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Analyticity of a thermoelastic plate with variable coefficients ☆

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ABSTRACT

A thermoelastic plate with variable coefficients in the absence of dissipative boundary mechanisms is considered. Using C_0-semigroup theory and some techniques in partial differential equations, it is shown that the corresponding C_0-semigroup is analytic and hence the exponential stability of the system is concluded.

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1. Introduction

Over the last decade, much effort has been made for the analyticity of thermoelastic plates with constant coefficients in the absence of dissipative boundary mechanisms, see [8,9,11,12,15], name just a few. The problem with dynamical mechanical boundary conditions was studied in [20] by semigroup theory. As a consequence, the exponential stability of the system was concluded.

Furthermore, the elastic plate on a Riemannian manifold was studied in [3,4] by Riemannian geometric method. This method was first introduced into the boundary control problem in [18] for proof of the exact controllability of wave equation and Euler–Bernoulli plate with variable coefficients [19]. Later, the thermoelastic plates with variable coefficients and dynamic dissipative boundary mechanisms was studied in [1] where the multiplier method was adopted in investigation.

On the other hand, the exact controllability and decay rate of the system without mechanical feedbacks have been established in [8,20] and [2] when there is thermal effect on the plate. The objective of this paper is to study the analyticity of a thermoelastic plate with variable coefficients in the absence of dissipative boundary mechanisms on a Riemannian manifold M of dimension 2 with metric $g = \langle\cdot,\cdot\rangle$. The dimension of the manifold is not essential in the sense that the computations on high dimensional manifold is a little bit more complicated. Using C_0-semigroup theory and some techniques in partial differential equations, it is shown that the corresponding C_0-semigroup is analytic and hence the exponential stability of the system is concluded.

We introduce some notations. It is noted that all definitions and notations in the sequel are standard in Riemannian geometry.

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Let (M, g) be a Riemannian manifold with metric $g = \langle\cdot,\cdot\rangle$. For each $x \in M$, M_x stands for the tangential space of M at x. Denote the set of all vector fields on M by $\mathcal{X}(M)$. The sets of all k-order tensor fields and all k-forms on M are denoted by $T^k(M)$ and $\Lambda^k(M)$ respectively, where k is a nonnegative integer. Then

$$\Lambda^k(M) \subset T^k(M). \tag{1.1}$$

In particular, $\Lambda^0(M) = T^0(M) = C^\infty(M)$ is the set of all C^∞ functions on M and

$$T^1(M) = T(M) = \Lambda(M) = \mathcal{X}(M), \tag{1.2}$$

where $\Lambda(M) = \mathcal{X}(M)$ is interpreted as an isomorphism: for any given $X \in \mathcal{X}(M)$, equation

$$U(Y) = \langle Y, X\rangle, \quad \forall Y \in \mathcal{X}(M), \tag{1.3}$$

determines a unique $U \in \Lambda(M)$.

It is well known that, for each $x \in M$, k-order tensor space T^k_x on M_x is an inner product space defined as follows: Let e_1, e_2 be an orthonormal basis of M_x; for any $\alpha, \beta \in T^k_x$, $x \in M$, the inner product is given by

$$\langle\alpha, \beta\rangle_{T^k_x} = \sum_{i_1, i_2, \ldots, i_k = 1}^{2} \alpha(e_{i_1}, \ldots, e_{i_k})\beta(e_{i_1}, \ldots, e_{i_k}) \quad \text{at } x. \tag{1.4}$$

Let Ω be a bounded region of M with a regular boundary Γ. Then $T^k(\Omega)$ is an inner product space with the following inner product

$$(T_1, T_2)_{T^k(\Omega)} = \int_\Omega \langle T_1, T_2\rangle_{T^k_x}\, dx, \quad \forall T_1, T_2 \in T^k(\Omega), \tag{1.5}$$

where dx is the volume element of M with respect to its Riemannian metric g.

The completion of $T^k(\Omega)$ with the inner product (1.5) is denoted by $L^2(\Omega, T^k)$. In particular, $L^2(\Omega, \Lambda) = L^2(\Omega, T)$. The completion of $C^\infty(\Omega)$ with inner below is $L^2(\Omega)$:

$$(f, h)_{L^2(\Omega)} = \int_\Omega f(x)h(x)\, dx, \quad \forall f, h \in C^\infty(\Omega). \tag{1.6}$$

Let D be the Levi–Civita connection on M in the metric g. For $U \in \mathcal{X}(M)$, DU is the covariant differential of U which is a 2-order covariant tensor field in the following sense:

$$DU(X, Y) = D_Y U(X) = \langle D_Y U, X\rangle, \quad \forall X, Y \in M_x,\ x \in M. \tag{1.7}$$

