1 > \frac{Q}{2}$ ,  $0 < \alpha \leq \beta \leq 1$  and let  $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$ . Then there exists a constant  $C > 0$  such that

$$|T_1^* f(x)| \leq C \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}, \quad f \in C_0^\infty(G),$$

where  $\gamma = 2(\beta - \alpha)$ .

**Theorem 4.** Suppose  $V \in B_{q_1}$  for some  $q_1 > \frac{Q}{2}$ ,  $0 < \alpha \leq \frac{1}{2} < \beta \leq 1$  and  $\beta - \alpha \geq \frac{1}{2}$ . And let

$$\frac{1}{q_2} = \begin{cases} 1 - \frac{\alpha}{q_1} & \text{if } q_1 > Q, \\ 1 - \frac{(\alpha+1)}{q_1} + \frac{1}{Q} & \text{if } \frac{Q}{2} < q_1 < Q. \end{cases}$$

Then there exists a constant  $C > 0$  such that

$$|T_2^* f(x)| \leq C \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}, \quad f \in C_0^\infty(G),$$

where  $\gamma = 2(\beta - \alpha) - 1$ .

The above theorems will yield the weighted  $L^p$  estimates for  $T_1$  and  $T_2$  which generalize the main results in [8] to the stratified Lie group.

**Corollary 1.** Assume that  $V \in B_\infty$ , and  $0 \leq \alpha \leq \beta \leq 1$ . Let  $1 < p < \frac{Q}{\gamma}$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{Q}$ , where  $\gamma = 2(\beta - \alpha)$ . We suppose  $w$  satisfies

- (A):  $\alpha > 0$ ,  $w \in A_{p,q}$ , and  $w^{-\frac{p}{p-1}} \in A_\infty$ ;  
 (B):  $\alpha = 0$ ,  $w \in A_{p,q}$ ,  $w^q$  and  $w^{-\frac{p}{p-1}} \in A_\infty$ .

Then there exists a positive constant  $C$  such that for any  $f \in C_0^\infty(G)$ ,

$$\|(T_1 f)w\|_{L^q(G)} \leq C \|f w\|_{L^p(G)}.$$

**Corollary 2.** Assume that  $V \in B_\infty$ ,  $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$  and  $\beta - \alpha \geq \frac{1}{2}$ . Let  $1 < p < \frac{Q}{\gamma}$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{Q}$ , where  $\gamma = 2(\beta - \alpha) - 1$ . We suppose  $w$  satisfies

- (A):  $\alpha > 0$ ,  $w \in A_{p,q}$ , and  $w^{-\frac{p}{p-1}} \in A_\infty$ ;  
 (B):  $\alpha = 0$ ,  $w \in A_{p,q}$ ,  $w^q$  and  $w^{-\frac{p}{p-1}} \in A_\infty$ .

Then there exists a positive constant  $C$  such that for any  $f \in C_0^\infty(G)$ ,

$$\|(T_2 f)w\|_{L^q(G)} \leq C \|f w\|_{L^p(G)}.$$

**Corollary 3.** Assume that  $V \in B_{q_1}$  for  $q_1 > \frac{Q}{2}$ , and  $0 \leq \alpha \leq \beta \leq 1$ . Let  $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\gamma}{Q}}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{Q}$  and  $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$ , where  $\gamma = 2(\beta - \alpha)$ . We suppose  $w$  satisfies

- (A):  $\alpha > 0$ ,  $w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}$  and  $w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty$ ;  
 (B):  $\alpha = 0$ ,  $w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}$ ,  $w^{-p'}$  and  $w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty$ ,

where  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . Then there exists a positive constant  $C$  such that for any  $f \in C_0^\infty(G)$ ,

$$\|(T_1 f)w\|_{L^q(G)} \leq C \|fw\|_{L^p(G)}.$$

**Corollary 4.** Assume that  $V \in B_{q_1}$  for  $q_1 > \frac{Q}{2}$ , and

$$\begin{cases} 0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1 & \text{if } q_1 > Q, \\ 0 \leq \alpha \leq \frac{1}{2} < \beta \leq 1 & \text{if } \frac{Q}{2} < q_1 < Q. \end{cases}$$

Let  $\gamma = 2(\beta - \alpha) - 1$  and  $\beta - \alpha \geq \frac{1}{2}$ , and let  $1 < p < \frac{1}{\frac{\alpha}{q_1} + \frac{\gamma}{Q}}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{Q}$ ,  $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$ , where

