



# Iterative schemes for computing fixed points of nonexpansive mappings in Banach spaces

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## ABSTRACT

Let  $X$  be a real Banach space with a normalized duality mapping uniformly norm-to-weak\* continuous on bounded sets or a reflexive Banach space which admits a weakly continuous duality mapping  $J_\phi$  with gauge  $\phi$ . Let  $f$  be an  $\alpha$ -contraction and  $\{T_n\}$  a sequence of nonexpansive mappings, we study the strong convergence of explicit iterative schemes

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \quad (1)$$

with a general theorem and then recover and improve some specific cases studied in the literature [K. Aoyoma, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed point of a countable family of nonexpansive mappings, *Nonlinear Anal.* 67 (8) (2007) 2350–2360; G. Lopez, V. Martin, H.-K. Xu, Perturbation techniques for nonexpansive mappings with applications, *Nonlinear Anal. Real World Appl.*, in press, available online 4 May 2008; H.-K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298 (1) (2004) 279–291; T.-H. Kim, H.-K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.* 61 (1–2) (2005) 51–60; Y. Song, R. Chen, Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings, *Appl. Math. Comput.* 180 (2006) 275–287; Y. Song, R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, *J. Math. Anal. Appl.* 321 (1) (2006) 316–326; J. Chen, L. Zhang, T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, *J. Math. Anal. Appl.* 334 (2) (2007) 1450–1461; Y. Kimura, W. Takahashi, M. Toyoda, Convergence to common fixed points of a finite family of nonexpansive mappings, *Arch. Math.* 84 (2005) 350–363].

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## 1. Introduction and preliminaries

Let  $X$  be a real Banach space,  $C$  a nonempty closed convex subset of  $X$ . Recall that a mapping  $T : C \mapsto C$  is *nonexpansive* if  $\|T(x) - T(y)\| \leq \|x - y\|$  for all  $x, y \in C$  and a mapping  $f : C \mapsto C$  is an  $\alpha$ -contraction if there exists  $\alpha \in (0, 1)$  such that  $\|f(x) - f(y)\| \leq \alpha \|x - y\|$  for all  $x, y \in C$ .

We denote by  $\text{Fix}(T)$  the set of fixed points of  $T$ , that is

$$\text{Fix}(T) \stackrel{\text{def}}{=} \{x \in C : Tx = x\} \quad (2)$$

and  $\Pi_C$  will denote the collection of contractions on  $C$ .

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Let  $X$  be a real Banach space. The (normalized) duality mapping  $J : X \rightrightarrows X^*$ , where  $X^*$  is the dual space of  $X$ , is defined by:

$$J(x) \stackrel{\text{def}}{=} \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

and we have the following inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \text{where } x, y \in X \text{ and } j(x + y) \in J(x + y).$$

Recall that if  $C$  and  $F$  are nonempty subsets of a Banach space  $X$  such that  $C$  is nonempty closed convex and  $F \subset C$ , then a mapping  $R : C \mapsto F$  is called a *retraction* from  $C$  onto  $F$  if  $R(x) = x$  for all  $x \in F$ . A retraction  $R : C \mapsto F$  is *sunny* provided that  $R(x + t(x - R(x))) = R(x)$  for all  $x \in C$  and  $t \geq 0$  whenever  $x + t(x - R(x)) \in C$ . A *sunny nonexpansive retraction* is a sunny retraction, which is also nonexpansive.

Suppose that  $F$  is the nonempty fixed point set of a nonexpansive mapping  $T : C \mapsto C$ , that is  $F = \text{Fix } T \neq \emptyset$ , and assume that  $F$  is closed. For a given  $u \in C$  and every  $t \in (0, 1)$  there exists a fixed point, denoted by  $x_t$ , of the  $(1 - t)$ -contraction  $tu + (1 - t)T$ . Then we define  $Q : C \mapsto F = \text{Fix}(T)$  by  $Q(u) \stackrel{\text{def}}{=} \sigma\text{-}\lim_{t \rightarrow 0} x_t$  when this limit exists ( $\sigma\text{-}\lim$  denotes the strong limit).  $Q$  will also be denoted by  $Q_{\text{Fix}(T)}$  when necessary and it is easy to check that, if it exists,  $Q$  is a nonexpansive retraction.

Consider now  $f$  an  $\alpha$ -contraction. Then  $Q_{\text{Fix}(T)} \circ f$  is also an  $\alpha$ -contraction and admits therefore a unique fixed point  $\tilde{x} = Q_T \circ f(\tilde{x})$ . We define by  $\mathbf{Q}(f)$  or  $\mathbf{Q}_{\text{Fix}(T)}(f)$  the mapping  $\mathbf{Q}(f) : \Pi_C \rightarrow \text{Fix}(T)$  such that:

$$\mathbf{Q}(f) \stackrel{\text{def}}{=} \tilde{x} \quad \text{where } \tilde{x} = (Q_{\text{Fix}(T)} \circ f)(\tilde{x}). \quad (3)$$

For  $t \in (0, 1)$  we can also find a fixed point, denoted by  $x_t^f$  of the  $(1 - (1 - t)\alpha)$ -contraction  $tf + (1 - t)T$ . If  $\lim_{t \rightarrow 0} x_t^f$  exists we can define a mapping  $\tilde{\mathbf{Q}} : \Pi_C \mapsto \text{Fix}(T)$  by:

$$\tilde{\mathbf{Q}}(f) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} x_t^f \quad \text{where } x_t^f = tf(x_t^f) + (1 - t)Tx_t^f. \quad (4)$$

We then gather known theorems under which  $Q$ ,  $\mathbf{Q}$  and  $\tilde{\mathbf{Q}}$  are defined and give relations between them.

When  $X$  is a uniformly smooth Banach space, denoted by  $\mathcal{B}_{\text{us}}$ , it is known [19, Theorem 4.1] that  $\mathbf{Q}(f)$  is well defined and is equal to  $\mathbf{Q}(f)$ . Furthermore  $\tilde{x} = \mathbf{Q}(f)$  is characterized by:

$$\langle \tilde{x} - f(\tilde{x}), J(\tilde{x} - p) \rangle \leq 0 \quad \text{for all } p \in F = \text{Fix}(T). \quad (5)$$

A special case is when  $f$  is a constant function  $\mathbf{u}(x) = u$ . Then [19, Theorem 4.1] shows that  $Q$  is well defined and that  $Q(u) = \mathbf{Q}(\mathbf{u}) = P_{\text{Fix } T}u$  (where  $P_S$  is the metric projection on  $S$ ). If  $X$  is a smooth Banach space,  $R : C \mapsto F$  is a sunny nonexpansive retraction [7] if and only if the following inequality holds:

$$\langle x - Rx, J(y - Rx) \rangle \leq 0 \quad \text{for all } x \in C \text{ and } y \in F. \quad (6)$$

$Q$  is thus the unique sunny nonexpansive retraction from  $C$  to  $\text{Fix } T$ . This result [19, Theorem 4.1] was already known for the case  $f$  constant and for Hilbert spaces ([19, Theorem 3.1] and [13, Theorem 2.1]).

On one hand, the same existence and characterization results can be found when  $X$  is a reflexive Banach space which admits a weakly continuous duality mapping  $J_\phi$  with gauge  $\phi$ , denoted by  $\mathcal{B}_{\text{rWSC}}$ , in [20, Theorem 3.1] (with  $f$  constant) and [16, Theorem 2.2] (where  $J$  is the (normalized) duality mapping). Note that the limitation of  $f$  constant in [20] can be relaxed with [17]. On the other hand, the same existence and characterization results can be found when  $X$  is a reflexive and a strictly convex Banach space with a uniformly Gâteaux differentiable norm, denoted by  $\mathcal{B}_{\text{rug}}$  [15, Theorem 3.1]. Note that in these three Banach spaces cases listed here, the normalized duality mapping is shown to be single valued.

The aim of this paper is to study the strong convergence of iterative schemes:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n \quad (7)$$

when  $X$  can be a  $\mathcal{B}_{\text{us}}$ , or a  $\mathcal{B}_{\text{rWSC}}$ , or a  $\mathcal{B}_{\text{rug}}$  real Banach space and  $\{T_n\}$  is a sequence of nonexpansive mappings which share at least a common fixed point. We give a general framework to show that  $\{x_n\}$  converges strongly to  $\tilde{x}$  where  $\tilde{x}$  is the unique solution of (5) for a fixed nonexpansive mapping  $T$  related to the sequence  $\{T_n\}$ . The key ingredient is Lemma 26 given in Section 3 which is valid in the three previous contexts. Then we show that by specifying the sequence  $T_n$ , we can recover and extend some known convergence theorems [1,4,9–11,15,16,19]. Note also that in Eq. (7),  $f$  is an  $\alpha$ -contraction, but following [17] it is easy to show that  $f$  can be replaced by a Meir–Keeler contraction (Lemma 31 in Section 3 is devoted to this extension). The paper is organized as follows: a key lemma and two theorems are proved in Section 3 using a set of lemmata which are postponed to the last section of the paper and which are verbatim or slight extensions of known results. Then in a collection of subsections, known convergence theorems are revisited with shorter proofs.

## 2. Main theorems

In the sequel a  $\mathcal{B}$  real Banach space, will denote when not specifically stated a real Banach space with a normalized duality mapping uniformly norm-to-weak\* continuous on bounded sets (which is the case for  $\mathcal{B}_{us}$  or  $\mathcal{B}_{rug}$ ) or a reflexive Banach space which admits a weakly continuous duality mapping  $J_\phi$  with gauge  $\phi$  ( $\mathcal{B}_{rwc}$ ).

