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Classification of bifurcation diagrams of a p-Laplacian Dirichlet problem with examples ☆

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ABSTRACT

We study bifurcation diagrams of positive solutions of the p-Laplacian Dirichlet problem

$$\begin{cases} (\varphi_p(u'(x)))' + f_\lambda(u(x)) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $\varphi_p(y) = |y|^{p-2}y$, $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, and $p > 1$ and $\lambda > 0$ are two bifurcation parameters. Assume that $f_\lambda(u) = \lambda g(u) - h(u)$ where $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy hypotheses (H1)–(H5) presented herein. For different values p with $1 < p \leqslant 2$ and with $p > 2$, we give a classification of totally six different bifurcation diagrams. We prove that, on the $(\lambda, \|u\|_\infty)$-plane, each possible bifurcation diagram consists of exactly one curve with exactly one turning point where the curve turns to the right. Hence we are able to determine the exact multiplicity of positive solutions. In addition, for $1 < p \leqslant 2$ and for $p > 2$, we give interesting examples $f_\lambda(u) = \lambda(ku^{p-1} + u^q) - u^r$ satisfying $r > q > p - 1$ and $k \geqslant 0$, and show complete evolution of bifurcation diagrams as evolution parameter k varies from 0 to ∞.

1. Introduction

In this paper we study bifurcation diagrams of positive solutions of the p-Laplacian Dirichlet problem

$$\begin{cases} (\varphi_p(u'(x)))' + f_\lambda(u(x)) = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\varphi_p(y) = |y|^{p-2}y$, $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, and $p > 1$ and $\lambda > 0$ are two bifurcation parameters. We assume that the nonlinearity

$$f_\lambda(u) = \lambda g(u) - h(u)$$

where functions $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy hypotheses (H1)–(H5):

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(H1) $g(0) = h(0) = 0$, $g(u)$, $h(u) > 0$ on $(0, \infty)$, and

$$0 = \lim_{u \to 0^+} \frac{h(u)}{u^{p-1}} \leqslant m_0^g \equiv \lim_{u \to 0^+} \frac{g(u)}{u^{p-1}} < \infty. \tag{1.2}$$

(H2) The positive function $h(u)/g(u)$ is strictly increasing on $(0, \infty)$, and

$$\lim_{u \to 0^+} \frac{h(u)}{g(u)} = 0, \qquad \lim_{u \to \infty} \frac{h(u)}{g(u)} = \infty.$$

(H3) $(p-2)g'(u) - ug''(u) < 0$ on $(0, \infty)$ and $(p-2)h'(u) - uh''(u) < 0$ on $(0, \infty)$.
(H4) The positive function $[(p-2)h'(u) - uh''(u)]/[(p-2)g'(u) - ug''(u)]$ is strictly increasing on $(0, \infty)$, and

$$\lim_{u \to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0, \qquad \lim_{u \to \infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

(H5) If $p > 2$, there exists a positive number $p^* > p-1$ such that $g(u)/u^{p^*}$ is strictly decreasing on $(0, \infty)$ and $h(u)/u^{p^*}$ is strictly increasing on $(0, \infty)$. In addition, for each fixed $s \in (0, 1)$,

$$\frac{h(su)}{u^{p-1}} \left[\frac{h(u)g(su)}{g(u)h(su)} - 1 \right]$$

is a strictly increasing function of u on $(0, \infty)$, and

$$\lim_{u \to \infty} \frac{h(u)g(su)}{g(u)h(su)} \in (1, \infty]. \tag{1.3}$$

(Note that (H5) is a technical hypothesis for the case $p > 2$.)

It is easy to see that, hypotheses (H1) and (H2) imply that, for each fixed $\lambda > 0$, $f_\lambda(0) = \lambda g(0) - h(0) = 0$ and there exists a unique positive number β_λ such that

$$\begin{cases} f_\lambda(u) = \lambda g(u) - h(u) > 0 \quad \text{on } (0, \beta_\lambda), \\ f_\lambda(\beta_\lambda) = \lambda g(\beta_\lambda) - h(\beta_\lambda) = 0, \\ f_\lambda(u) = \lambda g(u) - h(u) < 0 \quad \text{on } (\beta_\lambda, \infty). \end{cases} \tag{1.4}$$

Moreover, the number β_λ is a strictly increasing and continuous function of λ on $(0, \infty)$, and

$$\lim_{\lambda \to 0^+} \beta_\lambda = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \beta_\lambda = \infty. \tag{1.5}$$