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1} & \text{if } q_1 > Q, \\ \frac{(\alpha+1)}{q_1} - \frac{1}{Q} & \text{if } \frac{Q}{2} < q_1 < Q. \end{cases}$$

We suppose  $w$  satisfies

$$(A): \quad \alpha > 0, \quad w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}} \quad \text{and} \quad w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty;$$

$$(B): \quad \alpha = 0, \quad w^{-q_2} \in A_{\frac{q'}{q_2}, \frac{p'}{q_2}}, \quad w^{-p'} \quad \text{and} \quad w^{-\frac{q_2 q'}{q_2 - q'}} \in A_\infty,$$

where  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ . Then there exists a positive constant  $C$  such that for any  $f \in C_0^\infty(G)$ ,

$$\|(T_2 f)w\|_{L^q(G)} \leq C \|fw\|_{L^p(G)}.$$

**Remark 1.** For  $w \equiv 1$ , when  $(\alpha, \beta) = (1, 1)$  and  $(\frac{1}{2}, \frac{1}{2})$ , Corollary 1 and Corollary 3 have been proved in Theorem 4.1, Theorem 6.5 in [6], respectively. For  $w \equiv 1$ , Corollaries 2 and 4 have been also proved in Corollary 5.4 in [6] when  $(\alpha, \beta) = (\frac{1}{2}, 1)$ .

Throughout this paper, unless otherwise indicated, we will use  $C$  to denote constants, which are not necessarily the same at each occurrence. By  $A \sim B$ , we mean that there exist  $C > 0$  and  $c > 0$  such that  $c \leq \frac{A}{B} \leq C$ .

## 2. Preliminaries

First we briefly recall the definition of the auxiliary function  $m(x, V)$  and its basic properties on nilpotent Lie groups in [6].

Let  $V \in B_{q_1}$  for some  $q_1 > \frac{Q}{2}$ ,  $Q$  is the homogeneous dimension of  $G$ . Then the auxiliary function  $\rho(x, V) = \rho(x)$  is defined by

$$\rho(x) = \frac{1}{m(x, V)} \doteq \sup_{r>0} \left\{ r: \frac{1}{r^{Q-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in G.$$

**Lemma 1.** The measure  $V(x) dx$  satisfies the doubling condition, that is, there exists  $C > 0$  such that

$$\int_{B(x,2r)} V(y) dy \leq C \int_{B(x,r)} V(y) dy$$

for all balls  $B(x, r)$  in  $G$ .

**Lemma 2.** There exists  $C > 0$  such that, for  $0 < r < R < \infty$ ,

$$\frac{1}{r^{Q-2}} \int_{B(x,r)} V(y) dy \leq C \left( \frac{r}{R} \right)^{2-\frac{Q}{q_1}} \frac{1}{R^{Q-2}} \int_{B(x,R)} V(y) dy.$$

**Lemma 3.** If  $r = \rho(x)$ , then

$$\frac{1}{r^{Q-2}} \int_{B(x,r)} V(y) dy = 1.$$

Moreover,

$$\frac{1}{r^{Q-2}} \int_{B(x,r)} V(y) dy \sim 1 \quad \text{if and only if} \quad r \sim \rho(x).$$

**Lemma 4.** There exist  $C > 0$  and  $l_0 > 0$  such that, for any  $x$  and  $y$  in  $G$ ,

$$\frac{1}{C} \left( 1 + \frac{d(x,y)}{\rho(x)} \right)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C \left( 1 + \frac{d(x,y)}{\rho(x)} \right)^{\frac{l_0}{l_0+1}}.$$

In particular,  $\rho(x) \sim \rho(y)$  if  $d(x,y) < C\rho(x)$ .

**Lemma 5.** There exist  $C > 0$  and  $l_1 > 0$  such that

$$\int_{B(x,R)} \frac{V(y)}{d(x,y)^{Q-2}} dy \leq \frac{C}{R^{Q-2}} \int_{B(x,R)} V(y) dy \leq C \left( 1 + \frac{R}{\rho(x)} \right)^{l_1}.$$

See [6] for the proofs of Lemmas 1–5.