**H<sub>1,N</sub>**: For a fixed given  $N \geq 1$  and a given sequence  $\{\alpha_n\}$ , a sequence of mappings  $\{T_n\}$  will be said to verify **H<sub>1,N</sub>**, if for any bounded subset  $B$  of  $C$ , we have

$$\sup\{\|(1 - \alpha_{n+N})T_{n+N}z - (1 - \alpha_n)T_nz\| : z \in B\} \leq \delta_n M \quad (8)$$

with either (i)  $\sum_0^\infty |\delta_n| < \infty$  or (i')  $\limsup_{n \rightarrow \infty} \delta_n / \alpha_n \leq 0$  and  $M$  a constant.

**Remark 1.** Note that using Corollary 30  $\{\delta_n\}$  can be replaced by  $\{\mu_n + \rho_n\}$  where  $\{\mu_n\}$  satisfies (i) and  $\{\rho_n\}$  satisfies (i').

**Remark 2.** Note that when  $\alpha_n \in (0, 1)$  we have for  $z \in C$ :

$$\|(1 - \alpha_{n+N})T_{n+N}z - (1 - \alpha_n)T_nz\| \leq |\alpha_{n+N} - \alpha_n| \|T_{n+N}z\| + \|T_{n+N}z - T_nz\|. \quad (9)$$

Thus, when  $\{\alpha_n\}$  satisfies **H<sub>3,N</sub>** (given below), if for each bounded sequence  $\{z_n\}$ ,  $\{T_n z_n\}$  is bounded and either (vi)  $\sum_{n=0}^\infty \sup\{\|T_{n+N}z - T_nz\| : z \in B\} < \infty$  or (vi')  $\sup\{\|T_{n+N}z - T_nz\| : z \in B\} / \alpha_n \rightarrow 0$  for any bounded subset  $B$  of  $C$ , then **H<sub>1,N</sub>** is satisfied (again using previous remark about mixing between conditions with or without prime). In the previous case, **H<sub>1,N</sub>** is thus implied by **H'\_{1,N}** which is stated now.

**H'\_{1,N}**: For a fixed given  $N \geq 1$  and a given sequence  $\{\alpha_n\}$  which satisfies **H<sub>3,N</sub>** a sequence of mappings  $\{T_n\}$  will be said to verify **H'\_{1,N}**, if for any bounded subset  $B$  of  $C$ , we have  $\sup\{\|T_{n+N}z - T_nz\| : z \in B\} \leq \rho_n$  with either (vi)  $\sum_{n=0}^\infty \rho_n < \infty$  or (vi')  $\rho_n / \alpha_n \rightarrow 0$ .

**H<sub>2,p</sub>**: For a given  $p \in X$ , a sequence  $\{x_n\}$  will be said to verify **H<sub>2,p</sub>** if we have

$$\limsup_{n \rightarrow \infty} (f(p) - p, J(x_n - p)) \leq 0. \quad (10)$$

**H<sub>3,N</sub>**: For a fixed given  $N \geq 1$ , a sequence of real numbers  $\{\alpha_n\}$  will be said to verify **H<sub>3,N</sub>** if the sequence  $\{\alpha_n\}$  is such that (i)  $\alpha_n \in (0, 1)$ , (ii)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , (iii)  $\sum_{n=0}^\infty \alpha_n = \infty$  and either (iv)  $\sum_{n=0}^\infty |\alpha_{n+N} - \alpha_n| < \infty$  or (iv')  $\lim_{n \rightarrow \infty} (\alpha_{n+N} / \alpha_n) = 1$ .

We first start with a general lemma:

**Lemma 3.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T_n : C \mapsto C$  a sequence of nonexpansive mapping,  $T$  a nonexpansive mapping and  $f \in \Pi_C$ . We assume that  $\text{Fix}(T) \neq \emptyset$  and that for all  $n \in \mathbb{N}$ ,  $\text{Fix}(T) \subset \text{Fix}(T_n)$ . Let  $\{\alpha_n\}$  be a sequence of real numbers for which there exists a fixed  $N \geq 1$  such that **H<sub>3,N</sub>** is satisfied and suppose that there exists  $p \in \text{Fix}(T)$  such that **H<sub>2,p</sub>** is satisfied. Then the sequence  $\{x_n\}$  defined by (7) converges strongly to  $p$ .

**Proof.** The proof uses a set of lemmata which are given in Section 3. Since  $p$  is in  $\text{Fix}(T_n)$  for all  $n$  we can use Lemma 23 to obtain the boundedness of the sequence  $\{x_n\}$ . Thus we can conclude using Lemma 28.  $\square$

We now give a first theorem in which we assume that the sequence  $\{T_n\}$  converges to  $T$  uniformly on bounded subsets of  $C$ . Note that when  $f$  is constant,  $N = 1$  and **H'\_{1,N}** is satisfied with (i) the result is covered by [1, Theorem 3.4] (for the case of real Banach space with a normalized duality mapping uniformly norm-to-weak\* continuous on bounded sets) and by the forthcoming paper [11, Theorem 3.1] (for the case of reflexive Banach space which admits a weakly continuous duality mapping  $J_\phi$  with gauge  $\phi$ ). Note also that when  $N = 1$  and **H'\_{1,N}** is satisfied with (i) we can use [1, Lemma 3.2] to prove the existence of the limit  $T$  and not assume it existence.

**Theorem 4.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T_n : C \mapsto C$  a sequence of nonexpansive mapping,  $T$  a nonexpansive mapping and  $f \in \Pi_C$ . We assume that  $\text{Fix}(T) \neq \emptyset$  and that for all  $n \in \mathbb{N}$ ,  $\text{Fix}(T) \subset \text{Fix}(T_n)$ . Let  $\{\alpha_n\}$  be a sequence of real numbers and a fixed  $N \geq 1$  for which **H<sub>3,N</sub>** is satisfied. Suppose that for the same  $N$ , **H<sub>1,N</sub>** or **H'\_{1,N}** is satisfied and that for any bounded subset  $B$  of  $C$ ,  $\sup\{\|T_nz - Tz\| : z \in B\} \rightarrow 0$ . Then the sequence  $\{x_n\}$  defined by (7) converges strongly to  $p = Q(f)$ .

**Proof.** We just need to prove that **H<sub>2,p</sub>** is satisfied for  $p = Q(f)$  in order to conclude with Lemma 3. We first show that if **H'\_{1,N}** is satisfied then **H<sub>1,N</sub>** is also satisfied. As in Lemma 3,  $\{x_n\}$  is a bounded sequence. Then, let  $B$  a bounded subset of  $C$  which contains the sequence  $\{x_n\}$ , we have:

$$\|T_n x_n\| \leq \sup\{\|T_n z - Tz\| : z \in B\} + \|T x_n\|. \quad (11)$$

This implies that  $\{T_n(x_n)\}$  is bounded. Notice, thanks to Remark 2, that **H<sub>1,N</sub>** is implied by **H'\_{1,N}**. Using Lemma 24 and Corollary 25, we obtain the convergence of  $\|T x_n - x_n\|$ . We can then apply Lemma 26 to get **H<sub>2,p</sub>** for  $p = Q(f)$ .  $\square$

Theorem 4 can be extended as follows when a limit operator  $T$  cannot be found.

**Theorem 5.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T_n : C \mapsto C$  a sequence of nonexpansive mapping,  $T$  a nonexpansive mapping and  $f \in \Pi_C$ . We assume that  $\text{Fix}(T) \neq \emptyset$  and that for all  $n \in \mathbb{N}$ ,  $\text{Fix}(T) \subset \text{Fix}(T_n)$ . Let  $\{\alpha_n\}$  be a sequence of real numbers and a fixed  $N \geq 1$  for which  $\mathbf{H}_{3,N}$  is satisfied. Suppose that for the same  $N$ ,  $\mathbf{H}'_{1,N}$  is satisfied and that from each subsequence  $\sigma(n)$  we can extract a subsequence  $\mu(n)$  and find a fixed mapping  $T_\mu$  such that for any bounded subset  $B$  of  $C$ ,

$$\sup\{\|T_{\mu(n)}z - T_\mu z\| : z \in B\} \rightarrow 0.$$

If  $F = \text{Fix}(T_\mu)$  does not depend on  $\mu$  and is not empty, then the sequence  $\{x_n\}$  defined by (7) converges strongly to  $p = \mathbf{Q}_F(f)$ .

**Proof.** We first need to prove that  $\{T_n x_n\}$  is bounded. As in the previous theorem  $\{x_n\}$  is a bounded sequence. Suppose that  $\{T_n x_n\}$  is not bounded. Then, there exists a subsequence  $\sigma(n)$  such that  $\|T_{\sigma(n)} x_{\sigma(n)}\| \rightarrow +\infty$ . By hypothesis, we can then find a subsequence  $\mu(n)$  and find  $T_\mu$  such that  $\|T_{\mu(n)} x_{\mu(n)} - T_\mu x_{\mu(n)}\| \rightarrow 0$ . Since  $\|T_\mu x_{\mu(n)}\|$  is bounded, we should then have  $\|T_{\mu(n)} x_{\mu(n)}\| \rightarrow 0$  which is a contradiction.