Also, hypotheses (H1)–(H4) imply that, for each fixed $\lambda > 0$, the function $(p-2)f'_\lambda(u) - uf''_\lambda(u)$ changes sign exactly once on $(0, \beta_\lambda)$. More precisely, there exists a unique positive number $\gamma_\lambda < \beta_\lambda$ such that

$$\begin{cases} (p-2)f'_\lambda(u) - uf''_\lambda(u) < 0 \quad \text{on } (0, \gamma_\lambda), \\ (p-2)f'_\lambda(\gamma_\lambda) - \gamma_\lambda f''_\lambda(\gamma_\lambda) = 0, \\ (p-2)f'_\lambda(u) - uf''_\lambda(u) > 0 \quad \text{on } (\gamma_\lambda, \beta_\lambda). \end{cases} \tag{1.6}$$

(We omit the proofs of (1.4) and (1.6).) So, in particular when $p = 2$, $f_\lambda(u) = \lambda g(u) - h(u)$ is convex-concave on $(0, \beta_\lambda)$.

Note that, by a positive solution to p-Laplacian problem (1.1), we mean a positive function $u \in C^1[-1, 1]$ with $\varphi_p(u') \in C^1[-1, 1]$ satisfying (1.1). Let $Z = \{x \in [-1, 1]\colon\ u'(x) = 0\}$. We note that it is easy to show that, if $f \in C[0, \infty)$ and u is a positive solution of (1.1), then $u \in C^2[-1, 1]$ if $1 < p \leqslant 2$ and $u \in C^2([-1, 1] - Z)$ if $p > 2$. For the proof we refer to Addou [1, Lemma 6].

In this paper we are concerned only with positive solutions u_λ of (1.1) satisfying $\|u_\lambda\|_\infty < \beta_\lambda$.

The p-Laplacian problem (1.1) arises in the study of non-Newtonian fluids and nonlinear diffusion problems. The quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids (see, e.g., Díaz [3,4] and their bibliographies). The p-Laplacian also appears in the study of torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, see [8]), glacial sliding ($p \in (1, 4/3]$, see [12]) or flow through porous media ($p = 3/2$, see [18]). The reaction term of (1.1) $f_\lambda(u) = \lambda g(u) - h(u)$, for fixed $\lambda > 0$, consists of a source term $\lambda g(u)$ and an absorption term $h(u)$ which is dominated by the source term when u near 0^+ and dominates the source term when u near ∞. The model nonlinearity for p-Laplacian problem (1.1) is

$$f_\lambda(u) = \lambda g(u) - h(u) = \lambda\left(ku^{p-1} + u^q\right) - u^r \quad \text{with } r > q > p-1 \text{ and } k \geqslant 0.$$

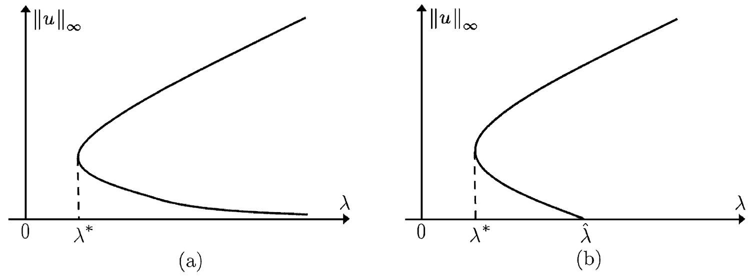

Fig. 1. Classified two bifurcation diagrams of (1.1) with $1 < p \leqslant 2$, $f_\lambda(u) = \lambda g(u) - h(u)$ and g, h satisfying (H1)–(H4); (a) $\hat{\lambda} = \infty$. (b) $0 < \hat{\lambda} < \infty$.

It is easy to check that functions $g(u) = ku^{p-1} + u^q$ and $h(u) = u^r$ with $r > q > p - 1$ and $k \geqslant 0$ satisfy hypotheses (H1)–(H5) with $p^* = (r + q)/2 > p - 1$. In particular, when $k = 0$, $g(u) = u^q$ and $h(u) = u^r$, we obtain that

$$0 < \gamma_\lambda = \left[\frac{q(q-p+1)}{r(r-p+1)}\lambda\right]^{1/(r-q)} < \beta_\lambda = \lambda^{1/(r-q)}.$$

See Corollary 2.2 for $1 < p \leqslant 2$ and Corollary 2.4 for $p > 2$ and their proofs given below.

