



Infinite horizon BSDEs with dissipative coefficients in Hilbert spaces and applications

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ARTICLE INFO

Article history:

Received 6 February 2007

Available online 21 February 2009

Submitted by U. Stadtmueller

Keywords:

Infinite horizon BSDEs

Dissipative mappings

Yosida approximation

Viscosity solution

ABSTRACT

In this paper we study a class of infinite horizon backward stochastic differential equations (BSDEs) of the form

$$dY(t) = \lambda Y(t) dt - f(t, Y(t), Z(t)) dt + Z(t) dW(t), \quad 0 \leq t < \infty,$$

in a real separable Hilbert space, where λ is a given real parameter and the coefficient f is dissipative in y and Lipschitz in z . By Yosida approximation to dissipative mappings we show existence and uniqueness of the solutions for these equations. This result is applied to construct unique viscosity solutions to semilinear elliptic partial differential equations (PDEs).

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1. Introduction

Consider the following infinite horizon BSDE in a real separable Hilbert space \mathbb{Y} :

$$dY(t) = AY(t) dt - f(t, Y(t), Z(t)) dt + Z(t) dW(t), \quad 0 \leq t < \infty, \quad (1)$$

where $\{W(t), t \geq 0\}$ is a cylindrical Wiener process in another real separable Hilbert space \mathbb{U} , defined on a probability space (Ω, \mathcal{F}, P) . We denote by $\{\mathcal{F}_t, t \geq 0\}$ its augmented natural filtration. A is a given operator and $f: [0, \infty) \times \Omega \times \mathbb{Y} \times \mathcal{L}_2(\mathbb{U}, \mathbb{Y}) \rightarrow \mathbb{Y}$ is a function which satisfies suitable conditions. Let $\mathcal{L}_2(\mathbb{U}, \mathbb{Y})$ be the Hilbert space consisting of Hilbert–Schmidt operators from \mathbb{U} to \mathbb{Y} endowed with the norm $\|\cdot\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}$. In \mathbb{Y} the norm is denoted $\|\cdot\|_{\mathbb{Y}}$ and scalar product $\langle \cdot, \cdot \rangle$.

We assume:

(H_f¹) $f(\cdot, y, z)$ is progressively measurable, for all $y \in \mathbb{Y}, z \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$;

(H_f²) $\|f(t, y, 0)\|_{\mathbb{Y}} \leq \|f(t, 0, 0)\|_{\mathbb{Y}} + r\|y\|_{\mathbb{Y}}$, P -a.s., for $t \geq 0, y \in \mathbb{Y}$, and some $r > 0$;

(H_f³) $\|f(t, y, z_1) - f(t, y, z_2)\|_{\mathbb{Y}} \leq k\|z_1 - z_2\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}$, P -a.s., for $t \geq 0, y \in \mathbb{Y}, z_1, z_2 \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$, and some $k > 0$;

(H_f⁴) $\langle y_1 - y_2, f(t, y_1, z) - f(t, y_2, z) \rangle \leq \mu\|y_1 - y_2\|_{\mathbb{Y}}^2$, P -a.s., for $t \geq 0, y_1, y_2 \in \mathbb{Y}, z \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$, and some $\mu \in \mathbb{R}$;

(H_f⁵) $y \mapsto f(t, y, z)$ is continuous, P -a.s., for all $t \geq 0, z \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$.

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When $\forall T > 0$, $Y(T)$ is a given \mathcal{F}_T -measurable \mathbb{Y} -valued random variable (called the terminal condition), Eq. (1) has appeared in many papers under the above assumptions. Let us recall some well-known results in this direction. In a finite-dimensional framework R.W.R. Darling and E. Pardoux [5] proved an existence and uniqueness result for Eq. (1) when $A = 0$,

$$\mathbb{E} \left[\|Y(T)\|_{\mathbb{Y}}^2 + \int_0^T \|f(t, 0, 0)\|_{\mathbb{Y}}^2 dt \right] < \infty. \quad (2)$$

At the same time they considered Eq. (1) on a random time interval as

$$\mathbb{E} \left[e^{2\beta T} \|Y(T)\|_{\mathbb{Y}}^2 + \int_0^T e^{2\beta t} \|f(t, 0, 0)\|_{\mathbb{Y}}^2 dt \right] < \infty, \quad (3)$$

for some $\beta > \mu + \frac{k^2}{2}$. If $A = \partial\varphi$, the subdifferential of the convex lower-semicontinuous function φ , in finite dimensions E. Pardoux and A. Rascanu [13] showed that Eq. (1) was solved in the special sense, when $Y(T)$ and the process $\{f(t, 0, 0), t \geq 0\}$ satisfied (2) and (3), respectively. In [14] they generalized the result on a fixed time interval to the case of infinite dimensions. After that, F. Confortola [3] replaced $-\partial\varphi$ with an infinitesimal generator of a strongly continuous contraction semigroup. In [3] she proved that Eq. (1) had a unique mild solution (Y, Z) with values in $L^2(\Omega; C([0, T], \mathbb{Y})) \times L^2(\Omega; L^2([0, T], \mathcal{L}_2(\mathbb{U}, \mathbb{Y})))$ by Yosida approximation of dissipative mappings.

Our work extends these results in a special direction. We consider $A = \lambda$, a real parameter and the following equation in the differential form:

$$dY(t) = \lambda Y(t) dt - f(t, Y(t), Z(t)) dt + Z(t) dW(t), \quad 0 \leq t < \infty, \quad (4)$$

and in the integral form: for $\forall T > 0$

$$Y(t) - Y(T) + \int_t^T Z(s) dW(s) + \lambda \int_t^T Y(s) ds = \int_t^T f(s, Y(s), Z(s)) ds, \quad 0 \leq t \leq T < \infty. \quad (5)$$

And we take off the terminal condition and assume again:

(H_f^6) for some $\beta \in \mathbb{R}$ and $p \in (2, \infty)$,

$$\mathbb{E} \left(\int_0^\infty e^{2\beta t} \|f(t, 0, 0)\|_{\mathbb{Y}}^2 dt \right)^{\frac{p}{2}} < \infty.$$

Under these assumptions the solution for Eq. (4) is a pair $\{(Y(t), Z(t)), 0 \leq t < \infty\}$ of progressively measurable processes with values in $L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{Y}))$ and $L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))$ (see Section 2 for details). The equation was previously considered in [6] and the drift f was required to be Lipschitz with respect to y . In [6] M. Fuhrman and G. Tessitore stated that the extension of non-Lipschitz in y could be based on the finite-dimensional result in [13] and [5] and the finite horizon result in [14] and [2]. However, when some results are proved by taking advantage of the finite-dimensional situation, approximation procedures by means of convolution with smooth kernels are commonly used. In this paper we follow the approach of [3] by providing a direct proof in the infinite-dimensional case based on the classical Yosida approximation of dissipative mappings.

It is well known that BSDE is applied to the study of deterministic PDE (see [1,5–7,11–13,15,16]). In the same way our theory of BSDE can be used to solve a class of semilinear elliptic PDEs in [6]. By means of forward and backward infinite-dimensional stochastic evolution equations with differentiable coefficients M. Fuhrman and G. Tessitore [6] gave mild solutions for the type of PDEs. Here we remove differentiability assumption on the coefficients, define their viscosity solutions and show that they indeed have unique viscosity solutions.

This paper is organized as follows. In Section 2, we give notations and preliminaries. Main result is placed in Section 3. In Section 4 we apply the result to semilinear elliptic PDEs.

2. Notations and preliminaries

$L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))$, defined for $\beta \in \mathbb{R}$ and $p \in [2, \infty)$, denotes the space of progressively measurable processes $\{Y(t), t \geq 0\}$ with continuous paths in \mathbb{Y} , such that the norm

$$\|Y\|_{L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))}^p = \mathbb{E} \left(\sup_{t \geq 0} e^{p\beta t} \|Y(t)\|_{\mathbb{Y}}^p \right)$$

is finite. Elements of $L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))$ are identified up to indistinguishability.

$L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathbb{Y}))$, defined for $\beta \in \mathbb{R}$ and $p \in [2, \infty)$, denotes the space of equivalence classes of processes $\{Y(t), t \geq 0\}$ with values in \mathbb{Y} , such that the norm

$$\|Y\|_{L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathbb{Y}))} = \mathbb{E} \left(\int_0^{\infty} e^{2\beta t} \|Y(t)\|_{\mathbb{Y}}^2 dt \right)^{\frac{p}{2}}$$

is finite, and Y admits a progressively measurable version. $L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))$ could be defined in the same way.