Let  $\Gamma(g, h, \lambda)$  denote the fundamental solution for the operator  $-\Delta_G + V + \lambda$ , where  $\lambda \geq 0$ . The following estimates of the fundamental solution for the Schrödinger operator on the nilpotent Lie group have been proved in [6].

**Lemma 6.** Let  $l > 0$  be an integer.

(1) Suppose  $V \in B_{\frac{Q}{2}}$ . Then there exists  $C_l > 0$  such that for  $x \neq y$ ,

$$|\Gamma(x, y, \lambda)| \leq \frac{C_l}{(1 + d(x, y) \lambda^{\frac{1}{2}})^l (1 + d(x, y) \rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2}}.$$

(2) Suppose  $V \in B_Q$ . Then there exists  $C_l > 0$  such that for  $x \neq y$ ,

$$|\nabla_{G,x} \Gamma(x, y, \lambda)| \leq \frac{C_l}{(1 + d(x, y) \lambda^{\frac{1}{2}})^l (1 + d(x, y) \rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-1}}.$$

In order to prove Corollaries 1–4, we need to introduce the theory of the weighted norm inequalities for fractional maximal operators and fractional integral operators on spaces of homogeneous type in [1].

Let  $(X, d, \mu)$  be a space of homogeneous type, where  $d$  is a quasi-distance and  $\mu$  is a positive measure defined on a  $\sigma$ -algebra of subsets of  $X$  and satisfies the doubling condition. It is easy to see that the stratified Lie group is also a space of homogeneous type. Let  $M_\delta$  be the fractional maximal operator on the space of homogeneous type  $X$  which is defined, for each  $\delta \in [0, 1)$ , by

$$M_\delta f(x) = \sup_{x \in B} \frac{1}{\mu(B)^{1-\delta}} \int_B |f(y)| d\mu(y), \quad f \in L^1_{\text{loc}}(X, d\mu).$$

Let  $I_\delta$  be the fractional integral operator on the space of homogeneous type  $X$  which is defined, for each  $\delta \in (0, 1)$ , by

$$I_\delta f(x) = \int_X \frac{f(y)}{\mu(B(y, d(x, y)))^{1-\delta}} d\mu(y), \quad f \in L^1(X, d\mu).$$

A weight  $\omega$  is a nonnegative function in  $L^1_{\text{loc}}(X, d\mu)$  and we shall use  $\omega(A)$  to denote  $\int_A \omega d\mu$ . We say that a weight  $\omega$  belongs to  $A_\infty$  if there exist positive constants  $C > 0$  and  $\delta > 0$  such that

$$\frac{\mu(E)}{\mu(B)} \leq C \left( \frac{\omega(E)}{\omega(B)} \right)^\delta,$$

for every ball  $B$  and every measurable set  $E \subseteq B$ .

**Proposition 1.**

(1) Suppose  $0 \leq \delta < 1$  and  $1 < p \leq q < \infty$ . Let  $(w, v)$  be a pair of weight with  $v^{-\frac{1}{p-1}} \in A_\infty$ . Then

$$\|M_\delta f\|_{L^q(X, w d\mu)} \leq C \|f\|_{L^p(X, v d\mu)},$$

if and only if

$$\frac{1}{\mu(B)^{(1-\delta)p}} \left( \int_B w d\mu \right)^{\frac{p}{q}} \left( \int_B v^{-\frac{1}{p-1}} d\mu \right)^{p-1} \leq C < \infty, \quad \text{for every ball } B \subseteq X.$$

(2) Suppose  $0 < \delta < 1$ ,  $(w, v)$  be a pair of weight with  $w \in A_\infty$  and  $v^{-\frac{1}{p-1}} \in A_\infty$ . Then

$$\|I_\delta f\|_{L^q(X, w d\mu)} \leq C \|f\|_{L^p(X, v d\mu)},$$

if and only if

$$\frac{1}{\mu(B)^{(1-\delta)p}} \left( \int_B w d\mu \right)^{\frac{p}{q}} \left( \int_B v^{-\frac{1}{p-1}} d\mu \right)^{p-1} \leq C < \infty, \quad \text{for every ball } B \subseteq X.$$

**3. The proof of the main results**

**Proof of Theorem 1.** By the functional calculus, we may write, for all  $0 < \beta < 1$ ,

$$(-\Delta_G + V)^{-\beta} = \frac{1}{\pi} \int_0^\infty \lambda^{-\beta} (-\Delta_G + V + \lambda)^{-1} d\lambda. \quad (5)$$