Takeuchi [15,16] and Takeuchi and Yamada [17] studied existence and multiplicity of positive solutions of a degenerate p-Laplacian elliptic problem with logistic reaction

$$\begin{cases} \Delta_p u + \lambda\left(u^q - u^r\right) = 0 & \text{in } \Omega \ \left(\Omega \subset \mathbb{R}^N,\ N \geqslant 1\right), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $p > 2$ and $r > q > p - 1$. When $N \geqslant 2$ and Ω is a connected, bounded open subset of $\mathbb{R}^N$ with $C^{2,\alpha}$ boundary $\partial\Omega$ and $\alpha \in (0,1)$, under the restriction $p > 2$, Takeuchi [15,16] proved that there exists a number $\lambda^* > 0$ such that (1.7) has at least two positive weak solutions for $\lambda > \lambda^*$, at least one positive weak solution for $\lambda = \lambda^*$, and no positive weak solution for $0 < \lambda < \lambda^*$. (Note that the existence of solutions u_λ of (1.7) with *flat core* $\mathcal{O}_\lambda = \mathcal{O}_\lambda(u_\lambda) \equiv \{x \in \Omega \mid u_\lambda(x) = 1\} \neq \emptyset$ was also considered in [15,16]. Also note that the case $p = 2$, $r = 3 > 2 = q$ has been previously studied in Rabinowitz [13].) Later Dong and Chen [7] proved the same multiplicity result for general $p > 1$ and more general nonlinearity. In particular, in the next theorem, when $N = 1$ and $\Omega = (-1,1)$, Takeuchi and Yamada [17, Lemma 3.1 and Fig. 2(iii)] obtained exact multiplicity of positive solutions u of (1.7) satisfying $\|u\|_\infty < 1$.

Theorem 1.1. *Consider* (1.7) *where $p > 2$ and $r > q > p - 1$. When $N = 1$ and $\Omega = (-1,1)$, there exist two positive numbers $\lambda^* < \tilde{\lambda}$ such that* (1.7) *has exactly two positive solutions u satisfying $\|u\|_\infty < 1$ for $\lambda^* < \lambda < \tilde{\lambda}$, exactly one positive solution u satisfying $\|u\|_\infty < 1$ for $\lambda = \lambda^*$ and $\lambda \geqslant \tilde{\lambda}$, and no positive solution u satisfying $\|u\|_\infty < 1$ for $0 < \lambda < \lambda^*$.*

It is interesting to note that Díaz and Hernández [5] studied the exact multiplicity of positive solutions of the p-Laplacian Dirichlet problem

$$\begin{cases} \left(\varphi_p\left(u'(x)\right)\right)' + \lambda u^q(x) - \sigma u^r(x) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.8}$$

where $p > 1$, $0 < r < q < p - 1$, σ is a positive constant, and $\lambda > 0$ is a bifurcation parameter. Define

$$\bar{\lambda} = \sigma^{(p-1-q)/(p-1-r)} \left\{ \left(\frac{p-1}{p}\right)^{1/p} \int_0^{(q/r)^{1/(q-r)}} \frac{du}{[-F(u)]^{1/p}} \right\}^{(p-1)(q-r)/(p-1-r)},$$

where $F(u) = \int_0^u f(t)\,dt$ and $f(u) = u^q - u^r$. They [5, Theorem 1] proved that there exists a positive number $\lambda^* < \bar{\lambda}$ such that (1.8) has exactly two positive solutions for $\lambda^* < \lambda \leqslant \bar{\lambda}$, exactly one positive solution for $\lambda = \lambda^*$ and $\lambda > \bar{\lambda}$, and no positive solution for $0 < \lambda < \lambda^*$. The results were extended very recently by Díaz, Hernández and Mancebo [6] from $p - 1 > q > r > 0$ to $p - 1 > q > r > -1$; see [6, Theorems 2–4] for details.

2. Main results

The main results in this paper are Theorem 2.1 and Corollary 2.2 for $1 < p \leqslant 2$ and Theorem 2.3 and Corollary 2.4 for $p > 2$. In Theorems 2.1 and 2.3, we give a classification of totally *six* different bifurcation diagrams of positive solutions u_λ for p-Laplacian problem (1.1) satisfying $\|u_\lambda\|_\infty < \beta_\lambda$ under hypotheses (H1)–(H5). Figs. 1(a)–(b) represent *two* different bifurcation diagrams for $1 < p \leqslant 2$ and Figs. 3(a)–(d) (drawn below) represent another *four* different bifurcation diagrams