\mathcal{K}^p_{β} denotes the space $L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathbb{Y})) \times L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))$. The norm of an element $(Y, Z) \in \mathcal{K}^p_{\beta}$ is

$$\|(Y, Z)\|_{\mathcal{K}^p_{\beta}} = \|Y\|_{L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathbb{Y}))} + \|Z\|_{L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))}.$$

For $\eta \in \mathbb{R}$ and $q \in [1, \infty)$, we define $\mathcal{H}^q_{\eta}(\mathbb{Y})$ as the space $L^q_{\mathcal{P}}(\Omega; L^q_{\eta}(\mathbb{Y})) \cap L^q_{\mathcal{P}}(\Omega; C_{\eta}(\mathbb{Y}))$, endowed with the norm

$$\|Y\|_{\mathcal{H}^q_{\eta}(\mathbb{Y})} = \|Y\|_{L^q_{\mathcal{P}}(\Omega; L^q_{\eta}(\mathbb{Y}))} + \|Y\|_{L^q_{\mathcal{P}}(\Omega; C_{\eta}(\mathbb{Y}))}.$$

And then we introduce a typical tool which allows us to treat dissipative functions.

Given f satisfying (\mathbf{H}^4_f) , (\mathbf{H}^2_f) , we note that $f - \mu I$ is a dissipative and continuous function with respect to y . Define its Yosida approximation, for $\varepsilon > 0$, by

$$\begin{aligned} J_{\varepsilon}(t, \omega, y, z) &:= (I - \varepsilon(f - \mu I))^{-1}(t, \omega, y, z), \\ f_{\varepsilon}(t, \omega, y, z) &:= (f(t, \omega, \cdot, z) - \mu I)(J_{\varepsilon}(t, \omega, y, z)), \end{aligned}$$

then for any $t \geq 0$, $x, y \in \mathbb{Y}$, $z \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$ (see [4] for details)

- (i) $\|J_{\varepsilon}(t, x, z) - J_{\varepsilon}(t, y, z)\|_{\mathbb{Y}} \leq \|x - y\|_{\mathbb{Y}}$.
- (ii) $\lim_{\varepsilon \rightarrow 0} J_{\varepsilon}(t, y, z) = y$.
- (iii) $\langle x - y, f_{\varepsilon}(t, \omega, x, z) - f_{\varepsilon}(t, \omega, y, z) \rangle \leq 0$.
- (iv) $\|f_{\varepsilon}(t, \omega, x, z) - f_{\varepsilon}(t, \omega, y, z)\|_{\mathbb{Y}} \leq \frac{2}{\varepsilon} \|x - y\|_{\mathbb{Y}}$.
- (v) $\|f_{\varepsilon}(t, \omega, y, z)\|_{\mathbb{Y}} \leq \|(f - \mu I)(t, \omega, y, z)\|_{\mathbb{Y}}$.
- (vi) $\langle x - y, f_{\varepsilon}(t, \omega, x, z) - f_{\sigma}(t, \omega, y, z) \rangle \leq (\varepsilon + \sigma)(\|f_{\varepsilon}(t, \omega, x, z)\|_{\mathbb{Y}} + \|f_{\sigma}(t, \omega, y, z)\|_{\mathbb{Y}})^2$, for any $\varepsilon, \sigma > 0$.

Because f is uniformly Lipschitz with respect to z , we get

$$(vii) \quad \|f_{\varepsilon}(t, \omega, y, z_1) - f_{\varepsilon}(t, \omega, y, z_2)\|_{\mathbb{Y}} \leq k \|z_1 - z_2\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}.$$

Throughout this paper the letter C will denote a trivial positive constant whose values may change in different occasions.

3. Main result

Theorem 3.1. Suppose that f satisfies (\mathbf{H}^1_f) – (\mathbf{H}^6_f) . Then for $\lambda > -(\beta - \mu - \frac{k^2}{2})$, Eq. (4) has a unique solution (Y, Z) such that

$$Y \in L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathbb{Y})), \quad Z \in L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathcal{L}_2(\mathbb{U}, \mathbb{Y}))).$$

Moreover, the following estimate holds

$$\|Y\|_{L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathbb{Y}))} + \|Z\|_{L^p_{\mathcal{P}}(\Omega; L^2_{\beta}(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))} + \|Y\|_{L^p_{\mathcal{P}}(\Omega; C_{\beta}(\mathbb{Y}))} \leq C \left(\mathbb{E} \left(\int_0^{\infty} e^{2\beta s} \|f(s, 0, 0)\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}},$$

for a suitable constant C .

Proof. (Existence) $\forall T > 0$ we write Eq. (5) again

$$Y(t) - Y(T) + \int_t^T Z(s) dW(s) + (\lambda - \mu) \int_t^T Y(s) ds = \int_t^T (f(s, Y(s), Z(s)) - \mu Y(s)) ds, \quad 0 \leq t \leq T < \infty.$$

Noting that $f - \mu I$ is dissipative and continuous with respect to y , for any $\varepsilon > 0$, we consider the following equation

$$Y_{\varepsilon}(t) - Y_{\varepsilon}(T) + \int_t^T Z_{\varepsilon}(s) dW(s) + (\lambda - \mu) \int_t^T Y_{\varepsilon}(s) ds = \int_t^T f_{\varepsilon}(s, Y_{\varepsilon}(s), Z_{\varepsilon}(s)) ds, \quad (6)$$

where $f_\varepsilon(t, y, z) = (f(t, \cdot, z) - \mu I)(J_\varepsilon(t, y, z))$ is the Yosida approximation of $f - \mu I$. By Lemma 2.6 in [17] and (iv), (vii) f_ε is progressively measurable and Lipschitz in y and z . Hence, in terms of Theorem 3.7 in [6] for $\lambda > -(\beta - \mu - \frac{k^2}{2})$ Eq. (6) has a unique solution $(Y_\varepsilon, Z_\varepsilon)$ such that

$$\|Y_\varepsilon\|_{L^p_P(\Omega; L^2_\beta(\mathbb{Y}))} + \|Z_\varepsilon\|_{L^p_P(\Omega; L^2_\beta(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))} + \|Y_\varepsilon\|_{L^p_P(\Omega; C_\beta(\mathbb{Y}))} \leq C \left(\mathbb{E} \left(\int_0^\infty e^{2\beta s} \|f(s, 0, 0)\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \quad (7)$$

where C is independent of ε .

Now if $(Y_\varepsilon, Z_\varepsilon)$ and (Y_σ, Z_σ) are two solutions of

$$\begin{cases} Y_\varepsilon(t) - Y_\varepsilon(T) + \int_t^T Z_\varepsilon(s) dW(s) + (\lambda - \mu) \int_t^T Y_\varepsilon(s) ds = \int_t^T f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) ds, \\ Y_\sigma(t) - Y_\sigma(T) + \int_t^T Z_\sigma(s) dW(s) + (\lambda - \mu) \int_t^T Y_\sigma(s) ds = \int_t^T f_\sigma(s, Y_\sigma(s), Z_\sigma(s)) ds, \end{cases}$$

respectively, then $(Y_\varepsilon - Y_\sigma, Z_\varepsilon - Z_\sigma)$ is the solution of

$$\begin{aligned} & (Y_\varepsilon(t) - Y_\sigma(t)) - (Y_\varepsilon(T) - Y_\sigma(T)) + \int_t^T (Z_\varepsilon(s) - Z_\sigma(s)) dW(s) + (\lambda - \mu) \int_t^T (Y_\varepsilon(s) - Y_\sigma(s)) ds \\ &= \int_t^T (f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) - f_\sigma(s, Y_\sigma(s), Z_\sigma(s))) ds. \end{aligned}$$

Applying the Itô formula to the process $e^{2\beta s} \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2$, $s \in [t, T]$, we obtain

$$\begin{aligned} & e^{2\beta t} \|Y_\varepsilon(t) - Y_\sigma(t)\|_{\mathbb{Y}}^2 - e^{2\beta T} \|Y_\varepsilon(T) - Y_\sigma(T)\|_{\mathbb{Y}}^2 \\ &+ \int_t^T e^{2\beta s} [2(\beta + \lambda - \mu) \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 + \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2] ds \\ &= -2 \int_t^T e^{2\beta s} \langle Y_\varepsilon(s) - Y_\sigma(s), (Z_\varepsilon(s) - Z_\sigma(s)) dW(s) \rangle \\ &+ 2 \int_t^T e^{2\beta s} \langle Y_\varepsilon(s) - Y_\sigma(s), f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) - f_\sigma(s, Y_\sigma(s), Z_\sigma(s)) \rangle ds. \end{aligned}$$