Let  $f \in C_0^\infty(G)$ . From  $(-\Delta_G + V + \lambda)^{-1} f(x) = \int_G \Gamma(x, y, \lambda) f(y) dy$ , it follows that

$$T_1 f(x) = \int_G K_1(x, y) V(x)^\alpha f(y) dy, \quad (6)$$

where

$$K_1(x, y) = \begin{cases} \frac{1}{\pi} \int_0^\infty \lambda^{-\beta} \Gamma(x, y, \lambda) d\lambda & \text{for } 0 < \beta < 1, \\ \Gamma(x, y, 0) & \text{for } \beta = 1. \end{cases} \quad (7)$$

By using Lemma 6(1), we conclude that for all  $0 < \beta \leq 1$  and all integer  $l \geq 2$ , there exists a constant  $C_l > 0$  such that

$$|K_1(x, y)| \leq \frac{C_l}{(1 + d(x, y)\rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta}}. \quad (8)$$

Let  $r = \rho(x)$ . Since  $V \in B_\infty$ , then  $V(x) \leq C m(x, V)^2 = \frac{C}{\rho(x)^2}$  a.e. on  $G$ . Therefore we have

$$\begin{aligned} |T_1 f(x)| &\leq \int_G \frac{C_l}{(1 + d(x, y)\rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta}} V(x)^\alpha |f(y)| dy \\ &\leq C C_l \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r < d(x, y) \leq 2^j r} \frac{1}{(1 + d(x, y)r^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta}} \frac{1}{r^{2\alpha}} |f(y)| dy \\ &\leq C C_l \sum_{j=-\infty}^{\infty} \frac{2^{2\alpha j}}{(1 + 2^{j-1})^l} \frac{1}{(2^j r)^{Q-2(\beta-\alpha)}} \int_{B(x, 2^j r)} |f(y)| dy \leq C M_\gamma(|f|)(x), \end{aligned}$$

where  $\gamma = 2(\beta - \alpha)$  and we choose  $l \geq 3$ .  $\square$

**Proof of Theorem 2.** Let  $f \in C_0^\infty(G)$ . As (6) and (7) in Theorem 1,  $T_2$  has the following form

$$T_2 f(x) = \int_G K_2(x, y) V(x)^\alpha f(y) dy, \quad (9)$$

where

$$K_2(x, y) = \begin{cases} \frac{1}{\pi} \int_0^\infty \lambda^{-\beta} \nabla_{G,x} \Gamma(x, y, \lambda) d\lambda & \text{for } \frac{1}{2} < \beta < 1, \\ \nabla_{G,x} \Gamma(x, y, 0) & \text{for } \beta = 1. \end{cases} \quad (10)$$

By Lemma 6(2), we conclude that for all  $\frac{1}{2} < \beta \leq 1$  and all integer  $l \geq 2$ , there exists a constant  $C_l$  such that

$$|K_2(x, y)| \leq \frac{C_l}{(1 + d(x, y)\rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta+1}}. \quad (11)$$

Then as in the proof of Theorem 1, we obtain

$$|T_2 f(x)| \leq CM_\gamma(|f|)(x), \quad (12)$$

where  $\gamma = 2(\beta - \alpha) - 1$ .  $\square$

**Proof of Theorem 3.** Let  $f \in C_0^\infty(G)$ . The adjoint of  $T_1$  is given by

$$T_1^* f(x) = \int_G \overline{K_1(y, x)} V(y)^\alpha f(y) dy.$$

By Lemma 6(1), for all  $0 < \beta \leq 1$  and all integer  $l \geq 2$ , there exists a constant  $C_l$  such that

$$|\overline{K_1(y, x)}| \leq \frac{C_l}{(1 + d(x, y)\rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta}}. \quad (13)$$

Let  $r = \rho(x)$ . It follows from Hölder's inequality that

$$\begin{aligned} |T_1^* f(x)| &\leq \int_G \frac{C_l}{(1 + d(x, y)\rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta}} V(y)^\alpha |f(y)| dy \\ &\leq C_l \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r < d(x, y) \leq 2^j r} \frac{1}{(1 + 2^{j-1})^l} \frac{1}{(2^{j-1}r)^{Q-2\beta}} V(y)^\alpha |f(y)| dy \\ &\leq CC_l \sum_{j=-\infty}^{\infty} \frac{(2^j r)^{2\beta}}{(1 + 2^{j-1})^l} \left\{ \frac{1}{(2^j r)^Q} \int_{B(x, 2^j r)} V(y)^{q_1} dy \right\}^{\frac{\alpha}{q_1}} \left\{ \frac{1}{(2^j r)^Q} \int_{B(x, 2^j r)} |f(y)|^{q_2} dy \right\}^{\frac{1}{q_2}}. \end{aligned}$$