We need now to prove that  $\mathbf{H}_{2,p}$  is satisfied for  $p = \mathbf{Q}_F(f)$ . Using Remark 2, notice that  $\mathbf{H}_{1,N}$  is implied by  $\mathbf{H}'_{1,N}$ . Using  $\mathbf{H}_{1,N}$ , we easily obtain that  $\|x_n - T_n x_n\| \rightarrow 0$  by an argument similar to Corollary 25. Then  $\mathbf{H}_{2,p}$  with  $p = \mathbf{Q}(f)$  follows from Corollary 27.  $\square$

We consider the case of composition. Assume that  $\{T_n^1\}$  and  $\{T_n^2\}$  satisfy  $\mathbf{H}'_{1,N}$  with sequences denoted by  $\{\rho_n^1\}$  and  $\{\rho_n^2\}$ . Assume also that for a bounded sequence  $\{z_n\}$  then the sequences  $\{T_{n+N}^2 z_n\}$  and  $\{T_{n+N}^1 T_{n+N}^2 z_n\}$  are also bounded. Then it is straightforward, since the mappings  $T_n^1$  are nonexpansive, that:

$$\|T_{n+N}^1 T_{n+N}^2 z_n - T_n^1 \cdots T_n^2 z_n\| \leq \rho_n^1 + \|T_{n+N}^2 z_n - T_n^2 z_n\|.$$

Thus the composition  $T_n^1 \circ T_n^2$  satisfies  $\mathbf{H}'_{1,N}$  with  $\rho_n \stackrel{\text{def}}{=} \rho_n^1 + \rho_n^2$ . This leads us to propose the following corollary:

**Corollary 6.** Assume that the hypotheses of Theorem 5 are satisfied for the sequence  $\{T_n^1\}$  with  $\mathbf{H}'_{1,N}$  and for  $\{T_n^2\}$  also with  $\mathbf{H}'_{1,N}$ . Then the conclusion of Lemma 3 remains for the sequence  $\{T_n^1 \circ T_n^2\}$  with  $p = \mathbf{Q}_F(f)$  and  $F = \text{Fix}(T_\mu^1 \circ T_\rho^2)$ .

**Proof.** As pointed out before the statement of the corollary, the composition  $T_n^1 \circ T_n^2$  satisfies  $\mathbf{H}'_{1,N}$ . Consider a subsequence  $\sigma(n)$ . We can find first a subsequence  $\mu(n)$  and a scalar  $\mu$  such that:

$$\|T_{\mu(n)}^2 x_{\mu(n)} - T_\mu^2 x_{\mu(n)}\| \rightarrow 0.$$

Then, using properties of the  $T_n^1$  sequence, there exists a subsequence  $\rho(n)$  and a scalar  $\rho$  such that:

$$\|T_{\rho(n)}^1 T_{\rho(n)}^2 x_{\rho(n)} - T_\rho^1 T_\rho^2 x_{\rho(n)}\| \rightarrow 0.$$

Since we have:

$$\|T_{\rho(n)}^1 T_{\rho(n)}^2 x_{\rho(n)} - T_\rho^1 T_\mu^2 x_{\rho(n)}\| \leq \|T_{\rho(n)}^1 T_{\rho(n)}^2 x_{\rho(n)} - T_\rho^1 T_{\rho(n)}^2 x_{\rho(n)}\| + \|T_{\rho(n)}^2 x_{\rho(n)} - T_\mu^2 x_{\rho(n)}\|,$$

we get the result for the composition.  $\square$

Recall that a mapping  $T$  is *attracting nonexpansive* if it is nonexpansive and satisfies:

$$\|Tx - p\| < \|x - p\| \quad \text{for all } x \notin \text{Fix } T \text{ and } p \in \text{Fix } T. \quad (12)$$

In particular a *firmly nonexpansive* mapping, i.e.  $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$  is attracting nonexpansive [7].

**Remark 7.** In the previous corollary, we obtain a fixed point of a composition. In practice the aim is to obtain a common fixed point of two mappings. If the mappings  $T_\mu^1$  and  $T_\rho^2$  are nonexpansive, have a common fixed point and  $T_\mu^1$  or  $T_\rho^2$  is attracting then we will have  $\text{Fix } T_\mu^1 \cap \text{Fix } T_\rho^2 = \text{Fix } T_\mu^1 \circ T_\rho^2$ . The proof is contained in [2, Proposition 2.10(i)] and given in Lemma 32 for completeness.

**Remark 8.** Note that if  $X$  is a strictly convex Banach space, then for  $\lambda \in (0, 1)$  the mapping  $T_\lambda \stackrel{\text{def}}{=} (1 - \lambda)I + \lambda T$  is attracting nonexpansive when  $T$  is nonexpansive. Extension to a set of  $N$  operators is immediate by induction. This gives a way to build attracting nonexpansive mappings. Combined with the previous remark it gives [18, Proposition 3.1].

**Remark 9.** Note also that, when  $X$  is strictly convex, an other way to obtain  $F = \bigcap_i \text{Fix}(T_i)$  for a sequence of nonexpansive mappings  $\{T_i\}$  is to use  $T = \sum_i \lambda_i T_i$  with a sequence  $\{\lambda_i\}$  of real positive numbers such that  $\sum_i \lambda_i = 1$  [3, Lemma 3].

### 2.1. Example 1

**Theorem 10.** (See [19, Theorem 4.2].) Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T : C \mapsto C$  a nonexpansive mapping with  $\text{Fix}(T) \neq \emptyset$ , and  $f$  an  $\alpha$ -contraction. If the sequence  $\{\alpha_n\}$  satisfies  $\mathbf{H}_{3,1}$ , then the sequence  $\{x_n\}$  defined by (35) with  $T_n \stackrel{\text{def}}{=} T$  converges strongly to  $\mathbf{Q}(f)$ .

**Proof.** The sequence  $T_n$  does not depend on  $n$ . We just apply Theorem 4 to get the result. Of course, if the sequence  $\{x_n\}$  is bounded then  $\{T_n(x_n) = Tx_n\}$  is bounded and Eq. (8) of  $\mathbf{H}_{1,1}$  is then satisfied with  $\delta_n = |\alpha_n - \alpha_{n+1}|$ . Since  $\{\alpha_n\}$  satisfies  $\mathbf{H}_{3,1}$ ,  $\{\delta_n\}$  satisfies  $\mathbf{H}_{1,1}$ . We also have  $\|T_n x_n - Tx_n\| = 0$  and the conclusion follows.  $\square$

**Remark 11.** Suppose now that  $T \stackrel{\text{def}}{=} \sum_i \lambda_i T_i$  where  $\{\lambda_i\}$  is a sequence of positive real numbers such that  $\sum_i \lambda_i = 1$  and the  $T_i$  mappings are all nonexpansive. Then, we can apply Theorem 10 to obtain the strong convergence of the sequence  $\{x_n\}$  to  $\mathbf{Q}_{\text{Fix } T}(f)$ . Moreover, if we assume that  $X$  is strictly convex, then using Remark 9 we obtain a strong convergence to  $\mathbf{Q}_F(f)$  with  $F \stackrel{\text{def}}{=} \bigcap_{i \in I} \text{Fix}(T_i)$ .

We recover [10, Theorem 4] by considering the case  $\lambda_i$  depending on  $n$ .

**Corollary 12.** Let  $X$  be a strictly convex  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T_i : C \mapsto C$  for  $i \in I$  a finite family of nonexpansive mappings with  $\bigcap_{i \in I} \text{Fix}(T_i) \neq \emptyset$ , and  $f$  an  $\alpha$ -contraction. For a sequence  $\{\alpha_n\}$  satisfying  $\mathbf{H}_{3,1}$ , we consider the sequence  $\{x_n\}$  defined by (35) with  $T_n \stackrel{\text{def}}{=} \sum_{i \in I} \lambda_{i,n} T_i$ . Assume that for all  $i$  and  $n$ ,  $\lambda_{i,n} \in [a, b]$  with  $a > 0$ ,  $b < \infty$ , and either  $\sum_n \lambda_{i,n} < \infty$  or  $\lambda_{i,n}/\alpha_n \rightarrow 0$ . Then  $\{x_n\}$  converges strongly to  $\mathbf{Q}_F(f)$  with  $F = \bigcap_{i \in I} \text{Fix}(T_i)$ .

**Proof.** The proof is a consequence of Theorem 5. Indeed since the  $\lambda_{i,n}$  are bounded,  $T_n x_n$  remains bounded for a bounded sequence  $x_n$ . Then  $T_n$  satisfies  $\mathbf{H}'_{1,1}$  with  $\rho_n = \sum_{i \in I} \lambda_{i,n}$ . By taking from each given subsequence  $\sigma(n)$  a subsequence  $\mu(n)$  such that  $\lim_{n \rightarrow \infty} \lambda_{i,\mu(n)} = \bar{\lambda}_i$  for all  $i \in I$ , we can use Theorem 5. For a strictly convex space  $X$ , the fixed points of  $T_{\bar{\lambda}} \stackrel{\text{def}}{=} \sum_{i \in I} \bar{\lambda}_i T_i$  do not depend on  $\bar{\lambda}$  and are equal to  $\bigcap_{i \in I} \text{Fix}(T_i)$ . This ends the proof.  $\square$

### 2.2. Example 1'

In [16] the following algorithm is considered:

$$y_{n+1} = P(\alpha_n f(y_n) + (1 - \alpha_n)Ty_n), \quad (13)$$

where  $P : X \mapsto C$  is a sunny nonexpansive retraction,  $f : C \mapsto X$  an  $\alpha$ -contraction and  $T : C \mapsto X$  a nonexpansive mapping such that  $\text{Fix}(T) \neq \emptyset$ .