For any $T \in T^2(M)$, the trace of T at x is defined by

$$\operatorname{tr} T = \sum_{i=1}^{2} T(e_i, e_i), \tag{1.8}$$

where e_1, e_2 is an orthonormal basis of M_x. It is obvious that $\operatorname{tr} T \in C^\infty(M)$ if $T \in T^2(M)$. The exterior derivative $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ satisfies $d^2 = 0$. There is a first-order differential operator $\delta : \Lambda^{k+1}(M) \to \Lambda^k(M)$, which is the formal adjoint of d and characterized by $(d\alpha, \beta)_{L^2(\Omega, \Lambda^{k+1})} = (\alpha, \delta\beta)_{L^2(\Omega, \Lambda^k)}$ for $\alpha \in \Lambda^k(\Omega)$ and $\beta \in \Lambda^{k+1}(\Omega)$ with compact support.

The Sobolev space $H^k(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm $\|f\|^2_{H^k(\Omega)} = \sum_{i=1}^k \|D^i f\|^2_{L^2(\Omega, T^i)} + \|f\|^2_{L^2(\Omega)}$ for any $f \in C^\infty(\Omega)$, where $D^i f$ is the ith covariant differential of f in metric g, and $\|\cdot\|_{L^2(\Omega, T^k)}$, $\|\cdot\|_{L^2(\Omega)}$ are the induced norms in the inner products (1.5), (1.6) respectively. For details on the Sobolev spaces on Riemannian manifolds, we refer to [5] or [16].

The following divergence formulae will be frequently used throughout the paper:

$$\int_\Omega \operatorname{div} X\, dx = \int_\Gamma \langle X, \nu\rangle\, d\Gamma, \tag{1.9}$$

where $\operatorname{div} X$ is the divergence of vector field X in Riemannian metric g, ν is the normal of Γ pointing towards the exterior of Γ (see e.g. [17]).

2. Main results

Let Ω be a bounded domain in Riemannian manifold (M, g) with smooth boundary Γ consisting of two disjoint parts $\Gamma_0 \cup \Gamma_1 = \Gamma$. We assume that the material undergoing obeys Hooke's law. Let us assume that, in addition to mechanical loads, a homogeneous plate is subject to an unknown temperature distribution. Consider the following thermoelastic system:

$$\begin{cases} y''(x,t) + \mathcal{A}y(x,t) + \alpha\Delta\theta(x,t) = 0 & \text{in } \Omega \times [0,\infty), \\ \beta\theta'(x,t) - \gamma\Delta\theta(x,t) + \sigma\theta(x,t) - \alpha\Delta y'(x,t) = 0 & \text{in } \Omega \times [0,\infty), \\ y(x,t) = \partial_\nu y(x,t) = 0 & \text{on } \Gamma_0 \times [0,\infty), \\ \Delta y(x,t) + (1-\mu)B_1 y(x,t) + \alpha\theta(x,t) = 0 & \text{on } \Gamma_1 \times [0,\infty), \\ \partial_\nu \Delta y(x,t) + (1-\mu)B_2 y(x,t) + \alpha\partial_\nu\theta(x,t) = 0 & \text{on } \Gamma_1 \times [0,\infty), \\ \partial_\nu\theta(x,t) = -\lambda\theta(x,t) & \text{on } \Gamma \times [0,\infty), \\ y(x,0) = y_0(x),\ y'(x,0) = y_1(x),\ \theta(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where operator $\mathcal{A}$ is defined by

$$\mathcal{A}y(x,t) = \Delta^2 y(x,t) - (1-\mu)\delta\big(k\,dy(x,t)\big), \tag{2.2}$$

ν is the unit normal along Γ pointing towards the exterior of Γ and Δ the Laplace operator in the Riemann metric g. k is the Gaussian curvature function on M, $\alpha = (1+\mu)/2$, d is the exterior derivative, δ is the formal adjoint of d, B_1, B_2 are the boundary operators defined by

$$B_1 y(x,t) = -D^2 y(\tau,\tau) \tag{2.3}$$

and

$$B_2 y(x,t) = \partial_\tau\big(D^2 y(\tau,\nu)\big) + k\partial_\nu y(x,t), \tag{2.4}$$

respectively, where $D^2 y$ is the Hessian of y, which is a 2-order tensor, and τ is the tangential along curve Γ, $\partial_\tau = \frac{\partial}{\partial\tau}$, $\partial_\nu = \frac{\partial}{\partial\nu}$. β, γ, σ and λ are positive constants. In the rest of the paper, we omit the variable (x,t) for brevity in notations without confusion.

Remark 1.1. The thermoelastic plate with variable coefficients and the dynamical boundary condition was considered in [1], where some decay rates of the energy of system were given under the dissipative boundary mechanisms. When there is no thermal effect on the plate, the plate model was established in [3]. Here we consider the system (2.1) for which the dynamical boundary conditions and the mechanical feedbacks in [1] are neglected.