It follows from the assumptions that  $0 \leq \frac{2(\beta-\alpha)q_2}{Q} < 1$ . Thus, letting  $\gamma = 2(\beta - \alpha)$  and using (3) we know that

$$|T_1^* f(x)| \leq CC_l \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{q_2} \sum_{j=-\infty}^{\infty} \frac{1}{(1 + 2^{j-1})^l} \left\{ \frac{(2^j r)^2}{|B(x, 2^j r)|} \int_{B(x, 2^j r)} V(y) dy \right\}^\alpha.$$

It follows from [6] that  $V(x) dx$  is a doubling measure. Then for the case  $j \geq 1$  there exists a constant  $C_0$  such that

$$\frac{(2^j r)^2}{|B(x, 2^j r)|} \int_{B(x, 2^j r)} V(y) dy \leq 2^{2j} C_0^j 2^{-jQ} \frac{(2^j r)^2}{|B(x, r)|} \int_{B(x, r)} V(y) dy \quad (14)$$

$$= C(2^j)^{k_0}, \quad (15)$$

where  $k_0 = 2 - Q + \log_2 C_0$ . For the case  $j \leq 0$ , by using Lemma 2 we see that

$$\frac{(2^j r)^2}{|B(x, 2^j r)|} \int_{B(x, 2^j r)} V(y) dy \leq C \left( \frac{r}{2^j r} \right)^{\frac{Q}{q_1}-2} \frac{r^2}{|B(x, r)|} \int_{B(x, r)} V(y) dy \quad (16)$$

$$= C(2^j)^{2-\frac{Q}{q_1}}. \quad (17)$$

Hence if we take  $l$  sufficiently large, we conclude that

$$|T_1^* f(x)| \leq C \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}. \quad \square$$

**Proof of Theorem 4.** Let  $f \in C_0^\infty(G)$ . The adjoint of  $T_2$  is also given by

$$T_2^* f(x) = \int_G \overline{K_2(y, x)} V(y)^\alpha f(y) dy.$$

Case  $q_1 \geq Q$ : By Lemma 6(2), for all  $0 < \beta \leq 1$  and all integer  $l \geq 2$ , there exists a positive constant  $C_l$  such that

$$|\overline{K_2(y, x)}| \leq \frac{C_l}{(1 + d(x, y)\rho(x)^{-1})^l} \frac{1}{d(x, y)^{Q-2\beta+1}}.$$

Let  $r = \rho(x)$ . Then similar to the proof of Theorem 3 we have

$$|T_2^* f(x)| \leq CC_l \sum_{j=-\infty}^{\infty} \frac{(2^j r)^{2\beta-1}}{(1 + 2^{j-1})^l} \left\{ \frac{1}{(2^j r)^Q} \int_{B(x, 2^j r)} V(y)^{q_1} dy \right\}^{\frac{\alpha}{q_1}} \left\{ \frac{1}{(2^j r)^Q} \int_{B(x, 2^j r)} |f(y)|^{q_2} dy \right\}^{\frac{1}{q_2}}.$$

Note that  $0 \leq (2(\beta - \alpha) - 1)q_2 < Q$ . Thus, letting  $\gamma = 2(\beta - \alpha) - 1$  and using (14) and (16) we know that

$$|T_2^* f(x)| \leq CC_l \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}.$$

Case  $\frac{Q}{2} < q_1 < Q$ : Fix  $x_0, y_0 \in G$ . Let  $R = \frac{|x_0^{-1}y_0|}{4}$ . By using (1.11) in [6] we get, for all positive integer  $l$ , there exists a positive constant  $C_l$  such that

$$|\nabla_{G,y} \Gamma(y_0, x_0, \lambda)| \leq \frac{C_l}{(1 + R\lambda^{\frac{1}{2}})^l (1 + R\rho(x_0)^{-1})^l} \left( \frac{1}{R^{Q-2}} \int_{B(y_0, \frac{1}{4}R)} \frac{V(y) dy}{|y_0^{-1}y|^{Q-1}} + \frac{1}{R^{Q-1}} \right).$$