Let the sequence  $x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)Ty_n$ . Then we have  $y_{n+1} = Px_{n+1}$  and thus

$$x_{n+1} = \alpha_n f(P(x_n)) + (1 - \alpha_n)T(P(x_n)). \quad (14)$$

Since  $f \circ P$  is an  $\alpha$ -contraction from  $X$  onto  $X$  and  $T \circ P$  a nonexpansive mapping from  $X$  onto  $X$ , we can use the previous theorem to get the strong convergence of the sequence  $\{x_n\}$  to  $x$ , a fixed point of  $T \circ P$ , such that  $x = P_{\text{Fix}(T \circ P)} f(T(x))$  ( $P_S$  is the metric projection on  $S$ ). This implies the strong convergence of the initial sequence  $\{y_n\}$  to  $y = P(x)$  and since  $x$  is a fixed point of  $T \circ P$ ,  $y$  is a fixed point of  $P \circ T$ .

If in addition  $X$  is such that the mapping  $J$  (or  $J_\phi$ ) is norm-to-weak\* continuous (i.e.  $X$  is smooth) and  $T$  satisfies the weakly inward condition, then we can use the result of [16, Lemma 1.2] which states that  $\text{Fix}(T) = \text{Fix}(P \circ T)$ , to conclude that  $y$  is in fact a fixed point of  $T$ . Then we recover the result of [16, Theorem 2.4].

### 2.3. Example 2

We consider now the example given in [9]. Let  $\{x_n\}$  be the sequence given by:

$$y_n = \beta_n x_n + (1 - \beta_n)Tx_n,$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n.$$

Let  $\{T_n\}$  a sequence of mappings defined by  $T_n x \stackrel{\text{def}}{=} \beta_n x + (1 - \beta_n)Tx$ . We rewrite the previous equations as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n. \quad (15)$$

**Theorem 13.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T : C \mapsto C$  a nonexpansive mapping with  $\text{Fix}(T) \neq \emptyset$ , and  $f$  an  $\alpha$ -contraction. We assume that the sequence  $\{\alpha_n\}$  satisfies  $\mathbf{H}_{3,1}$  and the sequence  $\{\beta_n\}$  converges to zero and satisfy either  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$  or  $|\beta_{n+1} - \beta_n|/\alpha_n \rightarrow 0$ . Then, the sequence  $\{x_n\}$  defined by (15) converges strongly to  $\mathbf{Q}(f)$ .

This theorem is very similar to [9, Theorem 1], where  $f$  was supposed to be constant. It could be covered by Corollary 12; if strict convexity was assumed.

**Proof.** We easily check that the fixed points  $p$  of  $T$  are fixed points of  $T_n$  for all  $n \in \mathbb{N}$  and  $T_n$  is nonexpansive for all  $n$ . Thus by Lemma 23 the sequence  $\{x_n\}$  is bounded. If the sequence  $\{x_n\}$  is bounded then  $\|T_n(x_n)\| \leq \max(\|x_n\|, \|Tx_n\|)$  is bounded too. Since:

$$\|T_n y_n - T y_n\| \leq \beta_n (\|y_n\| + \|T y_n\|), \quad (16)$$

we have  $\|T_n y_n - T y_n\| \rightarrow 0$  for each bounded sequence  $\{y_n\}$ . It is easily checked that  $\mathbf{H}_{1,1}$  is satisfied with  $\delta_n = |\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|$ . The conclusion follows from Theorem 4.  $\square$

#### 2.4. Example 3

We consider here the accretive operators example given in [9] or [20]:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad (17)$$

where  $T_n x \stackrel{\text{def}}{=} J_{r_n} x$  and  $J_\lambda$  is the resolvent of an  $m$ -accretive operator  $A$ ,  $J_\lambda x = (I + \lambda A)^{-1}$ . The following theorem is similar to [20, Theorems 4.2, 4.4] or [9, Theorem 2].

**Theorem 14.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $A$  an  $m$ -accretive operator in  $X$  such that  $A^{-1}(0) \neq \emptyset$ . We assume here that  $C \stackrel{\text{def}}{=} \overline{D(A)}$  where  $D(A)$  is the domain of  $A$  and suppose that  $C$  is convex. Suppose that  $\mathbf{H}_{3,1}$  is satisfied by the sequence  $\{\alpha_n\}$  and that the sequence  $r_n$  is such that  $r_n \geq \epsilon > 0$  and either  $\sum_{n=0}^{\infty} |1 - r_n/r_{n+1}| < \infty$  or  $|1 - r_n/r_{n+1}|/\alpha_n \rightarrow 0$ . Then the sequence  $\{x_n\}$  defined by (17) converges strongly to a zero of  $A$ .

**Proof.** Note that [20, p. 632], for  $\lambda > 0$ ,  $\text{Fix}(J_\lambda) = F$  where  $F$  is the set of zero of  $A$  and for an  $m$ -accretive operator  $A$ ,  $J_\lambda$  is nonexpansive from  $X \mapsto \overline{D(A)}$ . Using the resolvent identity  $J_\lambda x = J_\mu((\mu/\lambda)x + (1 - \mu/\lambda)J_\lambda x)$ , we obtain:

$$\|T_{n+1} z_n - T_n z_n\| \leq \left| 1 - \frac{r_n}{r_{n+1}} \right| (\|z_n\| + \|T_n z_n\|). \quad (18)$$

Since the sequence  $T_n y_n$  is bounded for a bounded sequence  $y_n$  (for  $p \in A^{-1}(0)$  we have  $\|T_n y_n - p\| \leq \|y_n - p\|$ ), we can use Remark 2 in order to obtain  $\mathbf{H}_{1,1}$ . We thus have  $\|x_{n+1} - x_n\| \rightarrow 0$  by Lemma 24. This implies  $\|x_n - T_n x_n\| \rightarrow 0$  by:

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \leq \|x_n - x_{n+1}\| + \alpha_n (\|f(x_n)\| + \|T_n(x_n)\|).$$

Take now  $r$  such that  $0 < r < \epsilon$  and define  $T \stackrel{\text{def}}{=} J_r$ . Then we have:

$$\|T_n x_n - T x_n\| \leq \left| 1 - \frac{r}{r_n} \right| \|x_n - T_n x_n\|. \quad (19)$$

We thus obtain that  $x_n - T x_n \rightarrow 0$  from:

$$\|x_n - T x_n\| \leq \|x_n - T_n x_n\| + \|T_n x_n - T x_n\|. \quad (20)$$

The conclusion is obtained thanks to Theorem 4.  $\square$

#### 2.5. Example 4

We consider here the example given in [15]

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \quad (21)$$

where  $T_n = Q_{n \bmod N}$ , with  $N \geq 1$  a fixed integer, and  $(Q_l)_{l=0, \dots, N-1}$  a family of nonexpansive mappings.

**Theorem 15.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $Q_l : C \mapsto C$  for  $l \in \{1, \dots, N\}$  a family of nonexpansive mappings such that  $F \stackrel{\text{def}}{=} \bigcap_{l=0}^{N-1} \text{Fix}(Q_l)$  is not empty and

$$\bigcap_{l=0}^{N-1} \text{Fix}(Q_l) = \text{Fix}(T_{n+N} T_{n+N-1} \cdots T_{n+1}) \quad \text{for all } n \in \mathbb{N} \quad (22)$$

and  $f$  an  $\alpha$ -contraction. When the sequence  $\{\alpha_n\}$  satisfies  $\mathbf{H}_{3,N}$  then the sequence  $\{x_n\}$  defined by (21) converges strongly to  $\mathbf{Q}_F(f)$ .

**Proof.** By Lemma 23, since the mappings  $T_n$  have a common fixed point, the sequence  $\{x_n\}$  is bounded. Since the sequence of mappings  $T_n$  is periodic, the sequence  $\{T_n x_n\}$  is bounded and Eq. (8) of  $\mathbf{H}_{1,N}$  is obtained for  $\delta_n = |\alpha_n - \alpha_{n+N}|$  using (9). Since  $\{\alpha_n\}$  satisfies  $\mathbf{H}_{3,N}$ ,  $\{\delta_n\}$  satisfies  $\mathbf{H}_{1,N}$ . Thus, using Lemma 24, we get that  $\|x_{n+N} - x_n\| \rightarrow 0$ . Since  $\|x_{n+1} - T_n x_n\| \leq \alpha_n (\|f(x_n)\| + \|T_n x_n\|)$ , we have  $\|x_{n+1} - T_n x_n\| \rightarrow 0$ . We introduce the sequence of mappings  $A_n^{(N,\alpha)} \stackrel{\text{def}}{=} T_{n+N-1} \cdots T_{n+\alpha}$  for  $\alpha \neq N$  and  $A_n^{(N,N)} = \text{Id}$ . Using Lemma 16, given below, we conclude that:  $\|x_{n+N} - A_n^{(N,0)} x_n\| \rightarrow 0$ . This combined with  $\|x_{n+N} - x_n\| \rightarrow 0$  gives  $\|x_{n+N} - A_n^{(N,0)} x_n\| \rightarrow 0$ . Note that the mappings  $A_n^{(N,0)}$  are in finite number. They are all nonexpansive and share common fixed points by hypothesis. Thus we can prove that  $\mathbf{H}_{2,p}$  is satisfied for  $p = \mathbf{Q}_F(f)$ . Let  $p = \mathbf{Q}_F(f)$ . Assume that  $\mathbf{H}_{2,p}$  is not satisfied. Then it is possible to extract a subsequence of  $\{x_{\sigma(n)}\}$  such that:

$$\lim_{n \rightarrow \infty} (f(p) - p, J(x_{\sigma(n)} - p)) \leq 0. \quad (23)$$

It is then possible to find  $q \in \{0, \dots, N-1\}$  and extract a new subsequence  $\mu(n)$  from  $\sigma(n)$  such that  $\mu(n) \bmod N = q$ . We thus have  $\|x_{\mu(n)} - T x_{\mu(n)}\| \rightarrow 0$ , with  $T \stackrel{\text{def}}{=} A_q^{(N,0)}$  which is now a fixed mapping and  $\text{Fix}(T) = F$ . Then  $\mathbf{H}_{2,p}$  should be true thanks to Lemma 26 and this leads to a contradiction. The conclusion follows by Lemma 28.  $\square$

**Lemma 16.** Let  $N \in \mathbb{N}$ ,  $\alpha \in \{0, \dots, N\}$ ,  $A_n^{(N,\alpha)} \stackrel{\text{def}}{=} T_{n+N-1} \cdots T_{n+\alpha}$  for  $\alpha \neq N$  and  $A_n^{(N,N)} = \text{Id}$ . Assume that  $\|x_{n+1} - T_n x_n\| \rightarrow 0$ . Then we have  $\|x_{n+N} - A_n^{(N,0)} x_n\| \rightarrow 0$ .