Remark 1.2. The term $(1-\mu)\delta(k\,dy)$ in system (2.1) comes from the curvedness of the metric. For the flat case where $M = R^2$ and $k = 0$, the equation in system (2.1) is similar as that in [6,7].

Let V be the Hilbert space:

$$V = \big\{ y \in H^2(\Omega) \;\big|\; y|_{\Gamma_0} = \partial_\nu y|_{\Gamma_0} = 0 \big\}$$

with the norm

$$\|y\|_V^2 = \mathcal{B}(y), \quad \forall y \in V,$$

where $\mathcal{B}(y) = \mathcal{B}(y,y)$ is defined by the following bilinear form:

$$\mathcal{B}(y_1,y_2) = \int_\Omega a(y_1,y_2)\,dx,$$

$$a(y_1,y_2) = (1-\mu)\big\langle D^2 y, D^2\overline{y_2}\big\rangle + \mu\,\mathrm{tr}\,D^2 y\,\mathrm{tr}\,D^2\overline{y_2}, \quad \forall y_1, y_2 \in V.$$

Define the Hilbert space H by

$$H = V \times L^2(\Omega) \times L^2(\Omega)$$

with the inner product induced norm

$$\|u\|_H = \big[\|y\|_V^2 + \|z\|_{L^2(\Omega)}^2 + \beta\|\theta\|_{L^2(\Omega)}^2\big]^{\frac{1}{2}}, \quad \forall u = (y,z,\theta) \in H.$$

Let A be an operator in H:

$$D(A) = \big\{u = (y, z, \theta) \in H \mid y \in H^4(\Omega) \cap V,\ z \in V,\ \theta \in H^2(\Omega),\ (\partial_\nu \theta + \lambda\theta)|_\Gamma = 0,$$
$$(\Delta y + (1-\mu)B_1 y + \alpha\theta)|_{\Gamma_1} = 0,\ (\partial_\nu \Delta y + (1-\mu)B_2 y + \alpha\partial_\nu\theta)|_{\Gamma_1} = 0\big\},$$
$$A(y, z, \theta) = \big(z, -\mathcal{A}y - \alpha\theta, (\gamma\Delta\theta - \sigma\theta + \alpha\Delta z)/\beta\big). \tag{2.5}$$

Then system (2.1) can be written as a first order evolution equation on H:

$$\begin{cases} u' = Au, \\ u = (y, y', \theta), \quad u(0) = u_0 \in H. \end{cases} \tag{2.6}$$

Our main results are stated as Theorems 2.1, 2.2 below.

Theorem 2.1. *Suppose Γ is smooth. Then*

(i) *A generates a C_0-semigroup of contractions e^{tA} on H.*
(ii) *The resolvent of A is compact on H, and $\{i\omega \mid \omega \in \mathbb{R}\} \subset \rho(A)$.*

Theorem 2.2. *e^{tA} is analytic on H.*

Define the energy of the system (2.1) by

$$E(t) = \frac{1}{2}\big[\|y'\|^2_{L^2(\Omega)} + \beta\|\theta\|^2_{L^2(\Omega)} + \mathcal{B}(y, y)\big]. \tag{2.7}$$

Then we have the following stability result, which is the consequence of Theorem 2.2.

Corollary 2.1. *e^{tA} is exponentially stable, i.e., there exist positive constants K and ω such that the energy of the system* (2.1) *satisfies*

$$E(t) \leqslant Ke^{-\omega t}E(0), \quad \forall t \geqslant 0.$$

3. Proof of main results

In this section, we give the proofs of our main results. First, we introduce the following Green's formulae

$$\int_\Omega \mathcal{A}y\bar{z}\,dx = \mathcal{B}(y, z) - \int_\Gamma \big[\Delta y + (1-\mu)B_1 y\big]\partial_\nu\bar{z}\,d\Gamma + \int_\Gamma \big[\partial_\nu(\Delta y) + (1-\mu)B_2 y\big]\bar{z}\,d\Gamma,$$

which was presented in [3]. It will be frequently used in the proofs of our main results.

Proof of Theorem 2.1. (i). For any $u = (y, z, \theta) \in D(A)$, it is easy to check that

$$\mathrm{Re}(Au, u)_H = -\gamma\int_\Omega |\nabla\theta|^2\,dx - \sigma\int_\Omega |\theta|^2\,dx - \gamma\lambda\int_\Gamma |\theta|^2\,d\Gamma \leqslant 0. \tag{3.1}$$

Thus A is dissipative. By the Lumer–Phillips theorem [14, Theorem 4.3], the proof will be accomplished if we can show that A^{-1} exists and is bounded on H.