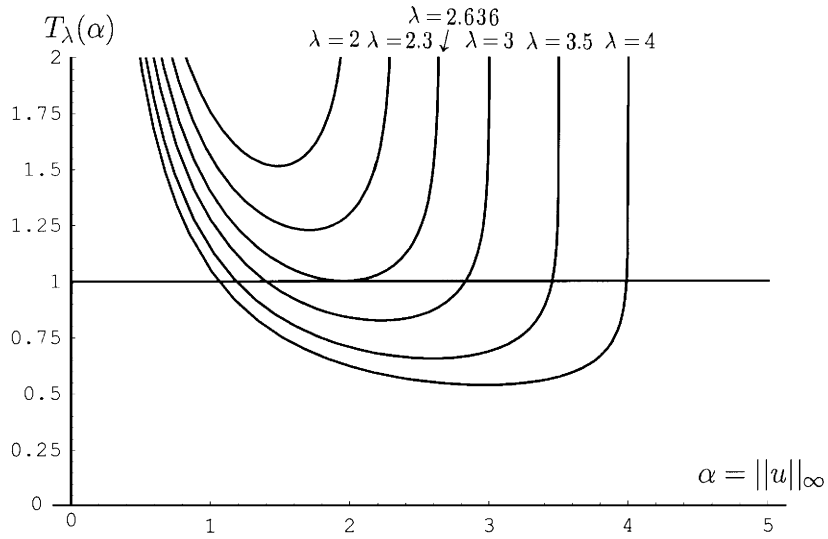

Fig. 2. Numerical simulations of $T_\lambda(\alpha)$ for $\alpha \in (0, \beta_\lambda)$: $p = 2$, $f_\lambda(u) = \lambda u^3 - u^4$, $\lambda = 2, 2.3, 2.636, 3, 3.5, 4$. Note that $\lambda^* \approx 2.636$, $\|u_{\lambda^*}\| \approx 1.957$, and $\lim_{\alpha\to 0^+} T_\lambda(\alpha) = \infty = \lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha)$.

for $p > 2$. We prove that, on the $(\lambda, \|u\|_\infty)$-plane, each bifurcation diagram consists of exactly one curve with exactly one turning point where the curve turns to the right. Hence we are able to determine the exact multiplicity of positive solutions by the values of λ^*, $\hat{\lambda}$, and $\tilde{\lambda}$, see Theorems 2.1 and 2.3 and Figs. 1 and 3.

We first define

$$\hat{\lambda} = \left(\frac{p-1}{m_0^g}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \in (0, \infty] \quad \text{for } 0 \leqslant m_0^g < \infty. \tag{2.1}$$

Theorem 2.1. (*See Fig.* 1.) *Let* $1 < p \leqslant 2$. *Consider* (1.1) *where* $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$ *satisfy* (H1)–(H4). *Consider positive solutions* u_λ *for* (1.1) *satisfying* $\|u_\lambda\|_\infty < \beta_\lambda$. *Then there exists a positive number* $\lambda^* < \hat{\lambda}$ $(\leqslant \infty)$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \hat{\lambda}$, *exactly one positive solution* v_λ *for* $\lambda = \lambda^*$ *and* $\lambda \geqslant \hat{\lambda}$, *and no positive solution for* $0 < \lambda < \lambda^*$. *Moreover, if we denote* $u_{\lambda^*} = v_{\lambda^*}$ *when* $\lambda = \lambda^*$, *then*:

(i) *For* $\lambda^* \leqslant \lambda_1 < \lambda_2 < \hat{\lambda}$, $\|u_{\lambda_2}\|_\infty < \|u_{\lambda_1}\|_\infty$.
(ii) *For* $\lambda^* \leqslant \lambda_1 < \lambda_2 < \infty$, $\|v_{\lambda_1}\|_\infty < \|v_{\lambda_2}\|_\infty$.
(iii) $\lim_{\lambda\to\hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\infty} \|v_\lambda\|_\infty = \infty$.

In the next Corollary 2.2 to Theorem 2.1, we give examples of polynomial nonlinearities for $f_\lambda(u) = \lambda g(u) - h(u) = \lambda(ku^{p-1} + u^q) - u^r$ satisfying $r > q > p - 1$ and $k \geqslant 0$. We give a classification of totally *two* bifurcation diagrams. We also show evolution phenomena of *two* different bifurcation diagrams as evolution parameter k varies from 0 to ∞.

Corollary 2.2. *Let* $1 < p \leqslant 2$.

(i) (*See Fig.* 1(a) *with* $\hat{\lambda} = \infty$.) *Let* $g(u) = u^q$ *and* $h(u) = u^r$ *with* $r > q > p - 1$. *Then* $g, h \in C[0, \infty) \cap C^2(0, \infty)$ *satisfy* (H1)–(H4) *with* $\hat{\lambda} = \infty$.
(ii) (*See Fig.* 1(b) *with* $0 < \hat{\lambda} < \infty$.) *Let* $g(u) = ku^{p-1} + u^q$ *and* $h(u) = u^r$ *with* $r > q > p - 1$ *and* $k > 0$. *Then* $g, h \in C[0, \infty) \cap C^2(0, \infty)$ *satisfy* (H1)–(H4) *with* $\hat{\lambda} = ((p-1)/k)((\pi/p)\csc(\pi/p))^p \in (0, \infty)$.