Then by (10), we see that there exists a positive constant  $C_l$  such that for all integer  $l \geq 2$ ,

$$|\overline{K_2(y_0, x_0)}| \leq \frac{C_l}{(1 + R\rho(x_0)^{-1})^l} \left( \frac{1}{R^{Q-2\beta}} \int_{B(y_0, \frac{1}{4}R)} \frac{V(y) dy}{|y_0^{-1}y|^{Q-1}} + \frac{1}{R^{Q-2\beta+1}} \right).$$

Let  $r = \rho(x)$  and choose  $p_1$  such that  $\frac{1}{p_1} = \frac{1}{q_1} - \frac{1}{Q}$ . Note that  $\frac{1}{p_1} + \frac{\alpha}{q_1} + \frac{1}{q_2} = 1$ . By Hölder inequality, we obtain

$$\begin{aligned} |T_2^* f(x)| &\leq \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r < d(x,y) \leq 2^j r} |\overline{K_2(y, x)}| V(y)^\alpha |f(y)| dy \\ &\leq \sum_{j=-\infty}^{\infty} (2^j r)^Q \left\{ \frac{1}{(2^j r)^Q} \int_{2^{j-1}r < d(x,y) \leq 2^j r} |\overline{K_2(y, x)}|^{p_1} dy \right\}^{\frac{1}{p_1}} \\ &\quad \times \left\{ \frac{1}{(2^j r)^Q} \int_{B(x, 2^j r)} V(y)^{q_1} dy \right\}^{\frac{\alpha}{q_1}} \left\{ \frac{1}{(2^j r)^Q} \int_{B(x, 2^j r)} |f(y)|^{q_2} dy \right\}^{\frac{1}{q_2}}. \end{aligned}$$

Using Minkowski's inequality and the well-known theorem on fractional integrals on the stratified Lie group (cf. [3]), we obtain

$$\begin{aligned} &(2^j r)^Q \left\{ \frac{1}{(2^j r)^Q} \int_{2^{j-1}r < d(x,y) \leq 2^j r} |\overline{K_2(y, x)}|^{p_1} dy \right\}^{\frac{1}{p_1}} \\ &\leq \frac{CC_l(2^j r)^Q}{(1 + 2^{j-3})^l} \left\{ \frac{1}{(2^j r)^{Q-2\beta-1}} \left[ \frac{1}{(2^j r)^Q} \int_{B(x, 2^{j-2}r)} V(y)^{q_1} dy \right]^{\frac{1}{q_1}} + \frac{1}{(2^j r)^{Q-2\beta+1}} \right\} \\ &\leq \frac{C'C_l(2^j r)^{2\beta-1}}{(1 + 2^{j-3})^l} \left[ \frac{1}{(2^{j-2}r)^{Q-2}} \int_{B(x, 2^{j-2}r)} V(y) dy + 1 \right]. \end{aligned}$$

For the case  $j \geq 1$ , using (14) we have

$$(2^j r)^Q \left\{ \frac{1}{(2^j r)^Q} \int_{2^{j-1}r < d(x,y) \leq 2^j r} |\overline{K_2(y, x)}|^{p_1} dy \right\}^{\frac{1}{p_1}} \leq C'_l \frac{2^{jk_0}(2^j r)^{2\beta-1}}{(1 + 2^{j-3})^l}.$$



For the case  $j \leq 0$ , using (16) we obtain

$$(2^j r)^Q \left\{ \frac{1}{(2^j r)^Q} \int_{2^{j-1}r < d(x,y) \leq 2^j r} |\overline{K_2(y,x)}|^{p_1} dy \right\}^{\frac{1}{p_1}} \leq C' C_l \frac{(2^j r)^{2\beta-1}}{(1+2^{j-3})^l}.$$

Then it follows that

$$|T_2^* f(x)| \leq C' C_l \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}} \left\{ \sum_{j=1}^{\infty} \frac{2^{jk_0}}{(1+2^{j-3})^l} + \sum_{j=-\infty}^0 \frac{1}{(1+2^{j-3})^l} \right\} \left[ \frac{1}{(2^j r)^{Q-2}} \int_{B(x, 2^{j-2}r)} V(y) dy \right]^{\alpha},$$

where  $\gamma = 2(\beta - \alpha) - 1$ . Combining (14) and (16) again and taking  $l$  sufficiently large, we get

$$|T_2^* f(x)| \leq C \{M_{\gamma q_2}(|f|^{q_2})(x)\}^{\frac{1}{q_2}}. \quad \square$$

**Proof of Corollary 1.** Case  $\alpha > 0$ : By using (1) of Proposition 1 and Theorem 1 we get the weighted  $L^p$ – $L^q$  estimates.