**Proof.** Using the fact that  $A_n^{(N,\alpha)}$  is nonexpansive we have for  $\alpha \in \{0, \dots, N-1\}$ :

$$\|A_n^{(N,\alpha+1)} x_{n+\alpha+1} - A_n^{(N,\alpha)} x_{n+\alpha}\| = \|A_n^{(N,\alpha+1)} x_{n+\alpha+1} - A_n^{(N,\alpha+1)} T_{n+\alpha} x_{n+\alpha}\| \leq \|x_{n+\alpha+1} - T_{n+\alpha} x_{n+\alpha}\|.$$

Thus we have:

$$\|x_{n+N} - A_n^{(N,0)} x_n\| \leq \sum_{\alpha=0}^{N-1} \|x_{n+\alpha+1} - T_{n+\alpha} x_{n+\alpha}\|$$

and the result follows.  $\square$

## 2.6. Example 5

Let  $\Gamma_n^{(j)}$  for  $j \in \{1, \dots, m\}$  be a sequence of mappings defined recursively as follows:

$$\Gamma_n^{(j)} x \stackrel{\text{def}}{=} \beta_n^{(j)} x + (1 - \beta_n^{(j)}) T_j \Gamma_n^{(j+1)} x \quad \text{and} \quad \Gamma_n^{(m+1)} x = x \quad (24)$$

where the sequences  $\{\beta_n^{(j)}\} \in (0, 1)$ , and  $\{T_j\}$  for  $j \in \{1, \dots, m\}$  are nonexpansive mappings. We want to prove the convergence of the sequence defined by:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \Gamma_n^{(1)} x_n. \quad (25)$$

**Theorem 17.** Let  $X$  be a  $\mathcal{B}$  real Banach space,  $C$  a closed convex subset of  $X$ ,  $T_j : C \mapsto C$  for  $j \in \{1, \dots, m\}$  a family of nonexpansive mappings such that  $\bigcap_{l=1}^m \text{Fix}(T_j)$  is not empty and  $f$  an  $\alpha$ -contraction. When the sequence  $\{\alpha_n\}$  satisfies  $\mathbf{H}_{3,N}$  and for  $j \in \{1, \dots, m\}$  the sequence  $\{\beta_n^{(j)}\}$  satisfies  $\lim_{n \rightarrow \infty} \beta_n^{(j)} = 0$  and either  $\sum_{n=0}^{\infty} |\beta_{n+1}^{(j)} - \beta_n^{(j)}| < \infty$  or  $|\beta_{n+1}^{(j)} - \beta_n^{(j)}|/\alpha_n \rightarrow 0$ . Then the sequence defined by (25) converges strongly to  $\mathbf{Q}_F(f)$  associated to  $F = \text{Fix}(T_1 \cdots T_m)$ .

**Proof.** By an elementary induction  $\Gamma_n^{(1)}$  is a nonexpansive mapping. If we assume that  $p$  is a common fixed point to the mappings  $T_i$  then  $p$  is a fixed point of the mappings  $\Gamma_n^{(j)}$ . By Lemma 23 the sequence  $\{x_n\}$  is bounded. Using Lemma 19 given below, combined with the boundedness of  $\{x_n\}$ , we get that  $\mathbf{H}_{1,1}$  is valid with

$$\delta_n = \sum_{p=1}^m |\beta_{n+1}^{(p)} - \beta_n^{(p)}| + |\alpha_{n+1} - \alpha_n|. \quad (26)$$

If we can prove that

$$\|\Gamma_n^{(1)}x_n - T_1T_2 \cdots T_mx_n\| \rightarrow 0, \quad (27)$$

then the conclusion will be given by Theorem 4. Assertion (27) can easily be obtained by induction on  $\|\Gamma_n^{(j)}x_n - T_j \cdots T_mx_n\|$ , since we have:

$$\begin{aligned} \|\Gamma_n^{(j)}x_n - T_j \cdots T_mx_n\| &\leq \beta_n^{(j)}(\|x_n\| + \|T_j \cdots T_mx_n\|) + (1 - \beta_n) \|T_j \Gamma_n^{(j+1)}x_n - T_j \cdots T_mx_n\| \\ &\leq \beta_n^{(j)}(\|x_n\| + \|T_j \cdots T_mx_n\|) + \|\Gamma_n^{(j+1)}x_n - T_{j+1} \cdots T_mx_n\|. \quad \square \end{aligned}$$

**Remark 18.** For  $m = 1$  we recover Theorem 13.

**Lemma 19.** Let  $\Gamma_n^{(j)}$  be the sequence of mappings defined by (24). Then we have for  $j \in \{1, \dots, m\}$ :

$$\|\Gamma_{n+1}^{(j)}x - \Gamma_n^{(j)}x\| \leq \left\{ \sum_{p=j}^m |\beta_{n+1}^{(p)} - \beta_n^{(p)}| \right\} K, \quad (28)$$

where  $K$  is a constant which depends on the mappings  $(T_p)_{p \geq j}$  and  $x$ .

**Proof.** Note that:

$$\|\Gamma_n^{(j)}x\| \leq \|x\| + \|T_j(\Gamma_n^{(j+1)}x)\|. \quad (29)$$

This implies that  $\|\Gamma_n^{(j)}x\|$  is bounded by a constant which depends on the mappings  $(T_p)_{p \geq j}$  and  $x$  but not on  $n$ . Then, using the definition of  $\Gamma_n^{(j)}$ , we have:

$$\|\Gamma_{n+1}^{(j)}x - \Gamma_n^{(j)}x\| \leq |\beta_{n+1}^{(j)} - \beta_n^{(j)}|(\|x\| + \|T_j \Gamma_n^{(j+1)}x\|) + \|T_j \Gamma_{n+1}^{(j+1)}(x) - T_j \Gamma_n^{(j+1)}(x)\|. \quad (30)$$

Since  $T_j$  is a nonexpansive mapping we get:

$$\|\Gamma_{n+1}^{(j)}x - \Gamma_n^{(j)}x\| \leq |\beta_{n+1}^{(j)} - \beta_n^{(j)}|(\|x\| + \|T_j \Gamma_n^{(j+1)}x\|) + \|\Gamma_{n+1}^{(j+1)}(x) - \Gamma_n^{(j+1)}(x)\|.$$

For  $j = m$  we have  $\|\Gamma_{n+1}^{(m+1)}(x) - \Gamma_n^{(m+1)}(x)\| = 0$  and we get the result by induction.  $\square$

Note that Lemma 19 remains valid for the sequence

$$\Gamma_n^{(j)}x \stackrel{\text{def}}{=} \beta_n^{(j)}g(x) + (1 - \beta_n^{(j)})T_j\Gamma_n^{(j+1)}x \quad \text{and} \quad \Gamma_n^{(m+1)}x = x \quad (31)$$

if  $g$  is a nonexpansive mapping.

## 2.7. Example 6

We consider here the example given in [4]:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n,$$

where  $T_n x \stackrel{\text{def}}{=} P_C(x - \lambda_n Ax)$  and  $P_C$  is the metric projection from  $X$  to  $C$ . The aim is to find a solution of the variational inequality problem that is to find  $x \in C$  such that  $\langle Ax, y - x \rangle \geq 0$  for all  $y \in C$ . The set of solutions of the variational inequality problem is denoted by  $\text{VI}(C, A)$ . The operator  $A$  is said to be  $\mu$ -inverse-strongly monotone if

$$\langle x - y, Ax - Ay \rangle \geq \mu \|Ax - Ay\|^2 \quad \text{for all } x, y \in C.$$

The next theorem is similar to [4, Proposition 3.1].

**Theorem 20.** Let  $X$  be a real Hilbert space,  $C$  a nonempty closed convex,  $f$  an  $\alpha$ -contraction, and let  $A$  be a  $\mu$ -inverse-strongly monotone mapping of  $H$  into itself such that  $\text{VI}(C, A) \neq \emptyset$ . Assume that  $\mathbf{H}_{3,1}$  is satisfied and that  $\{\lambda_n\}$  is chosen so that  $\lambda_n \in [a, b]$  for some  $a, b$  with  $0 < a < b < 2\mu$  and  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ . Then the sequence  $\{x_n\}$  generated by (7) converges strongly to  $\mathbf{Q}_F(f)$  associated to  $F = \text{Fix}(T_\lambda)$  where  $T_\lambda(x) \stackrel{\text{def}}{=} P_C(x - \lambda Ax)$ .  $F = \text{Fix}(T_\lambda)$  does not depend on  $\lambda$  for  $\lambda > 0$  and is equal to  $\text{VI}(C, A)$ .