Remark 1. For Corollary 2.2(i), in particular, let $g(u) = u^q$ and $h(u) = u^r$ with $r = 4 > 3 = q > p = 2$, then numerical simulations as given in Fig. 2 show that $\lambda^* \approx 2.636$ and $\|u_{\lambda^*}\| \approx 1.957$.

Theorem 2.3. *Let* $p > 2$. *Consider* (1.1) *where* $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$ *satisfy* (H1)–(H5). *Consider positive solutions* u_λ *for* (1.1) *satisfying* $\|u_\lambda\|_\infty < \beta_\lambda$. *Then there exist three positive numbers* $\lambda^* < \tilde{\lambda}$ *and* $\beta_{\tilde{\lambda}}$ *satisfying* $\lambda^* < \hat{\lambda}$ $(\leqslant \infty)$ *and*

$$f_{\tilde{\lambda}}(\beta_{\tilde{\lambda}}) = 0 \quad \textit{and} \quad \lim_{\alpha\to\beta_{\tilde{\lambda}}^-} T_{\tilde{\lambda}}(\alpha) = 1$$

(*where* $T_\lambda(\alpha)$ *is defined in* (3.14) *below*), *such that*:

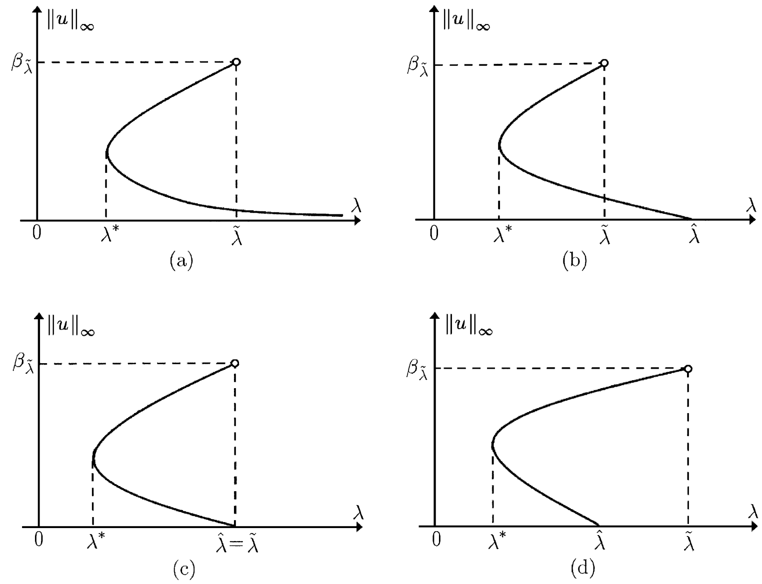

Fig. 3. Classified four bifurcation diagrams of (1.1) with $p > 2$, $f_\lambda(u) = \lambda g(u) - h(u)$ and g, h satisfying (H1)–(H5); (a) $0 < \tilde\lambda < \hat\lambda = \infty$. (b) $0 < \tilde\lambda < \hat\lambda < \infty$. (c) $0 < \hat\lambda = \tilde\lambda < \infty$. (d) $0 < \hat\lambda < \tilde\lambda < \infty$. The point $(\tilde\lambda, \beta_{\tilde\lambda})$ is defined by $f_{\tilde\lambda}(\beta_{\tilde\lambda}) = 0$ and it satisfies $\lim_{\alpha\to\beta_{\tilde\lambda}^-} T_{\tilde\lambda}(\alpha) = 1$.

(i) (*See Figs.* 3(a)–(b).) *If* $\tilde\lambda < \hat\lambda$ $(\leqslant \infty)$, *then* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \tilde\lambda$, *exactly one positive solution* u_λ *for* $\lambda = \lambda^*$ *and* $\tilde\lambda \leqslant \lambda < \hat\lambda$, *and no positive solution for* $0 < \lambda < \lambda^*$ *and for* $\lambda \geqslant \hat\lambda$ (*if* $\hat\lambda < \infty$).

(ii) (*See Figs.* 3(c)–(d).) *If* $\hat\lambda \leqslant \tilde\lambda$, *then* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \hat\lambda$, *exactly one positive solution* v_λ *for* $\lambda = \lambda^*$ *and for* $\hat\lambda \leqslant \lambda < \tilde\lambda$ (*if* $\tilde\lambda > \hat\lambda$), *and no positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \geqslant \tilde\lambda$.