By (vi), (vii) and the Young inequality we have

$$\begin{aligned} & 2 \langle Y_\varepsilon(s) - Y_\sigma(s), f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) - f_\sigma(s, Y_\sigma(s), Z_\sigma(s)) \rangle \\ &= 2 \langle Y_\varepsilon(s) - Y_\sigma(s), f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) - f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s)) \rangle \\ &\quad + 2 \langle Y_\varepsilon(s) - Y_\sigma(s), f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s)) - f_\sigma(s, Y_\sigma(s), Z_\sigma(s)) \rangle \\ &\leq 2(\varepsilon + \sigma) (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 \\ &\quad + 2k \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}} \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})} \\ &\leq 2(\varepsilon + \sigma) (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 \\ &\quad + \frac{k^2}{\rho} \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 + \rho \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2, \end{aligned}$$

where ρ is an arbitrary number in $(0, 1]$ which will be chosen later. Substituting in the previous equation yields

$$\begin{aligned} & e^{2\beta t} \|Y_\varepsilon(t) - Y_\sigma(t)\|_{\mathbb{Y}}^2 - e^{2\beta T} \|Y_\varepsilon(T) - Y_\sigma(T)\|_{\mathbb{Y}}^2 \\ &+ \int_t^T e^{2\beta s} \left[\left(2\beta + 2\lambda - 2\mu - \frac{k^2}{\rho} \right) \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 + (1 - \rho) \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 \right] ds \end{aligned}$$

$$\begin{aligned} &\leq -2 \int_t^T e^{2\beta s} \langle Y_\varepsilon(s) - Y_\sigma(s), (Z_\varepsilon(s) - Z_\sigma(s)) dW(s) \rangle \\ &\quad + 2(\varepsilon + \sigma) \int_t^T e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds. \end{aligned}$$

By the same method as Step 1 in the proof of Theorem 3.2 in [6] we get the following estimate

$$\mathbb{E} \sup_{t \geq 0} e^{p\beta t} \|Y_\varepsilon(t) - Y_\sigma(t)\|_{\mathbb{Y}}^p \leq C(2\varepsilon + 2\sigma)^{\frac{p}{2}} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}},$$

where C only depends on p .

The deduction similar to Step 2 in the proof of Theorem 3.2 in [6] gives

$$\begin{aligned} &\int_0^\infty e^{2\beta s} \left[\left(2\beta + 2\lambda - 2\mu - \frac{k^2}{\rho} \right) \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 + (1 - \rho) \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 \right] ds \\ &\leq -2 \int_0^\infty e^{2\beta s} \langle Y_\varepsilon(s) - Y_\sigma(s), (Z_\varepsilon(s) - Z_\sigma(s)) dW(s) \rangle \\ &\quad + 2(\varepsilon + \sigma) \int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds. \end{aligned}$$

Choosing $\rho < 1$ so close to 1 that $2\beta + 2\lambda - 2\mu - \frac{k^2}{\rho} > 0$, we obtain, for some constant $C > 0$,

$$\begin{aligned} &\int_0^\infty e^{2\beta s} (\|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 + \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2) ds \\ &\leq -2C \int_0^\infty e^{2\beta s} \langle Y_\varepsilon(s) - Y_\sigma(s), (Z_\varepsilon(s) - Z_\sigma(s)) dW(s) \rangle \\ &\quad + 2C(\varepsilon + \sigma) \int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds. \end{aligned}$$

The Burkholder inequality allows us to get

$$\begin{aligned} &\mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 + \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2) ds \right)^{\frac{p}{2}} \\ &\leq C \mathbb{E} \left(\int_0^\infty e^{4\beta s} \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^2 \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \right)^{\frac{p}{4}} \\ &\quad + C(\varepsilon + \sigma)^{\frac{p}{2}} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}} \\ &\leq C \mathbb{E} \left[\sup_{t \geq 0} e^{\frac{p\beta s}{2}} \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^{\frac{p}{2}} \left(\int_0^\infty e^{2\beta s} \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \right)^{\frac{p}{4}} \right] \\ &\quad + C(\varepsilon + \sigma)^{\frac{p}{2}} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}} \\ &\leq C \left(\mathbb{E} \sup_{t \geq 0} e^{p\beta s} \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^p \right)^{\frac{1}{2}} \left(\mathbb{E} \left(\int_0^\infty e^{2\beta s} \|Z_\varepsilon(s) - Z_\sigma(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{2}} \\ &\quad + C(\varepsilon + \sigma)^{\frac{p}{2}} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}} \end{aligned}$$

$$\begin{aligned}
& + C(\varepsilon + \sigma)^{\frac{p}{2}} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}} \\
& \leq C \left(\mathbb{E} \sup_{t \geq 0} e^{p\beta s} \|Y_\varepsilon(s) - Y_\sigma(s)\|_{\mathbb{Y}}^p \right)^{\frac{1}{2}} \\
& + C(\varepsilon + \sigma)^{\frac{p}{2}} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}},
\end{aligned}$$

where the last inequality follows from (7) and C is independent on ε, σ . And by (v) , (\mathbf{H}_f^2) , (\mathbf{H}_f^3) , (\mathbf{H}_f^6) and (7)

$$\sup_{\varepsilon, \sigma} \mathbb{E} \left(\int_0^\infty e^{2\beta s} (\|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}} + \|f_\sigma(s, Y_\sigma(s), Z_\varepsilon(s))\|_{\mathbb{Y}})^2 ds \right)^{\frac{p}{2}} < \infty.$$

Therefore

$$\|Y_\varepsilon - Y_\sigma\|_{L_{\mathcal{P}}^p(\Omega; L_\beta^2(\mathbb{Y}))} + \|Z_\varepsilon - Z_\sigma\|_{L_{\mathcal{P}}^p(\Omega; L_\beta^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))} + \|Y_\varepsilon - Y_\sigma\|_{L_{\mathcal{P}}^p(\Omega; C_\beta(\mathbb{Y}))} \rightarrow 0, \quad \varepsilon, \sigma \rightarrow 0.$$

Set

$$\begin{aligned}
Y &:= \lim_{\varepsilon \rightarrow 0} Y_\varepsilon, \\
Z &:= \lim_{\varepsilon \rightarrow 0} Z_\varepsilon
\end{aligned}$$

in $L_{\mathcal{P}}^p(\Omega; L_\beta^2(\mathbb{Y})) \cap L_{\mathcal{P}}^p(\Omega; C_\beta(\mathbb{Y}))$ and $L_{\mathcal{P}}^p(\Omega; L_\beta^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))$, respectively. And then the following estimate holds

$$\|Y\|_{L_{\mathcal{P}}^p(\Omega; L_\beta^2(\mathbb{Y}))} + \|Z\|_{L_{\mathcal{P}}^p(\Omega; L_\beta^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))} + \|Y\|_{L_{\mathcal{P}}^p(\Omega; C_\beta(\mathbb{Y}))} \leq C \left(\mathbb{E} \left(\int_0^\infty e^{2\beta s} \|f(s, 0, 0)\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}. \quad (8)$$