Let $f = (f_1, f_2, f_3) \in H$. Consider the following equation:

$$Au = f, \quad u = (y, z, \theta) \in D(A), \tag{3.2}$$

or

$$\begin{cases} z = f_1 & \text{in } V, \\ \mathcal{A}y + \alpha\Delta\theta = -f_2 & \text{in } L^2(\Omega), \\ \gamma\Delta\theta - \sigma\theta + \alpha\Delta z = \beta f_3 & \text{in } L^2(\Omega). \end{cases} \tag{3.3}$$

From the first and third equation of (3.3) and $u \in D(A)$, it follows that θ satisfies the following elliptic equation

$$\begin{cases} -\gamma\Delta\theta - \sigma\theta = -\alpha\Delta f_1 + \beta f_3 & \text{in } \Omega, \\ \partial_\nu\theta + \lambda\theta = 0 & \text{on } \Gamma. \end{cases} \tag{3.4}$$

By the regularity theory of elliptic equations [10], we know that there is a unique $\theta \in H^2(\Omega)$ satisfying (3.4). Furthermore, combine three equations of (3.3) and notice $u \in D(A)$, to obtain that y satisfies

$$\begin{cases} \mathcal{A}y = -f_2 - \dfrac{1}{\gamma}(\sigma\theta - \alpha\Delta f_1 + \beta f_3) & \text{in } \Omega, \\ y = \partial_\nu y = 0 & \text{on } \Gamma_0, \\ \Delta y + (1-\mu)B_1 y = -\alpha\theta & \text{on } \Gamma_1, \\ \partial_\nu\Delta y + (1-\mu)B_2 y = \alpha\lambda\theta & \text{on } \Gamma_1. \end{cases} \tag{3.5}$$

(3.5) is equivalent to the following variational equation:

$$\mathcal{B}(y,\phi)=\int_{\Omega}\left(-f_2-\frac{1}{\gamma}(\sigma\theta-\alpha\Delta f_1+\beta f_3)\right)\overline{\phi}\,dx-\int_{\Gamma_1}(\alpha\lambda\theta\overline{\phi}+\alpha\theta\partial_\nu\overline{\phi})\,d\Gamma,\quad\forall\phi\in V. \tag{3.6}$$

Thanks to the Lax–Milgram theorem, (3.6) admits a unique solution $y\in V$ for any $f\in H$ and $\theta\in H^{\frac{1}{2}}(\Omega)$. We have thus proved that for any $f\in H$, there exists a unique $u\in D(A)$ such that $Au=f$. The closed graph theorem implies that the inverse A^{-1} is bounded, and hence $0\in\rho(A)$.

(ii). By the regularity theory of elliptic equations again, it follows that the solution of (3.5) satisfies $y\in H^{\frac{5}{2}}(\Omega)$ and

$$\|y\|_{H^{\frac{5}{2}}(\Omega)}\leqslant C_1\big(\|f\|_H+\|\theta\|_{H^{\frac{1}{2}}(\Omega)}\big),\quad C_1>0. \tag{3.7}$$

Similarly, it follows from (3.4) that

$$\|\theta\|_{H^2(\Omega)}\leqslant C_2\|f\|_H,\quad C_2>0. \tag{3.8}$$

Combine (3.7), (3.8) and the first equation of (3.3) to yield

$$\big\|(y,z,\theta)\big\|_{H^{\frac{5}{2}}(\Omega)\times V\times H^2(\Omega)}\leqslant C\|f\|_H,\quad C>0. \tag{3.9}$$

Thus A^{-1} is compact. Therefore, the spectrum of A consists of isolated eigenvalues only. If $\{i\omega\mid\omega\in\mathbb{R}\}\subset\rho(A)$ is false, then there is a real number ω such that $i\omega$ is an eigenvalue of A. Suppose $u=(y,z,\theta)\in D(A)$, $\|u\|_H=1$ satisfies

$$(i\omega-A)u=0, \tag{3.10}$$

that is,

$$\begin{cases} i\omega y-z=0 & \text{in } V,\\ i\omega z+\mathcal{A}y+\alpha\Delta\theta=0 & \text{in } L^2(\Omega),\\ i\omega\beta\theta-\gamma\Delta\theta+\sigma\theta-\alpha\Delta z=0 & \text{in } L^2(\Omega).\end{cases} \tag{3.11}$$

Take the real part of the inner product of (3.10) with u and apply (3.1), to get

$$\int_{\Omega}|\nabla\theta|^2\,dx=\int_{\Omega}|\theta|^2\,dx=\int_{\Gamma}|\theta|^2\,d\Gamma=0, \tag{3.12}$$

which implies that

$$\theta=0,\qquad x\in\Omega. \tag{3.13}$$

From (3.11) and (3.13), it follows that

$$\begin{cases}\mathcal{A}y-\omega^2y=0 & \text{in } \Omega,\\ \Delta y=0 & \text{in } \Omega.\end{cases} \tag{3.14}$$