Moreover, if we denote $u_{\lambda^*} = v_{\lambda^*}$ *when* $\lambda = \lambda^*$, *then*:

(iii) *For* $\lambda^* \leqslant \lambda_1 < \lambda_2 < \hat\lambda$, $\|u_{\lambda_2}\|_\infty < \|u_{\lambda_1}\|_\infty < \beta_{\tilde\lambda}$.

(iv) *For* $\lambda^* \leqslant \lambda_1 < \lambda_2 < \tilde\lambda$, $\|v_{\lambda_1}\|_\infty < \|v_{\lambda_2}\|_\infty < \beta_{\tilde\lambda}$.

(v) $\lim_{\lambda\to\hat\lambda^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\tilde\lambda^-} \|v_\lambda\|_\infty = \beta_{\tilde\lambda}$.

We give next remark to Theorem 2.3.

Remark 2. (See Fig. 3 and cf. Fig. 1.) Let $p > 2$. Consider (1.1) where $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1)–(H5). Then it can be proved that:

(i) for each $\lambda \geqslant \tilde\lambda$, there exists a positive solution v_λ of (1.1) with *flat core* $\mathcal{O}_\lambda = \mathcal{O}_\lambda(v_\lambda) \equiv \{x \in (-1,1) \mid v_\lambda(x) = \beta_\lambda\} \neq \emptyset$,

(ii) on the $(\lambda, \|u\|_\infty)$-plane, for $\lambda \geqslant \tilde\lambda$, the solution branch of positive solutions v_λ with *flat core* $\mathcal{O}_\lambda \neq \emptyset$ starts at $(\tilde\lambda, \beta_{\tilde\lambda})$, it is a monotone curve for $\lambda \geqslant \tilde\lambda$, and $\|v_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$.

The above parts (i) and (ii) can be proved since β_λ is a strictly increasing and continuous function of λ on $(0,\infty)$, and by applying (1.5) and Lemma 3.4 stated below; we omit the details of the proofs due to space limitations.

In the next Corollary 2.4 to Theorem 2.3, we give examples of polynomial nonlinearities for $f_\lambda(u) = \lambda g(u) - h(u) = \lambda(ku^{p-1} + u^q) - u^r$ satisfying $r > q > p-1$ and $k \geqslant 0$. We give a classification of totally *four* bifurcation diagrams. We also show *complete* evolution phenomena of *four* different bifurcation diagrams as evolution parameter k varies from 0 to ∞.

Corollary 2.4. *Let* $p > 2$.

(i) (*See Fig.* 3(a) *with* $0 < \tilde\lambda < \hat\lambda = \infty$.) *Let* $g(u) = u^q$ *and* $h(u) = u^r$ *with* $r > q > p-1$. *Then* $g, h \in C[0,\infty) \cap C^2(0,\infty)$ *satisfy* (H1)–(H5) *with* $\hat\lambda = \infty$.

(ii) *Let* $g(u) = ku^{p-1} + u^q$ *and* $h(u) = u^r$ *with* $r > q > p-1$ *and* $k > 0$. *Then* $g, h \in C[0,\infty) \cap C^2(0,\infty)$ *satisfy* (H1)–(H5) *with* $\hat\lambda = ((p-1)/k)((\pi/p)\csc(\pi/p))^p \in (0,\infty)$. *More precisely, there exists a unique positive number* $k^* = k^*(p,q,r)$ *such that*:
   (a) (*See Fig.* 3(b) *with* $0 < \tilde\lambda < \hat\lambda < \infty$.) *If* $0 < k < k^*$, *then* $0 < \tilde\lambda(k) < \hat\lambda(k) < \infty$.
   (b) (*See Fig.* 3(c) *with* $0 < \hat\lambda = \tilde\lambda < \infty$.) *If* $k = k^*$, *then* $0 < \hat\lambda(k) = \tilde\lambda(k) < \infty$.
   (c) (*See Fig.* 3(d) *with* $0 < \hat\lambda < \tilde\lambda < \infty$.) *If* $k > k^*$, *then* $0 < \hat\lambda(k) < \tilde\lambda(k) < \infty$.