**Proof.** For  $\lambda > 0$ , let  $T_\lambda x \stackrel{\text{def}}{=} P_C(x - \lambda Ax)$ . As  $X$  is a Hilbert space we have  $\text{Fix}(T_\lambda) = \text{VI}(C, A)$ . Since  $A$  is  $\mu$ -inverse-strongly monotone then for,  $\lambda \leq 2\mu$ ,  $I - \lambda A$  is nonexpansive. Thus the mappings  $T_n$  are nonexpansive and  $\text{Fix}(T_n) = \text{VI}(C, A) \neq \emptyset$ . By Lemma 23, the sequence  $\{x_n\}$  is bounded. Since  $\|T_n z\| \leq K(\|z\| + 2\mu\|Az\|)$ , the sequence  $\{T_n x_n\}$  is bounded too. We also have  $\|T_{n+1} z_n - T_n z_n\| \leq |\lambda_{n+1} - \lambda_n| \|Az_n\|$  which gives  $\mathbf{H}_{1,\mathbf{N}}$  with  $\delta_n = |\lambda_{n+1} - \lambda_n| + |\alpha_{n+1} - \alpha_n|$  thanks to Remark 2. The result follows now from Theorem 5. Indeed, since  $\lambda_{\sigma(n)} \in [a, b]$  it is possible to extract a subsequence  $\lambda_{\mu(n)}$  which converges to  $\bar{\lambda} \in [a, b]$ . Then we have  $\|T_{\mu(n)} z - T_{\bar{\lambda}} z\| \leq |\lambda_{\mu(n)} - \bar{\lambda}| \|Az\|$ . This implies  $\|T_{\mu(n)} x_{\mu(n)} - T_{\bar{\lambda}} x_{\mu(n)}\| \rightarrow 0$ .  $\square$

**Remark 21.** Note that for  $\lambda < 2\alpha$ ,  $I - \lambda A$  is in fact attracting nonexpansive since:

$$\|(I - \lambda A)x - (I - \lambda A)y\| \leq \|x - y\| + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.$$

So is  $P_C \circ (I - \lambda A)$  [2]. For a nonexpansive mapping  $S$ , we can consider the previous theorem with  $T_\lambda x \stackrel{\text{def}}{=} S \circ P_C(x - \lambda Ax)$  and use Remark 7 (a Hilbert space is strictly convex) to obtain a strong convergence to a point in  $\text{Fix}(T_\lambda) = \text{Fix } S \cap \text{VI}(C, A)$ . Thus, we recover [4, Proposition 3.1].

## 2.8. Example 7

We consider the equilibrium problem for a bifunction  $F : C \times C \mapsto \mathbb{R}$  where  $C$  is a closed convex subset of a real Hilbert space  $X$ . The problem is to find  $x \in C$  such that  $F(x, y) \geq 0$  for all  $y \in C$ . The set of solutions is denoted by  $\text{EP}(F)$ . It is proved in [6] (see also [5]) that for  $r > 0$ , the mapping  $T_r : X \mapsto C$  defined as follows:

$$T_r(x) \stackrel{\text{def}}{=} \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \quad (32)$$

is singled valued, firmly nonexpansive (i.e.  $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$  for any  $x, y \in X$ ). Furthermore we have  $\text{Fix}(T_r) = \text{EP}(F)$ . Notice that  $\text{EP}(F)$  is closed and convex if the bifunction  $F$  satisfies  $(A_1)$   $F(x, x) = 0$  for all  $x \in C$ ,  $(A_2)$   $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in C$ , and  $(A_3)$  for each  $x, y, z \in C$ ,  $\lim_{t \rightarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ , and  $(A_4)$  for each  $x \in C$ ,  $y \mapsto F(x, y)$  is convex and lower semicontinuous.

We consider the sequence  $\{x_n\}$  given by:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n$$

where  $T_n \stackrel{\text{def}}{=} T_{r_n}$  for a given sequence of real numbers  $\{r_n\}$ .

**Theorem 22.** Let  $X$  be a real Hilbert space,  $C$  a nonempty closed convex,  $f$  an  $\alpha$ -contraction, assume that  $\text{EP}(F) \neq \emptyset$ ,  $\mathbf{H}_{3,1}$  is satisfied and the sequence  $\{r_n\}$  is such that  $\liminf_{n \rightarrow \infty} r_n > 0$  and either  $\sum_n |r_{n+1} - r_n| < \infty$  or  $|r_{n+1} - r_n|/\alpha_n \rightarrow 0$ . Then, the sequence  $\{x_n\}$  generated by (7) converges strongly to  $\mathbf{Q}_{\text{EP}(F)}(f)$ .

**Proof.** Since the  $r_n$  are strictly positive the mappings  $T_{r_n}$  are nonexpansive and share the same fixed points  $\text{EP}(F)$  which was supposed to be nonempty. By Lemma 23, the sequence  $\{x_n\}$  is bounded.

Using the definition of  $T_r(x)$ , the monotonicity of  $F$  and  $(A_2)$ , easy computations lead to the following inequality [14, p. 464]:

$$\|T_r(x) - T_s(y)\| \leq \|x - y\| + \left| 1 - \frac{s}{r} \right| \|T_r(y) - y\|. \quad (33)$$

Using  $r > 0$  such that  $r_n > r$  for all  $n \in \mathbb{N}$  and  $y \in \text{Fix}(T_r)$  we obtain  $\|T_{r_n}(x_n) - T_r(y)\| \leq \|x_n - y\|$  which implies that the sequence  $\{T_{r_n}(x_n)\}$  is bounded. Moreover, for a bounded sequence  $\{y_n\}$  we have:

$$\|T_{r_{n+1}}(y_n) - T_{r_n}(y_n)\| \leq \frac{|r_{n+1} - r_n|}{r} \|T_{r_n}(y_n) - y_n\|. \quad (34)$$

We thus obtain  $\mathbf{H}_{1,1}$  with  $\delta_n = |r_{n+1} - r_n| + |\alpha_{n+1} - \alpha_n|$  using Remark 2. The result follows now from Theorem 5. Indeed, since  $r_{\sigma(n)} > r$ , it is possible to find a subsequence  $r_{\mu(n)}$  converging to  $\bar{r} > r$ . Then we have  $\|T_{r_{\mu(n)}} z - T_{\bar{r}} z\| \leq |r_{\mu(n)} - \bar{r}| K$  and thus

$$\|T_{r_{\mu(n)}} x_{\mu(n)} - T_{\bar{r}} x_{\mu(n)}\| \rightarrow 0. \quad \square$$

### 3. A collection of lemmata

The first lemma can be used to get that the sequence  $\{x_n\}$  generated by (35) is bounded.

**Lemma 23.** *Let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \quad (35)$$

where  $f$  is contraction of parameter  $\alpha$ ,  $T_n$  is a family of nonexpansive mappings and  $\alpha_n$  is a sequence in  $(0, 1)$ . Suppose that there exists  $p$  a common fixed point of  $T_n$  for all  $n \in \mathbb{N}$ . Then, the sequence  $\{x_n\}$  is bounded.

**Proof.** The proof exactly follows the proof of [19, Theorem 3.2], but here the mappings  $T_n$  are indexed by  $n$ . Obviously we have:

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n) \|T_n x_n - p\| \\ &\leq \alpha_n (\alpha \|x_n - p\| + \|f(p) - p\|) + (1 - \alpha_n) \|x_n - p\| \\ &\leq (1 - \alpha_n(1 - \alpha)) \|x_n - p\| + \alpha_n(1 - \alpha) \frac{\|f(p) - p\|}{(1 - \alpha)} \\ &\leq \max\left(\|x_n - p\|, \frac{\|f(p) - p\|}{(1 - \alpha)}\right). \end{aligned}$$

And, by induction,  $\{x_n\}$  is bounded.  $\square$

The next lemma gives that the sequence  $\{x_n\}$  is asymptotically regular i.e. for a given  $N \geq 1$ , we have  $\|x_{n+N} - x_n\| \rightarrow 0$ .

**Lemma 24.** *With the same assumptions as in Lemma 23 and assuming that there exists  $N \geq 1$  such that  $\mathbf{H}_{1,N}$  and  $\mathbf{H}_{3,N}$  are fulfilled then, for the sequence  $\{x_n\}$  given by (35), we have  $\|x_{n+N} - x_n\| \rightarrow 0$ .*

**Proof.** Using the definition of  $\{x_n\}$ , we have:

$$\begin{aligned} x_{n+N+1} - x_{n+1} &= \alpha_{n+N}(f(x_{n+N}) - f(x_n)) + (\alpha_{n+N} - \alpha_n)f(x_n) + (1 - \alpha_{n+N})(T_{n+N}x_{n+N} - T_{n+N}x_n) \\ &\quad + ((1 - \alpha_{n+N})T_{n+N}x_n - (1 - \alpha_n)T_n x_n). \end{aligned}$$

By Lemma 23, the sequence  $\{x_n\}$  is bounded. We can therefore use  $\mathbf{H}_{1,N}$  with  $\{x_n\}$ . Since  $\{f(x_n)\}$  is bounded too, we can find three constants such that:

$$\begin{aligned} \|x_{n+N+1} - x_{n+1}\| &\leq \alpha_{n+N}\alpha \|x_{n+N} - x_n\| + |\alpha_{n+N} - \alpha_n|K_1 + (1 - \alpha_{n+N})\|x_{n+N} - x_n\| + \delta_n M \\ &\leq (1 - (1 - \alpha)\alpha_{n+N})\|x_{n+N} - x_n\| + (|\alpha_{n+N} - \alpha_n| + \delta_n)K_2. \end{aligned}$$

The proof then follows easily using the properties of  $\alpha_n$  i.e.  $\mathbf{H}_{3,N}$  and Corollary 30.  $\square$

The next step is to prove that we can find a fixed mapping  $T$  such that  $\|x_n - Tx_n\| \rightarrow 0$ . The next corollary gives a simple example for which the property can be derived from Lemma 24.