Since $y|_{\Gamma_0}=\partial_\nu y|_{\Gamma_0}=0$, it follows further, from the uniqueness continuation for the Cauchy problem for elliptic equations, that

$$y=0,\qquad x\in\Omega.$$

This together with the first equation of (3.11) gives $z=0$, $x\in\Omega$. This contradiction concludes the required result. □

Proof of Theorem 2.2. From (ii) of Theorem 2.1 and analyticity criterion for C_0-semigroup (see [13, Theorem 5.2]), we only need to verify that

$$\overline{\lim_{|\omega|\to\infty}}\big\|\omega(i\omega-A)^{-1}\big\|<\infty. \tag{3.15}$$

If (3.15) is false, then there exists $\{\omega_n\}\subset\mathbb{R}$ with $\omega_n\to\infty$ and a sequence of vector functions $u_n=(y_n,z_n,\theta_n)\in D(A^2)$ such that

$$\|u_n\|_H=1, \tag{3.16}$$

$$\lim_{n\to\infty}\left\|\left(i-\frac{1}{\omega_n}A\right)u_n\right\|_H=0. \tag{3.17}$$

From (3.17), we have

$$\begin{cases} f_{1,n} = iy_n - \dfrac{1}{\omega_n} z_n \to 0 & \text{in } V, \\ f_{2,n} = iz_n + \dfrac{1}{\omega_n} \mathcal{A} y_n + \dfrac{1}{\omega_n} \alpha \Delta \theta_n \to 0 & \text{in } L^2(\Omega), \\ f_{3,n} = i\beta\theta_n + \dfrac{1}{\omega_n}(-\gamma \Delta \theta_n - \alpha \Delta z_n + \sigma \theta_n) \to 0 & \text{in } L^2(\Omega). \end{cases} \tag{3.18}$$

Moreover, it follows from (3.16) and (3.17) that

$$\lim_{n\to\infty} \frac{1}{\omega_n} \operatorname{Re}(Au_n, u_n)_H = -\operatorname{Re}\left(\left(i - \frac{1}{\omega_n} A\right) u_n, u_n\right) = 0.$$

Using (3.1), we have

$$\lim_{n\to\infty} \frac{1}{|\omega_n|^{\frac{1}{2}}} \|\nabla \theta_n\|_{L^2(\Omega)} = \lim_{n\to\infty} \frac{1}{|\omega_n|^{\frac{1}{2}}} \|\theta_n\|_{L^2(\Omega)} = 0, \tag{3.19}$$

and

$$\lim_{n\to\infty} \frac{1}{|\omega_n|^{\frac{1}{2}}} \|\partial_\nu \theta_n\|_{L^2(\Gamma)} = \lim_{n\to\infty} \frac{1}{|\omega_n|^{\frac{1}{2}}} \|\theta_n\|_{L^2(\Gamma)} = 0. \tag{3.20}$$

The proof will be accomplished if we can prove the following lemmas for $u_n = (y_n, z_n, \theta_n) \in D(A^2)$ satisfying (3.16) and (3.17).

Lemma 3.1. $\lim_{n\to\infty} \|\theta_n\|_{L^2(\Omega)} = 0$.

Lemma 3.2. $\lim_{n\to\infty} \|y_n\|_V^2 = \lim_{n\to\infty} \|z_n\|_{L^2(\Omega)}^2 = \frac{1}{2}$.

Lemma 3.3. $\lim_{n\to\infty} \|\Delta y_n\|_{L^2(\Omega)} = 0$.

Lemma 3.4. $\lim_{n\to\infty} \|z_n\|_{L^2(\Omega)}^2 = 0$.

Actually, it is obvious that Lemma 3.2 contradicts Lemma 3.4. This contradiction concludes Theorem 1.2. □

Now we are in a position to prove Lemmas 3.1–3.4 one by one. For brevity in notation, we will use C to denote some positive constants independent of n.