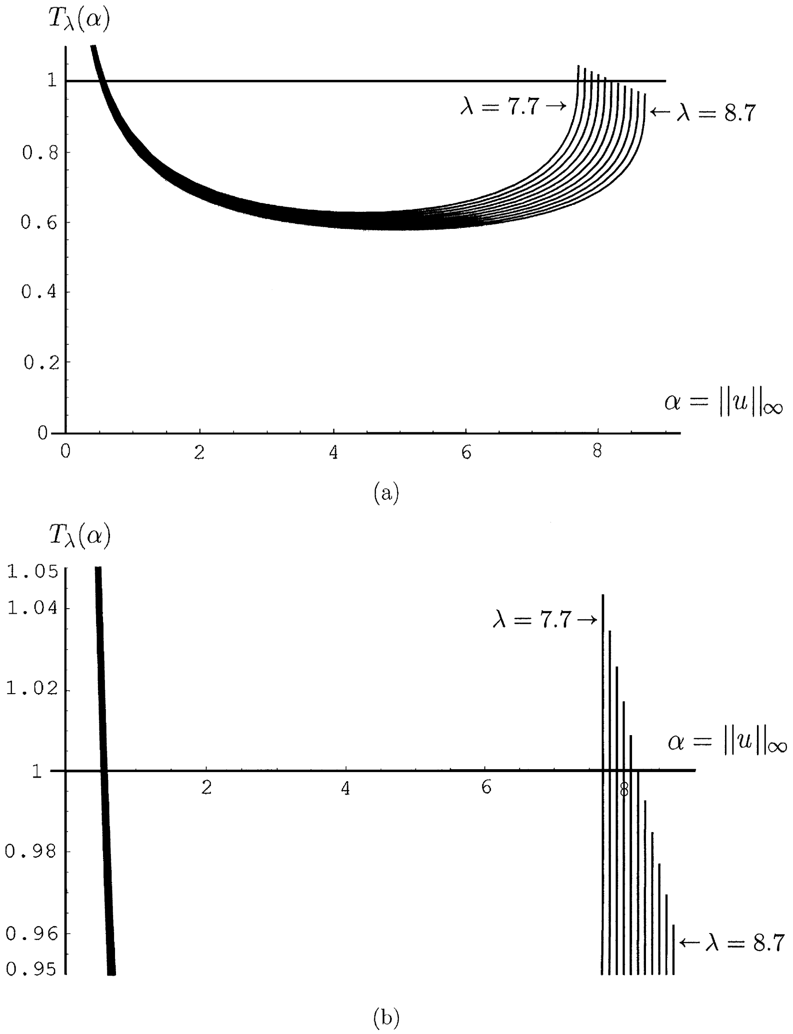

Fig. 4. (a) Numerical simulations of $T_\lambda(\alpha)$ for $\alpha \in (0, \beta_\lambda)$: $p = 3$, $f_\lambda(u) = \lambda u^3 - u^4$, $\lambda = 7.7, 7.8, 7.9, 8.0, 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7$. $\tilde{\lambda} \approx 8.2$, $\beta_{\tilde{\lambda}} \approx 8.2$. Note that, for $0 < \lambda_1 < \lambda_2$, $T_{\lambda_1}(\alpha) > T_{\lambda_2}(\alpha)$ for $\alpha \in (0, \beta_{\lambda_1})$, and $\lim_{\alpha \to 0^+} T_\lambda(\alpha) = \infty$ and $\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) \in (0, \infty)$. (b) A magnified strip region of numerical simulations of $T_\lambda(\alpha)$ for $\alpha \in (0, \beta_\lambda)$ with the range in [0.95, 1.05].

Remark 3. For Corollary 2.4(i), in particular, let $g(u) = u^q$ and $h(u) = u^r$ with $r = 4 > 3 = q = p$, then numerical simulations as given in Fig. 4 show that $\tilde{\lambda} \approx 8.2$, $\beta_{\tilde{\lambda}} \approx 8.2$ and the value $\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha)$ is strictly decreasing in $\lambda = 7.7, 7.8, 7.9, 8.0, 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7$; cf. Fig. 2 and Lemma 3.4 stated below.

Remark 4. In Corollary 2.2(ii) and Corollary 2.4(ii), by (2.1), it is easy to see that $\hat{\lambda} = \hat{\lambda}(k) = ((p-1)/k)((\pi/p)\csc(\pi/p))^p$ is a strictly decreasing and continuous function of k on $(0, \infty)$ and it satisfies $\lim_{k \to 0^+} \hat{\lambda}(k) = \infty$ and $\lim_{k \to \infty} \hat{\lambda}(k) = 0$.

Remark 5. For Corollary 2.4(ii), in particular, let $g(u) = ku^{p-1} + u^q$ and $h(u) = u^r$ with $r = 4 > 3 = q = p$, then numerical simulations show that $k^* \approx 0.47$. In addition, numerical simulations suggest that $\tilde{\lambda}(k)$ is a strictly decreasing function of k on $(0, \infty)$ and it satisfies $\lim_{k \to 0^+} \tilde{\lambda}(k) = \tilde{\lambda}(k = 0) \approx 8.2$ and $\lim_{k \to \infty} \tilde{\lambda}(k) = 0$. Further investigations are needed.