Because

$$\begin{aligned}
\mathbb{E} \int_0^T \|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}}^2 ds & \leq C \mathbb{E} \int_0^T e^{2\beta s} \|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}}^2 ds \\
& \leq C \left(\mathbb{E} \left(\int_0^T e^{2\beta s} \|f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \leq C \left(\mathbb{E} \left(\int_0^T e^{2\beta s} \|f(s, Y_\varepsilon(s), Z_\varepsilon(s)) - \mu Y_\varepsilon(s)\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \leq C \left(\mathbb{E} \left(\int_0^T e^{2\beta s} (\|f(s, 0, 0)\|_{\mathbb{Y}}^2 + (r + |\mu|)^2 \|Y_\varepsilon(s)\|_{\mathbb{Y}}^2 + k^2 \|Z_\varepsilon(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2) ds \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \leq C \left(\mathbb{E} \left(\int_0^\infty e^{2\beta s} \|f(s, 0, 0)\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{2}{p}},
\end{aligned}$$

where C is independent on ε , the family $f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s))$ is bounded in $L^2(\Omega \times [0, T], \mathcal{P}, dP \times dt; \mathbb{Y})$ and then uniformly integrable. In addition,

$$\begin{aligned}
& |f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) - (f(s, Y(s), Z(s)) - \mu Y(s))| \\
& \leq |f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) - f_\varepsilon(s, Y_\varepsilon(s), Z(s))| + |f_\varepsilon(s, Y_\varepsilon(s), Z(s)) - (f(s, Y(s), Z(s)) - \mu Y(s))| \\
& \leq k |Z_\varepsilon(s) - Z(s)| + |f(s, J_\varepsilon(s, Y_\varepsilon(s), Z(s)), Z(s)) - \mu J_\varepsilon(s, Y_\varepsilon(s), Z(s)) - (f(s, Y(s), Z(s)) - \mu Y(s))|
\end{aligned}$$

and

$$\begin{aligned} |J_\varepsilon(s, Y_\varepsilon(s), Z(s)) - Y(s)| &\leq |J_\varepsilon(s, Y_\varepsilon(s), Z(s)) - J_\varepsilon(s, Y(s), Z(s))| + |J_\varepsilon(s, Y(s), Z(s)) - Y(s)| \\ &\leq |Y_\varepsilon(s) - Y(s)| + |J_\varepsilon(s, Y(s), Z(s)) - Y(s)|, \end{aligned}$$

where we have used (i). By (ii) and (\mathbf{H}_f^5) we obtain $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) = f(s, Y(s), Z(s)) - \mu Y(s)$. Hence

$$f_\varepsilon(s, Y_\varepsilon(s), Z_\varepsilon(s)) \rightarrow f(s, Y(s), Z(s)) - \mu Y(s) \quad \text{in } L^1(\Omega \times [0, T], \mathcal{P}, dP \times dt; \mathbb{Y}).$$

By taking limits for Eq. (6) in $L^1(\Omega \times [0, T], \mathcal{P}, dP \times dt; \mathbb{Y})$, we get that (Y, Z) is a solution to Eq. (5).

(Uniqueness) Let (Y_1, Z_1) and (Y_2, Z_2) be two solutions to Eq. (5). Set for brevity $\bar{Y}(t) := Y_1(t) - Y_2(t)$, $\bar{Z}(t) := Z_1(t) - Z_2(t)$ and we get

$$\bar{Y}(t) - \bar{Y}(T) + \int_t^T \bar{Z}(s) dW(s) + \lambda \int_t^T \bar{Y}(s) ds = \int_t^T (f(s, Y_1(s), Z_1(s)) - f(s, Y_1(s) - \bar{Y}(s), Z_1(s) - \bar{Z}(s))) ds.$$

Using the Itô formula to the process $e^{2\beta s} \|\bar{Y}(s)\|_{\mathbb{Y}}^2$ for $s \in [t, T]$ gives

$$\begin{aligned} e^{2\beta t} \|\bar{Y}(t)\|_{\mathbb{Y}}^2 - e^{2\beta T} \|\bar{Y}(T)\|_{\mathbb{Y}}^2 + \int_t^T e^{2\beta s} [2(\beta + \lambda) \|\bar{Y}(s)\|_{\mathbb{Y}}^2 + \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2] ds \\ = 2 \int_t^T e^{2\beta s} \langle \bar{Y}(s), f(s, Y_1(s), Z_1(s)) - f(s, Y_1(s) - \bar{Y}(s), Z_1(s) - \bar{Z}(s)) \rangle ds - 2 \int_t^T e^{2\beta s} \langle \bar{Y}(s), \bar{Z}(s) dW(s) \rangle. \end{aligned}$$

It follows from (\mathbf{H}_f^3) , (\mathbf{H}_f^4) and the Young inequality that

$$\begin{aligned} 2 \langle \bar{Y}(s), f(s, Y_1(s), Z_1(s)) - f(s, Y_1(s) - \bar{Y}(s), Z_1(s) - \bar{Z}(s)) \rangle \\ = 2 \langle \bar{Y}(s), f(s, Y_1(s), Z_1(s)) - f(s, Y_1(s) - \bar{Y}(s), Z_1(s)) \rangle \\ + 2 \langle \bar{Y}(s), f(s, Y_1(s) - \bar{Y}(s), Z_1(s)) - f(s, Y_1(s) - \bar{Y}(s), Z_1(s) - \bar{Z}(s)) \rangle \\ \leq 2\mu \|\bar{Y}(s)\|_{\mathbb{Y}}^2 + 2k \|\bar{Y}(s)\|_{\mathbb{Y}} \cdot \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})} \\ \leq \left(2\mu + \frac{k^2}{\rho}\right) \|\bar{Y}(s)\|_{\mathbb{Y}}^2 + \rho \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2, \end{aligned}$$

where ρ is an arbitrary number in $(0, 1]$ which will be chosen later. Substituting in the previous equation yields

$$\begin{aligned} e^{2\beta t} \|\bar{Y}(t)\|_{\mathbb{Y}}^2 - e^{2\beta T} \|\bar{Y}(T)\|_{\mathbb{Y}}^2 + \int_t^T e^{2\beta s} \left[\left(2\beta + 2\lambda - 2\mu - \frac{k^2}{\rho}\right) \|\bar{Y}(s)\|_{\mathbb{Y}}^2 + (1 - \rho) \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 \right] ds \\ \leq -2 \int_t^T e^{2\beta s} \langle \bar{Y}(s), \bar{Z}(s) dW(s) \rangle. \end{aligned} \quad (9)$$

Choosing $\rho < 1$ so close to 1 that $2\beta + 2\lambda - 2\mu - \frac{k^2}{\rho} > 0$, we obtain

$$e^{2\beta t} \|\bar{Y}(t)\|_{\mathbb{Y}}^2 - e^{2\beta T} \|\bar{Y}(T)\|_{\mathbb{Y}}^2 \leq -2 \int_t^T e^{2\beta s} \langle \bar{Y}(s), \bar{Z}(s) dW(s) \rangle. \quad (10)$$

Because

$$\begin{aligned} \mathbb{E} \left(\int_t^T e^{4\beta s} \|\bar{Y}(s)\|_{\mathbb{Y}}^2 \cdot \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \right)^{\frac{1}{2}} &\leq \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} \|\bar{Y}(s)\|_{\mathbb{Y}} \left(\int_t^T e^{2\beta s} \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \right)^{\frac{1}{2}} \right) \\ &\leq \frac{1}{2} \mathbb{E} \left(\sup_{s \in [t, T]} e^{2\beta s} \|\bar{Y}(s)\|_{\mathbb{Y}}^2 \right) + \frac{1}{2} \mathbb{E} \int_t^T e^{2\beta s} \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \\ &\leq \frac{1}{2} \left(\mathbb{E} \left(\sup_{s \in [t, T]} e^{p\beta s} \|\bar{Y}(s)\|_{\mathbb{Y}}^p \right) \right)^{\frac{2}{p}} + \frac{1}{2} \left(\mathbb{E} \left(\int_t^T e^{2\beta s} \|\bar{Z}(s)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{2}{p}}, \end{aligned}$$

by (8) the stochastic integral is a martingale. Taking expectation on two sides for (10) one get

$$\mathbb{E}(e^{2\beta t} \|\bar{Y}(t)\|_{\mathbb{Y}}^2) - \mathbb{E}(e^{2\beta T} \|\bar{Y}(T)\|_{\mathbb{Y}}^2) \leq 0. \quad (11)$$

Since for $i = 1, 2$

$$\int_0^\infty \mathbb{E}(e^{2\beta t} \|Y_i(t)\|_{\mathbb{Y}}^2) dt = \mathbb{E}\left(\int_0^\infty e^{2\beta t} \|Y_i(t)\|_{\mathbb{Y}}^2 dt\right) \leq \left(\mathbb{E}\left(\int_0^\infty e^{2\beta t} \|Y_i(t)\|_{\mathbb{Y}}^2 dt\right)^{\frac{p}{2}}\right)^{\frac{2}{p}} < \infty,$$

we can find a sequence $T_n \rightarrow \infty$ such that $\mathbb{E}(e^{2\beta T_n} \|\bar{Y}(T_n)\|_{\mathbb{Y}}^2) \rightarrow 0$. Setting $T = T_n$ in (11) and letting $n \rightarrow \infty$, we obtain

$$\mathbb{E}(e^{2\beta t} \|\bar{Y}(t)\|_{\mathbb{Y}}^2) = 0.$$

From this $\bar{Y} = 0$ in $L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{Y}))$. Moreover, by (9) $\bar{Z} = 0$ in $L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))$. \square

Remark 3.2. In the scalar case $\mathbb{Y} = \mathbb{R}$, when $\sup_{t \geq 0} |f(t, 0, 0)| \leq C$, by special techniques, such as the Tanaka formula and the local time, it is possible to remove the assumption that the parameter λ is large as in [16].