Proof of Lemma 3.1. Take the inner product of the third equation of (3.18) with θ_n in $L^2(\Omega)$ and use (3.19) and (3.20), to obtain

$$i\beta \|\theta_n\|_{L^2(\Omega)}^2 - \frac{1}{\omega_n} \int_\Omega \gamma \Delta z_n \bar{\theta}_n \, dx \to 0. \tag{3.21}$$

By the Gagliardo–Nirenberg inequality, the first equation of (3.18), and (3.16), we have

$$\frac{1}{|\omega_n|^{\frac{1}{2}}} \|z_n\|_{H^1(\Omega)} \leqslant \frac{1}{|\omega_n|^{\frac{1}{2}}} \|z_n\|_V^{\frac{1}{2}} \|z_n\|_{L^2(\Omega)}^{\frac{1}{2}} \leqslant 1. \tag{3.22}$$

Combine (3.19) and (3.22) to yield

$$\left| \frac{1}{\omega_n} \int_\Omega \gamma \nabla z_n \nabla \bar{\theta}_n \, dx \right| \leqslant \frac{\gamma}{|\omega_n|^{\frac{1}{2}}} \|z_n\|_{H^1(\Omega)} \frac{1}{|\omega_n|^{\frac{1}{2}}} \|\nabla \theta_n\|_{L^2(\Omega)} \to 0. \tag{3.23}$$

By the trace theorem and the Gagliardo–Nirenberg inequality again, we have

$$\begin{aligned} \left| \frac{1}{\omega_n} \int_{\Gamma_1} \partial_\nu z_n \bar{\theta}_n \, d\Gamma \right| &\leqslant \frac{1}{|\omega_n|} \|\partial_\nu z_n\|_{L^2(\Gamma_1)} \|\theta_n\|_{L^2(\Gamma_1)} \\ &\leqslant \frac{1}{|\omega_n|^{\frac{1}{2}}} \|z_n\|_V^{\frac{1}{2}} \frac{1}{|\omega_n|^{\frac{1}{4}}} \|z_n\|_{H^1(\Omega)}^{\frac{1}{2}} \frac{1}{|\omega_n|^{\frac{1}{4}}} \|\theta_n\|_{H^1(\Omega)}^{\frac{1}{2}} \|\theta_n\|_{L^2(\Omega)}^{\frac{1}{2}} \to 0. \end{aligned} \tag{3.24}$$

By (3.23) and (3.24), it follows that

$$\left|\frac{1}{\omega_n}\int_\Omega \gamma\,\Delta z_n\bar{\theta}_n\,dx\right| \leqslant \left|\frac{1}{\omega_n}\int_\Omega \gamma\nabla z_n\nabla\bar{\theta}_n\,dx\right| + \left|\frac{1}{\omega_n}\int_{\Gamma_1}\gamma\,\partial_\nu z_n\bar{\theta}_n\,d\Gamma\right| \to 0. \tag{3.25}$$

Combing (3.25) and (3.21) proves Lemma 3.1. □

Proof of Lemma 3.2. Take the inner product of the first equation (3.18) with y_n in V, to get

$$i\mathcal{B}(y_n) + \frac{1}{\omega_n}\mathcal{B}(y_n, z_n) \to 0. \tag{3.26}$$

Furthermore, take the inner product of the second equation of (3.18) with z_n in $L^2(\Omega)$ and notice the fact $u_n \in D(A^2)$, to produce

$$\begin{aligned}
0 &\leftarrow i\int_\Omega |z_n|^2\,dx + \frac{1}{\omega_n}\int_\Omega \mathcal{A}y_n\bar{z}_n\,dx + \frac{1}{\omega_n}\int_\Omega \alpha\Delta\theta_n\bar{z}_n\,dx\\
&= i\int_\Omega |z_n|^2\,dx + \frac{1}{\omega_n}\Bigg[\mathcal{B}(y_n, z_n) - \int_{\Gamma_1}\big[\Delta y_n + (1-\mu)B_1 y_n\big]\partial_\nu\bar{z}_n\,d\Gamma\\
&\quad + \int_{\Gamma_1}\big[\partial_\nu\Delta y_n + (1-\mu)B_2 y_n\big]\bar{z}_n\,d\Gamma - \int_\Omega\langle\nabla\theta_n, \nabla\bar{z}_n\rangle\,dx + \int_\Gamma \alpha\,\partial_\nu\theta_n\bar{z}_n\,d\Gamma\Bigg]\\
&= i\int_\Omega |z_n|^2\,dx + \frac{1}{\omega_n}\Bigg[\mathcal{B}(y_n, z_n) - \int_\Omega\langle\nabla\theta_n, \nabla\bar{z}_n\rangle\,dx + \int_\Gamma \alpha\theta_n\partial_\nu\bar{z}_n\,d\Gamma\Bigg].
\end{aligned} \tag{3.27}$$

Lemma 3.2 then follows from (3.16), (3.23), (3.24), (3.26) and (3.27). □

Proof of Lemma 3.3. Substitute the first equation into the third equation in (3.18) and then take the inner product with Δy_n in $L^2(\Omega)$, to obtain

$$-i\alpha\|\Delta y_n\|^2_{L^2(\Omega)} - \frac{\alpha}{\omega_n}(\Delta\theta_n, \Delta y_n)_{L^2(\Omega)} + \left(i\beta + \frac{\sigma}{\omega_n}\right)(\theta_n, \Delta y_n)_{L^2(\Omega)} \to 0. \tag{3.28}$$