**Corollary 25.** *Using the same hypothesis as in Lemma 24 and assuming that  $\{T_n x_n\}$  is bounded and that  $\|T_n x_n - Tx_n\| \rightarrow 0$  we also have  $\|x_n - Tx_n\| \rightarrow 0$ .*

**Proof.** We have:

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \leq \|x_n - x_{n+1}\| + \alpha_n K_1 + (1 - \alpha_n)\|T_n x_n - Tx_n\|$$

and the result follows.  $\square$

The next lemma gives assumptions to obtain  $\mathbf{H}_{2,p}$  for a given  $p$ .

**Lemma 26.** *Suppose that  $X$  is a  $\mathcal{B}$  real Banach space. Let  $T$  be a nonexpansive mapping with  $\text{Fix}(T) \neq \emptyset$ ,  $f$  an  $\alpha$ -contraction and  $\{x_n\}$  a bounded sequence such that  $\|Tx_n - x_n\| \rightarrow 0$ . Then for  $\tilde{x} = Q(f)$ , we have:*

$$\limsup_{n \rightarrow \infty} \langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x}) \rangle \leq 0. \quad (36)$$

**Proof.** When  $X$  is a  $\mathcal{B}_{\text{us}}$  or a  $\mathcal{B}_{\text{rug}}$  the key point is the fact that  $J$  is uniformly norm-to-weak\* continuous on bounded sets.

The proof of this lemma can be found in the proof of theorem ([19, Theorem 4.2] or [15, Theorem 3.1]). We just give the main arguments. Let  $\tilde{x} \stackrel{\text{def}}{=} \sigma\text{-}\lim_{t \rightarrow 0} x_t$  where  $x_t$  solves  $x_t = tf(x_t) + (1-t)Tx_t$ , we thus have:

$$\begin{aligned} \|x_t - x_n\|^2 &\leq (1-t)^2 \|Tx_t - x_n\|^2 + 2t \langle f(x_t) - x_n, J(x_t - x_n) \rangle \\ &\leq (1-t)^2 (\|Tx_t - Tx_n\| + \|Tx_n - x_n\|)^2 + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle + 2t \|x_t - x_n\|^2 \\ &\leq (1+t^2) \|x_t - x_n\|^2 + a_n(t) + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle \end{aligned} \quad (37)$$

where  $a_n(t) = 2\|Tx_n - x_n\| \|x_t - x_n\| + \|Tx_n - x_n\|^2 \rightarrow 0$  when  $n$  tends to infinity. Thus we get:

$$\langle f(x_t) - x_t, J(x_n - x_t) \rangle \leq \frac{a_n(t)}{2t} + \frac{t}{2} \|x_t - x_n\|^2 \quad (38)$$

and we have:

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \langle f(x_t) - x_t, J(x_n - x_t) \rangle \leq 0. \quad (39)$$

We consider a sequence  $t_p \rightarrow 0$  and  $y_p \stackrel{\text{def}}{=} x_{t_p}$ . We have  $y_p \rightarrow \tilde{x}$ . With  $g(x) \stackrel{\text{def}}{=} (x) - x$  we get

$$\langle g(\tilde{x}), J(x_n - \tilde{x}) \rangle \leq \langle g(y_p), J(x_n - y_p) \rangle + |\langle g(\tilde{x}), J(x_n - \tilde{x}) - J(x_n - y_p) \rangle| + (1+\alpha) \|\tilde{x} - y_p\| \|x_n - y_p\|.$$

Since  $J$  is uniformly norm-to-weak\* continuous on bounded sets and  $y_p \rightarrow \tilde{x}$ , for  $\epsilon > 0$ , we can find  $\tilde{p}$  such that for all  $p \geq \tilde{p}$  and all  $n \in \mathbb{N}$  we have:

$$\langle g(\tilde{x}), J(x_n - \tilde{x}) \rangle \leq \langle g(y_p), J(x_n - y_p) \rangle + \epsilon(1+\alpha) \|\tilde{x} - y_p\| \|x_n - y_p\|. \quad (40)$$

Thus we get:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle g(\tilde{x}), J(x_n - \tilde{x}) \rangle &\leq \limsup_{n \rightarrow \infty} \langle g(y_p), J(x_n - y_p) \rangle + \epsilon + \|\tilde{x} - y_p\| K \\ &\leq \lim_{p \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} \langle g(y_p), J(x_n - y_p) \rangle + \epsilon \|\tilde{x} - y_p\| K \right) \leq \epsilon. \end{aligned}$$

Suppose that  $X$  is a  $\mathcal{B}_{\text{rwsc}}$ . We follow the proof of [16, Theorem 2.2] or [20, Theorem 3.1]. Let  $\tilde{x} = \mathbf{Q}(f)$  and consider a subsequence  $\{x_{\sigma(n)}\}$  such that

$$\limsup_{n \rightarrow \infty} \langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x}) \rangle = \lim_{n \rightarrow \infty} \langle f(\tilde{x}) - \tilde{x}, J(x_{\sigma(n)} - \tilde{x}) \rangle.$$

It is then possible to re-extract a subsequence  $x_{\mu(n)}$  weakly converging to  $x^*$ . Since we have  $x_{\mu(n)} - Tx_{\mu(n)} \rightarrow 0$  then we get  $x^* \in \text{Fix}(T)$  using the key property that  $X$  satisfies Opial's condition [8, Theorem 1] and the fact that  $I - T$  is demi-closed at zero [15, Lemma 2.2]. Thus by definition of  $\tilde{x}$  we must have  $\langle f(\tilde{x}) - \tilde{x}, J(x^* - \tilde{x}) \rangle \leq 0$ .  $\square$

**Corollary 27.** Suppose that  $X$  is a  $\mathcal{B}_{\text{us}}$ , or a  $\mathcal{B}_{\text{rug}}$ , or a  $\mathcal{B}_{\text{rwsc}}$ . Let  $f$  be a contraction and  $\{x_n\}$  be a bounded sequence such that  $x_n - Tx_n \rightarrow 0$ . From each subsequence  $\sigma(n)$  we can extract a subsequence  $\mu(n)$  and find a fixed mapping  $T_\mu$  such that  $\|T_{\mu(n)}x_{\mu(n)} - T_\mu x_{\mu(n)}\| \rightarrow 0$ . Then, if  $F = \text{Fix } T_\mu$  does not depend on  $\mu$ , for  $\tilde{x} = \mathbf{Q}(f)$  associated to  $F$ , we have:

$$\limsup_{n \rightarrow \infty} \langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x}) \rangle \leq 0. \quad (41)$$

**Proof.** The proof is by contradiction using Lemma 26. Assume that the result is false, then we can find a subsequence  $\sigma(n)$  such that

$$\limsup_{n \rightarrow \infty} \langle f(\tilde{x}) - \tilde{x}, J(x_{\mu(n)} - \tilde{x}) \rangle \geq \epsilon > 0. \quad (42)$$

By hypothesis we can extract from  $\sigma(n)$  a subsequence  $\mu(n)$  such that  $\|T_{\mu(n)}x_{\mu(n)} - Tx_{\mu(n)}\| \rightarrow 0$ . Thus, since

$$\|x_{\mu(n)} - Tx_{\mu(n)}\| \leq \|x_{\mu(n)} - T_{\mu(n)}x_{\mu(n)}\| + \|T_{\mu(n)}x_{\mu(n)} - Tx_{\mu(n)}\|,$$

we have  $x_{\mu(n)} - Tx_{\mu(n)} \rightarrow 0$ . We can then apply Lemma 26 to the sequence  $\{x_{\mu(n)}\}$  and the mapping  $T_\mu$  to derive that:

$$\limsup_{n \rightarrow \infty} \langle f(\tilde{x}) - \tilde{x}, J(x_{\mu(n)} - \tilde{x}) \rangle \leq 0$$

for  $\tilde{x} = \mathbf{Q}(f)$  corresponding to  $F = \text{Fix } T_\mu$ . Since  $F$  does not depend on  $\mu$ , this gives a contradiction with (42).  $\square$

**Lemma 28.** Assume that the sequence  $\{x_n\}$  given by iterations (35) is bounded and assume that for  $p$ , a common fixed point of the mappings  $T_n$ ,  $\mathbf{H}_{2,p}$  is satisfied and that (i)–(iii) of  $\mathbf{H}_{3,N}$  are also satisfied.<sup>1</sup> Then the sequence  $\{x_n\}$  converges to  $p$ .

**Proof.**

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)^2 \|T_n x_n - p\|^2 + 2\alpha_n \langle f(x_n) - p, J(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle f(x_n) - f(p), J(x_{n+1} - p) \rangle + 2\alpha_n \langle f(p) - p, J(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \alpha \|x_n - p\| \|x_{n+1} - p\| + 2\alpha_n \langle f(p) - p, J(x_{n+1} - p) \rangle.\end{aligned}$$

Note that  $\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha_n K$ . Thus:

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \alpha \|x_n - p\|^2 + 2\alpha_n^2 K + 2\alpha_n \langle f(p) - p, J(x_{n+1} - p) \rangle \\ &\leq (1 - \alpha_n(1 - \alpha) + \alpha_n^2) \|x_n - p\|^2 + 2\alpha_n^2 K + 2\alpha_n \langle f(p) - p, J(x_{n+1} - p) \rangle.\end{aligned}\quad (43)$$

We conclude using Lemma 29.  $\square$

**Lemma 29.** (See [9, Lemma 2.1].) Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying the property

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n \quad \text{for } n \geq 0,$$

where  $\alpha_n \in (0, 1)$  and  $\beta_n$  are sequences of real numbers such that: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ; (ii) either  $\limsup_{n \rightarrow \infty} \beta_n \leq 0$  or  $\sum_{n=0}^{\infty} |\alpha_n \beta_n| < \infty$ . Then  $\{s_n\}$  converges to zero.