Case  $\alpha = 0$ : It follows from (8) that the kernel  $K_1(x, y)$  of the operator  $T_1$  satisfies

$$|K_1(x, y)| \leq \frac{C}{d(x, y)^{Q-2\beta}},$$

where  $0 < \beta \leq 1$ . From Proposition 1(2), it is easy to deduce the weighted  $L^p$ – $L^q$  estimates.  $\square$

**Proof of Corollary 2.** Case  $\alpha > 0$ : Using Theorem 2 and following the same idea of the proof of Corollary 1, we arrive at the weighted  $L^p$ – $L^q$  estimates.

Case  $\alpha = 0$  and  $\frac{1}{2} < \beta \leq 1$ : It follows from (11) that the kernel  $K_2(x, y)$  of the operator  $T_2$  satisfies

$$|K_2(x, y)| \leq \frac{C}{d(x, y)^{Q-2\beta+1}},$$

where  $\frac{1}{2} < \beta \leq 1$ . By using Proposition 1(2) again we obtain the weighted  $L^p$ – $L^q$  estimates.  $\square$

**Proof of Corollary 3.** Case  $\alpha > 0$ : Let  $\gamma = 2(\beta - \alpha)$  and  $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$ . For  $q'$  such that  $q_2 < q' < \frac{Q}{\gamma}$  and  $\frac{1}{p'} = \frac{1}{q'} - \frac{\gamma}{Q}$ , then it follows from the assumptions that

$$0 < \gamma q_2 < Q, \quad 1 < \frac{q'}{q_2} < \frac{Q}{\gamma q_2}, \quad \frac{1}{p'/q_2} = \frac{1}{q'/q_2} - \frac{\gamma q_2}{Q}.$$

By Theorem 3 and Proposition 1(1), there exists a positive constant  $C$  such that for any  $f \in C_0^\infty(G)$ ,

$$\|(T_1^* f) w^{-1}\|_{L^{p'}(G)} \leq C \|f w^{-1}\|_{L^{q'}(G)}.$$

Since  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ , the desired estimate follows by duality.

Case  $\alpha = 0$ : Since  $q_2 = 1$ , so the condition for  $w$  is  $w^{-1} \in A_{p', q'}$ , which is equivalent to  $w \in A_{p, q}$ . Then following the same idea of the proof of Corollary 1, we get the desired estimate.  $\square$

**Proof of Corollary 4.** Case  $\alpha > 0$ : Let  $\gamma = 2(\beta - \alpha) - 1$  and  $\frac{1}{q_2} = 1 - \frac{1}{p_1}$ ,

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1} & \text{if } q_1 > Q, \\ \frac{(\alpha+1)}{q_1} - \frac{1}{Q} & \text{if } \frac{Q}{2} < q_1 < Q. \end{cases}$$

For  $q'$  such that  $q_2 < q' < \frac{Q}{\gamma}$  and  $\frac{1}{p'} = \frac{1}{q'} - \frac{\gamma}{Q}$ , then it follows from the assumptions that

$$0 < \gamma q_2 < Q, \quad 1 < \frac{q'}{q_2} < \frac{Q}{\gamma q_2}, \quad \frac{1}{p'/q_2} = \frac{1}{q'/q_2} - \frac{\gamma q_2}{Q}.$$

By Theorem 4 and Proposition 1(1), there exists a positive constant  $C$  such that for any  $f \in C_0^\infty(G)$ ,

$$\|(T_2^* f) w^{-1}\|_{L^{p'}(G)} \leq C \|f w^{-1}\|_{L^{q'}(G)}.$$

Since  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ , so the desired estimate follows by duality.

Case  $\alpha = 0$  and  $\frac{1}{2} < \beta \leq 1$ : Using the estimates of the kernel  $K_2(x, y)$  and following the same idea of the proof of Corollary 3, we get the desired estimate.  $\square$

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