From Lemmas 2.1 and 2.2, it follows that

$$\lim_{n\to\infty}(\theta_n, \Delta y_n)_{L^2(\Omega)} \to 0. \tag{3.29}$$

Combine (3.28) and (3.29) to yield

$$\lim_{n\to\infty}\|\Delta y_n\|^2_{L^2(\Omega)} \leqslant \lim_{n\to\infty}\frac{C}{|\omega_n|}\big(\|\nabla\Delta y_n\|_{L^2(\Omega)}\|\nabla\theta_n\|_{L^2(\Omega)} + \|\Delta y_n\|_{L^2(\Gamma)}\|\partial_\nu\theta_n\|_{L^2(\Gamma)}\big). \tag{3.30}$$

On the other hand, from the second and third equations of (3.18) and (3.16), one has

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{|\omega_n|}\big\|\Delta^2 y_n\big\|^2_{L^2(\Omega)} &\leqslant C\lim_{n\to\infty}\left(\left(\beta + \frac{\sigma}{|\omega_n|}\right)\|\theta_n\|_{L^2(\Omega)} + \|z_n\|_{L^2(\Omega)} + \|\Delta y_n\|_{L^2(\Omega)}\right)\\
&\leqslant C.
\end{aligned} \tag{3.31}$$

Apply the interpolation theorem and (3.20), to get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{|\omega_n|}\big\|\partial_\nu\Delta y_n + (1-\mu)B_2 y_n\big\|_{H^{\frac{1}{2}}(\Gamma_1)}\\
&\quad = \lim_{n\to\infty}\frac{\lambda}{|\omega_n|}\|\theta_n\|_{H^{\frac{1}{2}}(\Gamma_1)} \leqslant C\lim_{n\to\infty}\lambda\frac{1}{|\omega_n|^{\frac{1}{2}}}\|\partial_\nu\theta_n\|^{\frac{1}{2}}_{L^2(\Gamma)}\frac{1}{|\omega_n|^{\frac{1}{2}}}\|\theta_n\|^{\frac{1}{2}}_{L^2(\Gamma)} \leqslant C.
\end{aligned} \tag{3.32}$$

By (3.18) and the fact $u_n \in D(A^2)$, θ_n satisfies

$$\begin{cases}
\gamma\Delta\left(\dfrac{\theta_n}{\omega_n}\right) - \sigma\left(\dfrac{\theta_n}{\omega_n}\right) = -f_{3,n} + i\beta\theta_n - \alpha\Delta\left(\dfrac{z_n}{\omega_n}\right) & \text{in } \Omega,\\
\partial_\nu\left(\dfrac{\theta_n}{\omega_n}\right) + \lambda\left(\dfrac{\theta_n}{\omega_n}\right) = 0 & \text{on } \Gamma.
\end{cases} \tag{3.33}$$

Applying the regularity theory of elliptic equations, (3.16) and (3.18), we obtain

$$\left\|\frac{\theta_n}{\omega_n}\right\|_{H^2(\Omega)} \leqslant \left\|-f_{3,n}+i\beta\theta_n-\alpha\Delta\left(\frac{z_n}{\omega_n}\right)\right\|_{L^2(\Omega)} \leqslant C. \tag{3.34}$$

By the trace theorem and (3.34), it follows that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{|\omega_n|}\left\|\Delta y_n+(1-\mu)B_1y_n\right\|_{H^{\frac{3}{2}}(\Gamma_1)}\\&\quad=\lim_{n\to\infty}\frac{\alpha}{|\omega_n|}\|\theta_n\|_{H^{\frac{3}{2}}(\Gamma_1)}\leqslant C\lim_{n\to\infty}\frac{1}{|\omega_n|}\|\theta_n\|_{H^2(\Omega)}\leqslant C.\end{aligned} \tag{3.35}$$

Since $\dfrac{1}{\omega_n}y_n$ is a solution of the following elliptic boundary value problem:

$$\begin{cases}\mathcal{A}\left(\dfrac{y_n}{\omega_n}\right)=f_{2,n}+iz_n+\dfrac{\alpha}{\omega_n}\Delta\theta_n & \text{in } \Omega,\\ \dfrac{y_n}{\omega_n}=\partial_\nu\left(\dfrac{y_n}{\omega_n}\right)=0 & \text{on } \Gamma_0,\\ \Delta\left(\dfrac{y_n}{\omega_n}\right)+(1-\mu)B_1\left(\dfrac{y_n}{\omega_n}\right)=-\dfrac{\alpha}{\omega_n}\theta_n & \text{on } \Gamma_1,\\ \partial_\nu\Delta\left(\dfrac{y_n}{\omega_n}\right)+(1-\mu)B_2\left(\dfrac{y_n}{\omega_n}\right)=\dfrac{\alpha\lambda}{\omega_n}\theta_n & \text{on } \Gamma_1,\end{cases} \tag{3.36}$$

it follows from the regularity theory of elliptic equations that

$$\lim_{n\to\infty}\frac{1}{|\omega_n|}\|y_n\|_{H^4(\Omega)}\leqslant C. \tag{3.37}$$