**Corollary 30.** Let  $\{s_n\}$  be a sequence of nonnegative real numbers satisfying the property

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n + \alpha_n \gamma_n \quad \text{for } n \geq 0,$$

where  $\alpha_n \in (0, 1)$ ,  $\beta_n$  and  $\gamma_n$  are sequences of real numbers such that: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ; (ii)  $\limsup_{n \rightarrow \infty} \beta_n \leq 0$  and (iv)  $\sum_{n=0}^{\infty} |\alpha_n \gamma_n| < \infty$ . Then  $\{s_n\}$  converges to zero.

**Proof.** The proof is similar to the proof of Lemma 29 [9, Lemma 2.1]. Fix  $\epsilon > 0$  and  $N$  such that  $\beta_n \leq \epsilon/2$  for  $n \geq N$  and  $\sum_{j=N}^{\infty} |\alpha_j \gamma_j| \leq \epsilon/2$ . Then following [9] we have for  $n > N$ :

$$s_{n+1} \leq \prod_{j=N}^n (1 - \alpha_j) s_N + \frac{\epsilon}{2} \left( 1 - \prod_{j=N}^n (1 - \alpha_j) \right) + \sum_{j=N}^n |\alpha_j \gamma_j| \leq \prod_{j=N}^n (1 - \alpha_j) s_N + \frac{\epsilon}{2} \left( 1 - \prod_{j=N}^n (1 - \alpha_j) \right) + \frac{\epsilon}{2}.\quad (44)$$

Taking the limit sup when  $n \rightarrow \infty$ , we obtain  $\limsup_{n \rightarrow \infty} s_{n+1} \leq \epsilon$ .  $\square$

A contraction is said to be a *Meir-Keeler contraction* (MKC) if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\|x - y\| < \epsilon + \delta$  implies  $\|\Phi(x) - \Phi(y)\| < \epsilon$ .

**Lemma 31.** (See [17].) Suppose that the sequence  $\{x_n\}$  defined by Eq. (35) strongly converges for an  $\alpha$ -contraction  $f$  (or a constant function  $f$ ) to the fixed point of  $P_F \circ f$  then the results remain valid for a Meir-Keeler contraction  $\Phi$ .

**Proof.** Suppose that we have proved that the sequence defined by (35) converges for an  $\alpha$ -contraction  $f$  to the fixed point of  $P_F \circ f$ . Then indeed, the result is true when  $f$  is a constant mapping. Let  $\Phi$  be a Meir-Keeler contraction, fix  $y \in C$ , when  $f$  is constant and equal to  $\Phi(y)$  then  $\{x_n\}$  defined by (35) converges to  $P_F(\Phi(y))$ . If  $\Phi$  is a MKC then since  $P_F$  is nonexpansive  $P_F \circ \Phi$  is also MKC (Proposition 3 of [17]) and has a unique fixed point [12]. We can consider  $z = P_F(\Phi(z))$  and consider two sequences:

$$x_{n+1} = \alpha_n \Phi(x_n) + (1 - \alpha_n) T_n x_n, \quad (45)$$

$$y_{n+1} = \alpha_n \Phi(z) + (1 - \alpha_n) T_n y_n. \quad (46)$$

Of course  $\{y_n\}$  converges strongly to  $z$ . We prove that  $\{x_n\}$  also converges strongly to  $z$  following [17]. Fix  $\epsilon > 0$ , by Proposition 2 of [17], we can find  $r \in (0, 1)$  such that  $\|x - y\| \leq \epsilon$  implies  $\|\Phi(x) - \Phi(y)\| \leq r\|x - y\|$ . Choose now  $N$  such that  $\|y_n - z\| \leq \epsilon(1 - r)/r$ . Assume now that for all  $n \geq N$  we have  $\|x_n - y_n\| > \epsilon$  then

$$\|x_{n+1} - y_{n+1}\| \leq (1 - \alpha_n) \|x_n - y_n\| + \alpha_n \|\Phi(x_n) - \Phi(y_n)\| + \alpha_n \|\Phi(y_n) - z\| \leq (1 - \alpha_n(1 - r)) \|x_n - y_n\| + \alpha_n \epsilon.$$

<sup>1</sup> Note that (i)–(iii) of  $\mathbf{H}_{3,N}$  do not use the value of  $N$ .

We cannot use here directly Lemma 29 but following the proof of this lemma we obtain that  $\limsup \|x_n - y_n\| \leq \epsilon$ . Assume that for a given value of  $n$  we have  $\|x_n - y_n\| \leq \epsilon$ . Since  $\Phi$  is a MKC we have  $\|\Phi(x) - \Phi(y)\| \leq \max(r\|x - y\|, \epsilon)$  and since we have

$$r\|x_n - z\| \leq r\|x_n - y_n\| + r\|y_n - z\| \leq \epsilon, \quad (47)$$

we get

$$\|x_{n+1} - y_{n+1}\| \leq (1 - \alpha_n)\|T_n x_n - T_n y_n\| + \alpha_n \max(r\|x_n - z\|, \epsilon) \leq \epsilon. \quad (48)$$

Thus we have in both cases  $\limsup_{n \rightarrow \infty} \|x_n - y_n\| \leq \epsilon$  and the conclusion follows.  $\square$

**Lemma 32.** (See [2, Proposition 2.10(i)].) Suppose that  $X$  is strictly convex,  $T_1$  an attracting nonexpansive mapping and  $T_2$  a nonexpansive mapping which have a common fixed point. Then we have:

$$\text{Fix}(T_1 \circ T_2) = \text{Fix}(T_2 \circ T_1) = \text{Fix}(T_2) \cap \text{Fix}(T_1).$$

**Proof.** We have  $\text{Fix}(T_2) \cap \text{Fix}(T_1) \subset \text{Fix}(T_2 \circ T_1)$  and  $\text{Fix}(T_2) \cap \text{Fix}(T_1) \subset \text{Fix}(T_1 \circ T_2)$ . Let  $x$  be a common fixed point of  $T_1$  and  $T_2$ . If  $y$ , a fixed point of  $T_1 \circ T_2$ , is such that  $y \notin \text{Fix}(T_2)$  then since  $T_1$  is attracting nonexpansive, we have:

$$\|y - x\| = \|T_1 \circ T_2(y) - x\| < \|T_2(y) - x\| \leq \|y - x\|.$$

This gives a contradiction. Thus  $y$  is a fixed point of  $T_2$  and then also of  $T_1$ . If now  $y$  a fixed point of  $T_2 \circ T_1$  and assume that  $y \notin \text{Fix}(T_1)$ , then we have

$$\|y - x\| = \|T_2 \circ T_1(y) - x\| \leq \|T_1(y) - x\| < \|y - x\|.$$

This gives also a contradiction and we deduce the same conclusion.  $\square$

## References

- [1] K. Aoyoma, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed point of a countable family of nonexpansive mappings, *Nonlinear Anal.* 67 (8) (2007) 2350–2360.
- [2] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.* 38 (3) (1996) 367–426.
- [3] R.E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces, *Trans. Amer. Math. Soc.* 179 (1973) 251–262.
- [4] J. Chen, L. Zhang, T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, *J. Math. Anal. Appl.* 334 (2) (2007) 1450–1461.
- [5] P. Combettes, S. Histoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.* 6 (1) (2005) 117–136.
- [6] S. Flåm, A. Antipin, Equilibrium programming using proximal-like algorithms, *Math. Program.* 77 (1997) 29–41.
- [7] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Dekker, 1984.
- [8] J.-P. Gossez, E.L. Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, *Pacific J. Math.* 40 (3) (1972) 565–573.
- [9] T.-H. Kim, H.-K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.* 61 (1–2) (2005) 51–60.
- [10] Y. Kimura, W. Takahashi, M. Toyoda, Convergence to common fixed points of a finite family of nonexpansive mappings, *Arch. Math.* 84 (2005) 350–363.
- [11] G. Lopez, V. Martin, H.-K. Xu, Perturbation techniques for nonexpansive mappings with applications, *Nonlinear Anal. Real World Appl.*, in press, available online 4 May 2008.
- [12] A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.* 28 (1969) 326–329.
- [13] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* 241 (2000) 46–55.
- [14] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.* 336 (1) (2007) 455–469.
- [15] Y. Song, R. Chen, Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings, *Appl. Math. Comput.* 180 (2006) 275–287.
- [16] Y. Song, R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, *J. Math. Anal. Appl.* 321 (1) (2006) 316–326.
- [17] T. Suzuki, Moudafi's viscosity approximations with Meir–Keeler contractions, *J. Math. Anal. Appl.* 325 (1) (2007) 342–352.
- [18] W. Takahashi, T. Tamura, M. Toyoda, Approximation of common fixed points of a family of finite nonexpansive mappings in Banach spaces, *Sci. Math. Jpn.* 56 (3) (2002) 475–480.
- [19] H.-K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298 (1) (2004) 279–291.
- [20] H.-K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, *J. Math. Anal. Appl.* 314 (2006) 631–643.