4. Applications

Let us now consider another BSDE

$$Y(t) - Y(T) + \int_t^T Z(s) dW(s) + \lambda \int_t^T Y(s) ds = \int_t^T \psi(X(s), Y(s), Z(s)) ds, \quad 0 \leq t \leq T < \infty, \quad (12)$$

where λ is a real number, $\psi: \mathbb{X} \times \mathbb{Y} \times \mathcal{L}_2(\mathbb{U}, \mathbb{Y}) \mapsto \mathbb{Y}$ is Borel measurable and X is a progressively measurable process with values in some real separable Hilbert space \mathbb{X} .

We assume:

- (\mathbf{H}_{ψ}^1) $\|\psi(x, y, 0)\|_{\mathbb{Y}} \leq \|\psi(x, 0, 0)\|_{\mathbb{Y}} + r\|y\|_{\mathbb{Y}}$, for $x \in \mathbb{X}$, $y \in \mathbb{Y}$, and some $r > 0$;
- (\mathbf{H}_{ψ}^2) $\|\psi(x, y, z_1) - \psi(x, y, z_2)\|_{\mathbb{Y}} \leq k\|z_1 - z_2\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{Y})}$, for $x \in \mathbb{X}$, $y \in \mathbb{Y}$, $z_1, z_2 \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$, and some $k > 0$;
- (\mathbf{H}_{ψ}^3) $\langle y_1 - y_2, \psi(x, y_1, z) - \psi(x, y_2, z) \rangle \leq \mu\|y_1 - y_2\|_{\mathbb{Y}}^2$, for $x \in \mathbb{X}$, $y_1, y_2 \in \mathbb{Y}$, $z \in \mathcal{L}_2(\mathbb{U}, \mathbb{Y})$, and some $\mu \in \mathbb{R}$;
- (\mathbf{H}_{ψ}^4) $\psi(x, y, z)$ is separately continuous in (x, y) ;
- (\mathbf{H}_{ψ}^5) $\|\psi(x, 0, 0)\|_{\mathbb{Y}} \leq \|\psi(0, 0, 0)\|_{\mathbb{Y}} + l\|x\|_{\mathbb{X}}$, for $x \in \mathbb{X}$ and some $l > 0$.

Proposition 4.1. Under these assumptions (\mathbf{H}_{ψ}^1)–(\mathbf{H}_{ψ}^5), let $p > 2$ and $\beta < 0$ be given and choose $q \geq p$, $\beta < \eta < 0$. Then the following hold:

- (i) For $X \in L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))$ and $\lambda > -(\beta - \mu - \frac{k^2}{2})$, Eq. (12) has a unique solution $(Y(X), Z(X)) \in \mathcal{K}_{\beta}^p$. Moreover $Y(X) \in L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))$.
- (ii) $\|Y(X)\|_{L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{Y}))} + \|Z(X)\|_{L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))} + \|Y(X)\|_{L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))} \leq C(1 + \|X\|_{L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))})$, where C is a positive constant.
- (iii) The map $X \mapsto (Y(X), Z(X))$ is continuous from $L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))$ to \mathcal{K}_{β}^p and $X \mapsto Y(X)$ is continuous from $L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))$ to $L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))$.
- (iv) The statements of points (i)–(iii) still hold if the space $L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))$ is replaced by the space $L_{\mathcal{P}}^q(\Omega; C_{\eta}(\mathbb{X}))$.

Proof. By (\mathbf{H}_{ψ}^5) and the Hölder inequality we get

$$\begin{aligned} \mathbb{E}\left(\int_0^\infty e^{2\beta s} \|\psi(X(s), 0, 0)\|_{\mathbb{Y}}^2 ds\right)^{\frac{p}{2}} &\leq \mathbb{E}\left(\int_0^\infty e^{2\beta s} (\|\psi(0, 0, 0)\|_{\mathbb{Y}} + l\|X(s)\|_{\mathbb{X}})^2 ds\right)^{\frac{p}{2}} \\ &\leq C + C\mathbb{E}\left(\int_0^\infty e^{2\beta s} \|X(s)\|_{\mathbb{X}}^2 ds\right)^{\frac{p}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq C + C \mathbb{E} \int_0^\infty e^{p\beta's} \|X(s)\|_{\mathbb{X}}^p ds \\
&\leq C + C \left(\mathbb{E} \int_0^\infty e^{q\eta s} \|X(s)\|_{\mathbb{X}}^q ds \right)^{\frac{p}{q}},
\end{aligned} \tag{13}$$

where β' is chosen to satisfy $\beta < \beta' < \eta < 0$. And then (i) and (ii) follow from Theorem 3.1.

In order to deal with (iii), we define a Nemytskii operator

$$N: L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X})) \times L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{Y})) \times L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y}))) \mapsto L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{Y})),$$

as

$$N(X, Y, Z)(t) = \psi(X(t), Y(t), Z(t)).$$

By (\mathbf{H}_{ψ}^1) , (\mathbf{H}_{ψ}^2) , (\mathbf{H}_{ψ}^5) and estimates analogous to (13) we can prove that it is well defined. Continuity of N follows in a similar way by adopting the classical argument that proves continuity of Nemytskii operators in this framework (see [8, Theorem 3.4.4, p. 407]).

To (iii), one takes $X_n, X \in L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))$ satisfying $X_n \rightarrow X$ in $L_{\mathcal{P}}^q(\Omega; L_{\eta}^q(\mathbb{X}))$ for $n \rightarrow \infty$. Let $(Y(X_n), Z(X_n))$ and $(Y(X), Z(X))$ be the corresponding solutions. Then $(\bar{Y}, \bar{Z}) := (Y(X) - Y(X_n), Z(X) - Z(X_n))$ solves the equation

$$\begin{aligned}
&\bar{Y}(t) - \bar{Y}(T) + \int_t^T \bar{Z}(s) dW(s) + \lambda \int_t^T \bar{Y}(s) ds \\
&= \int_t^T (\psi(X(s), Y(X)(s), Z(X)(s)) - \psi(X_n(s), Y(X)(s) - \bar{Y}(s), Z(X)(s) - \bar{Z}(s))) ds.
\end{aligned}$$

The estimate of Theorem 3.1 gives for $\lambda > -(\beta - \mu - \frac{k^2}{2})$

$$\begin{aligned}
&\|\bar{Y}\|_{L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{Y}))} + \|\bar{Z}\|_{L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathcal{L}_2(\mathbb{U}, \mathbb{Y})))} + \|\bar{Y}\|_{L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{Y}))} \\
&\leq C \left(\mathbb{E} \left(\int_0^\infty e^{2\beta s} \|\psi(X(s), Y(X)(s), Z(X)(s)) - \psi(X_n(s), Y(X)(s) - \bar{Y}(s), Z(X)(s) - \bar{Z}(s))\|_{\mathbb{Y}}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, by continuity of the operator N , the right-hand side of the above inequality converges to zero as $n \rightarrow \infty$, and then we get (iii).

(iv) follows trivially from the previous ones, since $L_{\mathcal{P}}^q(\Omega; C_{\eta}(\mathbb{X})) \subset L_{\mathcal{P}}^q(\Omega; L_{\eta-\varepsilon}^q(\mathbb{X}))$ for every $\varepsilon > 0$. \square

Now we introduce a forward stochastic evolution equation

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s)) ds + \int_0^t e^{(t-s)A} G(X(s)) dW(s), \quad t \geq 0, \tag{14}$$

and assume that:

(HF1) The operator A is the infinitesimal generator of a C_0 -semigroup $\{e^{tA}\}$ of linear bounded operators on \mathbb{X} . We denote by M , a two positive constants such that $\|e^{tA}\| \leq Me^{at}$ for $t \geq 0$.