Applying Gagliardo–Nirenberg inequality, we have, from Lemma 3.2 and (3.37), that

$$\lim_{n\to\infty}\frac{1}{|\omega_n|^{\frac{1}{2}}}\|y_n\|_{H^3(\Omega)}\leqslant\lim_{n\to\infty}\frac{1}{|\omega_n|^{\frac{1}{2}}}\|y_n\|^{\frac{1}{2}}_{H^4(\Omega)}\|y_n\|^{\frac{1}{2}}_{V}\leqslant C. \tag{3.38}$$

By the trace theorem, the Gagliardo–Nirenberg inequality again and (3.38), we have

$$\lim_{n\to\infty}\frac{1}{|\omega_n|^{\frac{1}{4}}}\|\Delta y_n\|_{L^2(\Gamma)}\leqslant\lim_{n\to\infty}\frac{1}{|\omega_n|^{\frac{1}{4}}}\|y_n\|^{\frac{1}{2}}_{H^3(\Omega)}\|y_n\|^{\frac{1}{2}}_{V}\leqslant C. \tag{3.39}$$

Substituting (3.19), (3.38) and (3.39) into (3.30) proves Lemma 3.3. □

Proof of Lemma 3.4. Take the inner product of the second equation of (3.18) with z_n, to obtain

$$\begin{aligned}&i\|z_n\|^2_{L^2(\Omega)}+\frac{1}{\omega_n}\mathcal{B}(y_n,z_n)-\frac{1}{\omega_n}\int_{\Gamma_1}\left[\Delta y_n+(1-\mu)B_1y_n\right]\partial_\nu\bar{z}_n\,d\Gamma\\&\quad+\frac{1}{\omega_n}\int_{\Gamma_1}\left[\partial_\nu\Delta y_n+(1-\mu)B_2y_n\right]\bar{z}_n\,d\Gamma+\frac{1}{\omega_n}\int_\Omega\alpha\Delta\theta_n\bar{z}_n\,dx\to0.\end{aligned} \tag{3.40}$$

The same arguments as (3.23)–(3.25) yield

$$\frac{1}{\omega_n}\int_\Omega\alpha\Delta\theta_n\bar{z}_n\,dx\to0. \tag{3.41}$$

From the first equation of (3.18), it follows from Lemma 3.3 that

$$\lim_{n\to\infty}\frac{1}{\omega_n}(\Delta y_n,\Delta z_n)_{L^2(\Omega)}=-i\lim_{n\to\infty}\|\Delta y_n\|^2_{L^(\Omega)}=0. \tag{3.42}$$

On the other hand, by the first equation of (3.18), using the equivalent norm on V and (3.42), we have

$$\begin{aligned}\left|\frac{1}{\omega_n}\mathcal{B}(y_n,z_n)\right|&=\left|\mathcal{B}(y_n,iy_n-f_{1,n})\right|\\&\leqslant\left[\mathcal{B}(y_n,y_n)\right]^{\frac{1}{2}}\left[\mathcal{B}(y_n,y_n)+\mathcal{B}(f_{1,n},f_{1,n})\right]^{\frac{1}{2}}\\&\leqslant\left[C\|\Delta y_n\|^2\right]^{\frac{1}{2}}\left[C\|\Delta y_n\|^2+\mathcal{B}(f_{1,n},f_{1,n})\right]^{\frac{1}{2}}\to0.\end{aligned} \tag{3.43}$$

Substitute (3.41), (3.43) into (3.40) and notice $u_n \in D(A^2)$, to get

$$i\|z_n\|^2_{L^2(\Omega)} + \frac{1}{\omega_n}\int_{\Gamma_1} \alpha\theta_n \partial_\nu \bar{z}_n \, d\Gamma - \frac{1}{\omega_n}\int_{\Gamma_1} \alpha \partial_\nu \theta_n \bar{z}_n \, d\Gamma \to 0. \tag{3.44}$$

Finally, by (3.16), the first equation of (3.18) and (3.20), it is easy to check that

$$\frac{1}{\omega_n}\int_{\Gamma_1} \alpha \partial_\nu \theta_n \bar{z}_n \, d\Gamma \to 0. \tag{3.45}$$

Lemma 3.4 then follows from (3.24), (3.44) and (3.45). □

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