(HF2) The mapping $F: \mathbb{X} \mapsto \mathbb{X}$ satisfies, for some $l > 0$,

$$\|F(x) - F(y)\|_{\mathbb{X}} \leq l\|x - y\|_{\mathbb{X}}, \quad x, y \in \mathbb{X}.$$

(HF3) There exists $l > 0$ such that

$$\|G(x) - G(y)\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{X})} \leq l\|x - y\|_{\mathbb{X}}, \quad x, y \in \mathbb{X}.$$

From Proposition 4.6 in [6] we know that

Proposition 4.2. Suppose that (HF1)–(HF3) hold. Then for all $q \in [1, \infty)$ there exists a constant $\eta(q) < 0$, depending also on l , a and M , with the following properties:

- (i) For all $x \in \mathbb{X}$, the process $X(x)$, solution of Eq. (14), is in $\mathcal{H}_{\eta(q)}^q(\mathbb{X})$.
(ii) For a suitable constant $C > 0$, we have

$$\|X(x)\|_{L_{\mathcal{P}}^q(\Omega; L_{\eta(q)}^q(\mathbb{X}))} + \|X(x)\|_{L_{\mathcal{P}}^q(\Omega; C_{\eta(q)}(\mathbb{X}))} \leq C(1 + \|x\|_{\mathbb{X}}). \quad (15)$$

- (iii) The mapping $x \mapsto X(x)$ is continuous.

And then consider the system of stochastic differential equations:

$$\begin{cases} X(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s)) ds + \int_0^t e^{(t-s)A} G(X(s)) dW(s), & t \geq 0, \\ Y(t) - Y(T) + \int_t^T Z(s) dW(s) + \lambda \int_t^T Y(s) ds = \int_t^T \psi(X(s), Y(s), Z(s)) ds, & 0 \leq t \leq T < \infty. \end{cases} \quad (16)$$

Proposition 4.3. Suppose that (HF1)–(HF3) hold and ψ satisfies the conditions (\mathbf{H}_{ψ}^1) – (\mathbf{H}_{ψ}^5) (with $\mathbb{Y} = \mathbb{R}$). For $q \geq p > 2$, $\beta < \eta(q) < 0$ and $\lambda > -(\beta - \mu - \frac{k^2}{2})$. Then the following hold:

- (i) For every $x \in \mathbb{X}$ there exists a unique solution $(X(x), Y(x), Z(x))$ of Eq. (16) such that $X(x) \in \mathcal{H}_{\eta(q)}^q(\mathbb{X})$, $(Y(x), Z(x)) \in \mathcal{K}_{\beta}^p$. Moreover, $Y(x) \in L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{R}))$.
(ii) $\|Y(x)\|_{L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathbb{R}))} + \|Z(x)\|_{L_{\mathcal{P}}^p(\Omega; L_{\beta}^2(\mathcal{L}_2(\mathbb{U}, \mathbb{R})))} + \|Y(x)\|_{L_{\mathcal{P}}^p(\Omega; C_{\beta}(\mathbb{R}))} \leq C(1 + \|x\|_{\mathbb{X}})$.
(iii) Setting $u(x) := Y(0, x)$, then $u(x)$ is a continuous function.

Proof. (i) and (ii) follow from Proposition 4.1(i), (ii), (iv) and Proposition 4.2(i), (ii). Because $Y(x)$ is adapted, it follows that $Y(0, x)$ is deterministic. And from Proposition 4.1(iii) and Proposition 4.2(iii) we get (iii). \square

Next, we address solvability of the semilinear elliptic PDE

$$\mathcal{L}u(x) - \lambda u(x) + \psi(x, u(x), Du(x)G(x)) = 0, \quad x \in \mathbb{X}. \quad (18)$$

The operator \mathcal{L} associated with Eq. (14) is defined by

$$\mathcal{L}u(x) = \frac{1}{2} \text{Trace}(G(x)G(x)^* D^2 u(x)) + \langle Ax + F(x), Du(x) \rangle,$$

where $Du(x)$ and $D^2 u(x)$ correspond respectively to the first and second Fréchet derivatives of u at point $x \in \mathbb{X}$. Since we will always identify \mathbb{X} with its dual \mathbb{X}^* , $Du(x)$ may be identified with an element of \mathbb{X} while $D^2 u(x)$ may be identified indifferently with a bounded self-adjoint operator on \mathbb{X} or a symmetric bounded bilinear form on \mathbb{X} . $\mathcal{L}(\mathbb{X}, \mathbb{X})$ denotes the space of all symmetric bounded bilinear form on \mathbb{X} . In addition, we assume that A is bounded in order to make the operator \mathcal{L} meaningful. λ is a given real number and the function $\psi: \mathbb{X} \times \mathbb{R} \times \mathbb{U}^* \mapsto \mathbb{R}$ satisfies the conditions (\mathbf{H}_{ψ}^1) – (\mathbf{H}_{ψ}^5) (with $\mathbb{Y} = \mathbb{R}$). Note that, for $x \in \mathbb{X}$, $Du(x)$ belongs to \mathbb{X}^* , so that $Du(x)G(x)$ is in \mathbb{U}^* .

Definition 4.4. $u \in \mathcal{C}(\mathbb{X})$ is called a viscosity subsolution (supersolution) of Eq. (18) if, for any $\varphi \in \mathcal{C}^2(\mathbb{X})$, the following inequality holds at each local maximum (minimum) point x of $u - \varphi$:

$$-\mathcal{L}\varphi(x) + \lambda u(x) - \psi(x, u(x), D\varphi(x)G(x)) \leq (\geq) 0.$$

$u \in \mathcal{C}(\mathbb{X})$ is said to be a viscosity solution of Eq. (18) if it is both a viscosity subsolution and a viscosity supersolution.

To study the uniqueness, we introduce alternative definitions of viscosity sub- and supersolutions. For $u \in \mathcal{C}(\mathbb{X})$ and $x_0 \in \mathbb{X}$, let

$$\begin{aligned} D_+^2 u(x_0) &:= \left\{ (X, p) \in \mathcal{L}(\mathbb{X}, \mathbb{X}) \times \mathbb{X}; u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), x - x_0 \rangle + o(\|x - x_0\|_{\mathbb{X}}^2), x \in \mathbb{X} \right\}, \\ \bar{D}_+^2 u(x_0) &:= \left\{ (X, p) \in \mathcal{L}(\mathbb{X}, \mathbb{X}) \times \mathbb{X}; \exists \{(X_n, p_n, x_n)\} \subset \mathcal{L}(\mathbb{X}, \mathbb{X}) \times \mathbb{X} \times \mathbb{X} \text{ such that } (X_n, p_n) \in D_+^2 u(x_n), n \in \mathbb{N}, \right. \\ &\quad \left. \text{and } (X_n, p_n, u(x_n), x_n) \rightarrow (X, p, u(x_0), x_0) \text{ as } n \rightarrow \infty \right\}. \end{aligned}$$

Similarly, one defines D_-^2 of \bar{D}_-^2 for $u \in \mathcal{C}(\mathbb{X})$ on \mathbb{X} by setting

$$D_-^2 u(x_0) = -D_+^2 (-u)(x_0), \quad \bar{D}_-^2 u(x_0) = -\bar{D}_+^2 (-u)(x_0).$$

Definition 4.5. $u \in \mathcal{C}(\mathbb{X})$ is called a viscosity subsolution (supersolution) of Eq. (18) if for all $x \in \mathbb{X}$ and all $(X, p) \in \bar{D}_+^2 u(x)$ ($\bar{D}_-^2 u(x)$)

$$-\frac{1}{2} \text{Trace}(G(x)G(x)^*X) - \langle Ax + F(x), p \rangle + \lambda u(x) - \psi(x, u(x), pG(x)) \leq (\geq) 0.$$

By [9], Definitions 4.4 and 4.5 are equivalent.

Moreover, we also assume

$$(\mathbf{H}_\psi^6) \quad |\psi(x, r, p) - \psi(y, r, p)| \leq C\|x - y\|_{\mathbb{X}},$$

for all $x, y \in \mathbb{X}$, $r \in \mathbb{R}$, $p \in \mathbb{U}^*$.

Theorem 4.6. Suppose that $(\mathbf{HF1})$ – $(\mathbf{HF3})$ hold and ψ satisfies the conditions (\mathbf{H}_ψ^1) – (\mathbf{H}_ψ^6) (with $\mathbb{Y} = \mathbb{R}$). For $q \geq p > 2$, $\beta < \eta(q) < 0$ and $\lambda > -(\beta - \mu - \frac{k^2}{2})$. Then $u(x)$ is the unique viscosity solution of Eq. (18), among those functions which grow at most like $C|x|$ at infinity for some $C > 0$.

Proof. (Existence) We prove only that u is a viscosity subsolution, the proof of the other statement being similar.

For $\forall T > 0$ and $0 \leq t \leq T < \infty$, Proposition 4.1 shows that

$$Y(0, x) - Y(t, x) + \int_0^t Z(r, x) dW(r) + \lambda \int_0^t Y(r, x) dr = \int_0^t \psi(X(r, x), Y(r, x), Z(r, x)) dr. \quad (19)$$

By Markov property of the process X it holds $Y(t, x) = u(X(t, x))$.

Let now $\varphi \in \mathcal{C}^2(\mathbb{X})$ satisfying $\varphi(x) = u(x)$, $\varphi \geq u$. We can without loss of generality assume that φ , $D\varphi$, $D^2\varphi$ have at most polynomial growth at infinity.

Let $(\hat{Y}(x), \hat{Z}(x))$ be the unique solution (Proposition 4.3 in [7]) of the following BSDE:

$$\hat{Y}(s, x) - \varphi(X(t, x)) + \int_s^t \hat{Z}(r, x) dW(r) + \lambda \int_s^t Y(r, x) dr = \int_s^t \psi(X(r, x), Y(r, x), \hat{Z}(r, x)) dr, \quad s \in [0, t]. \quad (20)$$

Note that, from the Itô formula in [4, Theorem 4.17, p. 105]

$$\varphi(X(s, x)) - \varphi(X(t, x)) + \int_s^t D\varphi(X(r, x))G(X(r, x)) dW(r) + \int_s^t \mathcal{L}\varphi(X(r, x)) dr = 0. \quad (21)$$

Subtracting Eq. (21) from Eq. (20) we get that

$$\begin{aligned} \hat{Y}(s, x) - \varphi(X(s, x)) + \int_s^t (\hat{Z}(r, x) - D\varphi(X(r, x))G(X(r, x))) dW(r) \\ = \int_s^t (\mathcal{L}\varphi(X(r, x)) - \lambda Y(r, x) + \psi(X(r, x), Y(r, x), \hat{Z}(r, x))) dr, \quad s \in [0, t]. \end{aligned} \quad (22)$$

Taking expectation on both sides for $s = 0$ gives

$$\hat{Y}(0, x) - \varphi(x) = \mathbb{E} \int_0^t (\mathcal{L}\varphi(X(r, x)) - \lambda Y(r, x) + \psi(X(r, x), Y(r, x), \hat{Z}(r, x))) dr.$$

Since $u(X(t, x)) \leq \varphi(X(t, x))$, we may apply Corollary 4.4.2 in [5] to Eqs. (19) and (20) (think of $\psi(X(r, x), Y(r, x), z) - \lambda Y(r, x)$ as a random function of the z -argument only) to deduce that $u(x) = Y(0, x) \leq \hat{Y}(0, x)$, and since $\varphi(x) = u(x)$, we see that $\hat{Y}(0, x) - \varphi(x) \geq 0$. Now

$$\frac{1}{t} \mathbb{E} \int_0^t (\mathcal{L}\varphi(X(r, x)) - \lambda Y(r, x) + \psi(X(r, x), Y(r, x), \hat{Z}(r, x))) dr \geq 0.$$

By (\mathbf{H}_ψ^2) we deduce

$$\begin{aligned} & \frac{1}{t} \mathbb{E} \int_0^t (\mathcal{L}\varphi(X(r, x)) - \lambda Y(r, x) + \psi(X(r, x), Y(r, x), D\varphi(X(r, x))G(X(r, x)))) dr \\ & + \frac{k}{t} \mathbb{E} \int_0^t \|\hat{Z}(r, x) - D\varphi(X(r, x))G(X(r, x))\|_{\mathbb{U}^*} dr \geq 0. \end{aligned} \quad (23)$$

We claim that

$$\frac{k}{t} \mathbb{E} \int_0^t \|\hat{Z}(r, x) - D\varphi(X(r, x))G(X(r, x))\|_{\mathbb{U}^*} dr \rightarrow 0, \quad t \rightarrow 0.$$

And then taking the limit as $t \rightarrow 0$ in (23), we can obtain by dominated convergence theorem

$$\mathcal{L}\varphi(x) - \lambda u(x) + \psi(x, u(x), D\varphi(x)G(x)) \geq 0.$$

Now we conclude the claim. From the Itô formula and Eq. (22), we see that for $\delta > 0$ and $s < t$

$$\begin{aligned} & \mathbb{E} e^{\delta s} |\tilde{Y}(s, x)|^2 + \mathbb{E} \int_s^t e^{\delta r} (\delta |\tilde{Y}(r, x)|^2 + \|\tilde{Z}(r, x)\|_{\mathbb{U}^*}^2) dr \\ & = 2 \mathbb{E} \int_s^t e^{\delta r} \tilde{Y}(r, x) (\mathcal{L}\varphi(X(r, x)) - \lambda Y(r, x) + \psi(X(r, x), Y(r, x), \tilde{Z}(r, x) + D\varphi(X(r, x))G(X(r, x)))) dr, \end{aligned} \quad (24)$$

where $\tilde{Y}(r, x) := \hat{Y}(r, x) - \varphi(X(r, x))$, $\tilde{Z}(r, x) := \hat{Z}(r, x) - D\varphi(X(r, x))G(X(r, x))$. By those estimates (15), (17) and (\mathbf{H}_ψ^1) , (\mathbf{H}_ψ^2) , (\mathbf{H}_ψ^5) , we find that the right-hand side is less than or equal to

$$(k^2 + 1) \mathbb{E} \int_s^t e^{\delta r} |\tilde{Y}(r, x)|^2 dr + \mathbb{E} \int_s^t e^{\delta r} \|\tilde{Z}(r, x)\|_{\mathbb{U}^*}^2 dr + C \int_s^t e^{\delta r} dr.$$

Taking $\delta = k^2 + 1$ proves that

$$\mathbb{E} e^{(k^2+1)s} |\tilde{Y}(s, x)|^2 \leq C(t-s). \quad (25)$$

Working from (24) again with $\delta = 0$ and $s = 0$ gives that by the Hölder inequality and the Young inequality

$$\mathbb{E} \int_0^t \|\tilde{Z}(r, x)\|_{\mathbb{U}^*}^2 dr \leq \frac{1}{2} \mathbb{E} \int_0^t \|\tilde{Z}(r, x)\|_{\mathbb{U}^*}^2 dr + Ct^{\frac{1}{2}} \left(\mathbb{E} \int_0^t |\tilde{Y}(r, x)|^2 dr \right)^{\frac{1}{2}} + C \mathbb{E} \int_0^t |\tilde{Y}(r, x)|^2 dr.$$

Hence

$$\begin{aligned} \frac{1}{t} \mathbb{E} \int_0^t \|\tilde{Z}(r, x)\|_{\mathbb{U}^*} dr & \leq t^{-\frac{1}{2}} \left(\mathbb{E} \int_0^t \|\tilde{Z}(r, x)\|_{\mathbb{U}^*}^2 dr \right)^{\frac{1}{2}} \\ & \leq Ct^{-\frac{1}{2}} \left(t^{\frac{1}{2}} \left(\int_0^t (t-r) dr \right)^{\frac{1}{2}} + \int_0^t (t-r) dr \right)^{\frac{1}{2}} \\ & \leq C(t^{\frac{1}{2}} + t)^{\frac{1}{2}}. \end{aligned}$$

(Uniqueness) Set u_1, u_2 be two viscosity solution of Eq. (18) and then u_1 (resp. u_2) is a viscosity subsolution (resp. supersolution) of Eq. (18). We only need to prove $u_1 \leq u_2$.

Assume

$$M = \sup_{x \in \mathbb{X}} (u_1(x) - u_2(x)) > 0.$$

For $\delta > 0$, define

$$\Phi(x, y) = u_1(x) - u_2(y) - \frac{1}{2\delta} \|x - y\|_{\mathbb{X}}^2.$$

We may use the abstract optimization results to obtain (\hat{x}, \hat{y}) such that (\hat{x}, \hat{y}) is a unique strict maximum point of $\Phi(x, y)$ (see [10] for details). And then

$$\Phi(\hat{x}, \hat{y}) \geq M > 0.$$

Because \mathbb{X} is a real separable Hilbert space, there exists an increasing sequence of finite-dimensional subspaces \mathbb{X}_N of \mathbb{X} such that $\bigcup_N \mathbb{X}_N$ is dense in \mathbb{X} . P_N and Q_N denote the orthogonal projections onto \mathbb{X}_N and \mathbb{X}_N^\perp , respectively. By [10, Lemma 4] there exist $X = P_N X P_N$ and $Y = P_N Y P_N$ such that

$$\begin{aligned} \left(X + \frac{2}{\delta} Q_N, \frac{\hat{x} - \hat{y}}{\delta}\right) &\in \bar{D}_+^2 u_1(\hat{x}), \\ \left(Y - \frac{2}{\delta} Q_N, \frac{\hat{x} - \hat{y}}{\delta}\right) &\in \bar{D}_-^2 u_2(\hat{y}), \\ -\frac{2}{\lambda} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{1}{\lambda} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \end{aligned} \quad (26)$$

where $\lambda > 0$ is a constant. It follows from Definition 4.5 that

$$\begin{aligned} -\frac{1}{2} \text{Trace} \left(G(\hat{x}) G(\hat{x})^* \left(X + \frac{2}{\delta} Q_N \right) \right) - \left\langle A\hat{x} + F(\hat{x}), \frac{\hat{x} - \hat{y}}{\delta} \right\rangle + \lambda u_1(\hat{x}) - \psi \left(\hat{x}, u_1(\hat{x}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{x}) \right) &\leq 0, \\ -\frac{1}{2} \text{Trace} \left(G(\hat{y}) G(\hat{y})^* \left(Y - \frac{2}{\delta} Q_N \right) \right) - \left\langle A\hat{y} + F(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} \right\rangle + \lambda u_2(\hat{y}) - \psi \left(\hat{y}, u_2(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{y}) \right) &\geq 0. \end{aligned}$$

By simple calculation we obtain

$$J \leq I_1 + I_2 + I_3 + I_4 + I_5, \quad (27)$$

where

$$\begin{aligned} J &:= \lambda(u_1(\hat{x}) - u_2(\hat{y})) - \left[\psi \left(\hat{x}, u_1(\hat{x}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{x}) \right) - \psi \left(\hat{x}, u_2(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{x}) \right) \right], \\ I_1 &:= \frac{1}{2} \text{Trace} (G(\hat{x}) G(\hat{x})^* X - G(\hat{y}) G(\hat{y})^* Y), \\ I_2 &:= \frac{1}{2} \text{Trace} \left(G(\hat{x}) G(\hat{x})^* \frac{2}{\delta} Q_N \right) + \frac{1}{2} \text{Trace} \left(G(\hat{y}) G(\hat{y})^* \frac{2}{\delta} Q_N \right), \\ I_3 &:= \left\langle A(\hat{x} - \hat{y}) + F(\hat{x}) - F(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} \right\rangle, \\ I_4 &:= \left[\psi \left(\hat{x}, u_2(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{x}) \right) - \psi \left(\hat{x}, u_2(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{y}) \right) \right], \\ I_5 &:= \left[\psi \left(\hat{x}, u_2(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{y}) \right) - \psi \left(\hat{y}, u_2(\hat{y}), \frac{\hat{x} - \hat{y}}{\delta} G(\hat{y}) \right) \right]. \end{aligned}$$

To J , using (\mathbf{H}_ψ^3) gives

$$J \cdot (u_1(\hat{x}) - u_2(\hat{y})) \geq (\lambda - \mu)(u_1(\hat{x}) - u_2(\hat{y}))^2.$$

Because $u_1(\hat{x}) > u_2(\hat{y})$

$$J \geq (\lambda - \mu)(u_1(\hat{x}) - u_2(\hat{y})). \quad (28)$$

For I_1 , let $\{e_1, e_2, \dots\}$ be an orthonormal basis of \mathbb{U} ,

$$\begin{aligned} I_1 &= \frac{1}{2} \text{Trace} (G(\hat{x})^* X G(\hat{x}) - G(\hat{y})^* Y G(\hat{y})) \\ &= \frac{1}{2} \sum_{i=1}^{\infty} [\langle X G(\hat{x}) e_i, G(\hat{x}) e_i \rangle - \langle Y G(\hat{y}) e_i, G(\hat{y}) e_i \rangle] \\ &\leq \frac{1}{2\lambda} \sum_{i=1}^{\infty} \|G(\hat{x}) e_i - G(\hat{y}) e_i\|_{\mathbb{X}}^2 \\ &= \frac{1}{2\lambda} \|G(\hat{x}) - G(\hat{y})\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{X})}^2 \\ &\leq C \frac{1}{\lambda} \|\hat{x} - \hat{y}\|_{\mathbb{X}}^2, \end{aligned} \quad (29)$$

where we have used (26) and $(\mathbf{HF3})$.

Next, we treat I_2 . Firstly, $I_2 = I_{21} + I_{22}$, where $I_{21} := \frac{1}{2} \text{Trace}(G(\hat{x})G(\hat{x})^* \frac{2}{\delta} Q_N)$ and $I_{22} := \frac{1}{2} \text{Trace}(G(\hat{y})G(\hat{y})^* \frac{2}{\delta} Q_N)$. For I_{21} , we note that if $\{h_1, h_2, \dots\}$ is the orthonormal basis of \mathbb{X} consisting of the eigenvectors of $G(\hat{x})G(\hat{x})^*$ then $G(\hat{x})G(\hat{x})^*x = \sum_{j=1}^{\infty} \kappa_j \langle x, h_j \rangle h_j$, where $\sum_{j=1}^{\infty} \kappa_j < \infty$ and therefore

$$I_{21} \leq \frac{1}{\delta} \sum_{j=1}^{\infty} \langle Q_N G(\hat{x})G(\hat{x})^* h_j, h_j \rangle = \frac{1}{\delta} \sum_{j=1}^{\infty} \kappa_j \langle Q_N h_j, h_j \rangle \leq \frac{1}{\delta} \sum_{j=1}^{\infty} \kappa_j \|Q_N h_j\|_{\mathbb{X}}.$$

Because $\sum_{j=1}^{\infty} \kappa_j < \infty$ and $\lim_{N \rightarrow \infty} \|Q_N h_j\|_{\mathbb{X}} = 0$ for every j ,

$$I_{21} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (30)$$

In the same way as I_{21} one obtain

$$I_{22} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (31)$$

(HF2) admits us to get

$$I_3 \leq C \frac{1}{\delta} \|\hat{x} - \hat{y}\|_{\mathbb{X}}^2. \quad (32)$$

By (H_{ψ}^2) and (HF3) we can attain

$$I_4 \leq k \frac{1}{\delta} \|\hat{x} - \hat{y}\|_{\mathbb{X}} \|G(\hat{x}) - G(\hat{y})\|_{\mathcal{L}_2(\mathbb{U}, \mathbb{X})} \leq C \frac{1}{\delta} \|\hat{x} - \hat{y}\|_{\mathbb{X}}^2. \quad (33)$$

Finally we deal with I_5 . It holds by (H_{ψ}^6)

$$I_5 \leq C \|\hat{x} - \hat{y}\|_{\mathbb{X}}. \quad (34)$$

Combining with (27)–(34) and letting $N \rightarrow \infty$, we have

$$(\lambda - \mu)(u_1(\hat{x}) - u_2(\hat{y})) \leq C \frac{1}{\lambda} \|\hat{x} - \hat{y}\|_{\mathbb{X}}^2 + C \frac{1}{\delta} \|\hat{x} - \hat{y}\|_{\mathbb{X}}^2 + C \|\hat{x} - \hat{y}\|_{\mathbb{X}}. \quad (35)$$

By the similar method to Lemma 6.9 in [11] we may deduce $\frac{1}{\delta} \|\hat{x} - \hat{y}\|_{\mathbb{X}}^2 \rightarrow 0$ and $u_1(\hat{x}) - u_2(\hat{y}) \rightarrow M$ as $\delta \rightarrow 0$. When δ is small enough, the left side of (35) is bounded from below by $(\lambda - \mu) \frac{M}{2}$ and the right side is arbitrarily small. This is a contradiction. Thus the proof is completed. \square

Acknowledgments

The author would like to thank Jicheng Liu and Xicheng Zhang for their valuable discussions and also wish to thank referees for many suggestions and improvements.

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