3. Lemmas

To prove Theorems 2.1 and 2.3, we need the following four lemmas. First, we consider the p-Laplacian Dirichlet problem

$$\begin{cases} (\varphi_p(u'(x)))' + \mu f(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{3.1}$$

where $p > 1$. We assume that $f \in C[0, \infty) \cap C^2(0, \infty)$ and there exists a positive number $\tilde{\beta}$ such that f satisfies

$$\begin{cases} f(0)=0, \\ f(u)>0 \quad \text{on } (0,\tilde{\beta}), \\ f(\tilde{\beta})=0, \\ f(u)<0 \quad \text{on } (\tilde{\beta},\infty). \end{cases} \tag{3.2}$$

Let $F(u) \equiv \int_0^u f(t)\,dt$. The time map formula which we apply to study the p-Laplacian problem (3.1) takes the form as follows:

$$\mu^{1/p} = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\alpha} \left[F(\alpha)-F(u)\right]^{-1/p} du \equiv T(\alpha) \quad \text{for } 0<\alpha<\tilde{\beta}, \tag{3.3}$$

see, e.g., [2, Lemmas 2.1 and 2.2] and [10, Lemma 2.4] for the derivation of the time map formula $T(\alpha)$ for (3.1). So positive solutions u of (3.1) satisfying $\|u\|_\infty < \tilde{\beta}$ correspond to $\|u\|_\infty = \alpha$ and $T(\alpha)=\mu^{1/p}$. Thus to study the number of positive solutions of (3.1) is equivalent to study the shape of the time map $T(\alpha)$ on $(0,\tilde{\beta})$.

The next Lemma 3.1 for $1<p\leqslant 2$ and Lemma 3.2 for $p>2$ are of independent interest. In particular, Lemma 3.1 extends [19, Theorem] for (3.1) from $p=2$ to $1<p\leqslant 2$.

Lemma 3.1. *Consider* (3.1) *with* $1<p\leqslant 2$. *Suppose that* $f\in C[0,\infty)\cap C^2(0,\infty)$ *and there exists a positive number* $\tilde{\beta}$ *such that* f *satisfies* (3.2). *Assume that*

$$\tilde{m}_0 \equiv \lim_{u\to 0^+} \frac{f(u)}{u^{p-1}} \in [0,\infty) \quad \text{and} \quad \lim_{u\to\tilde{\beta}^-} \frac{f(u)}{(\tilde{\beta}-u)^{p-1}} \in [0,\infty), \tag{3.4}$$

and there exists a positive number $\tilde{\gamma}<\tilde{\beta}$ *such that*

$$\begin{cases} (p-2)f'(u)-uf''(u)<0 \quad \text{on } (0,\tilde{\gamma}), \\ (p-2)f'(\tilde{\gamma})-\tilde{\gamma}f''(\tilde{\gamma})=0, \\ (p-2)f'(u)-uf''(u)>0 \quad \text{on } (\tilde{\gamma},\tilde{\beta}). \end{cases} \tag{3.5}$$

Then

$$\lim_{\alpha\to 0^+} T(\alpha) = \left(\frac{p-1}{\tilde{m}_0}\right)^{1/p} \frac{\pi}{p} \csc\frac{\pi}{p} \in (0,\infty], \qquad \lim_{\alpha\to\tilde{\beta}^-} T(\alpha)=\infty, \tag{3.6}$$

and $T(\alpha)$ *has exactly one critical point, a minimum, on* $(0,\tilde{\beta})$.

Proof. First, we obtain

$$\lim_{\alpha\to 0^+} T(\alpha) = \left(\frac{p-1}{\tilde{m}_0}\right)^{1/p} \frac{\pi}{p} \csc\frac{\pi}{p} \in (0,\infty]$$

by (3.4) and a slight generalization of [9, Theorems 2.9 and 2.10]. In addition, by (3.4), we obtain

$$\lim_{u\to\tilde{\beta}^-} \frac{f(u)}{(\tilde{\beta}-u)^{p-1}(\ln\frac{1}{\tilde{\beta}-u})^p} = 0.$$

So $\lim_{\alpha\to\tilde{\beta}^-} T(\alpha)=\infty$ by applying [11, Theorem 2.1] and hence (3.6) holds.

Secondly, by (3.3), for $\alpha\in(0,\tilde{\beta})$, we compute that

$$T'(\alpha) = \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha} \int_0^{\alpha} \frac{\Delta\theta}{(\Delta F)^{(p+1)/p}}\,du \tag{3.7}$$

and

$$T''(\alpha) = \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha^2} \int_0^{\alpha} \frac{-\frac{p+1}{p}(\Delta\theta)(\Delta\tilde{f}) + \Delta F(\Delta\tilde{\theta}')}{(\Delta F)^{(2p+1)/p}}\,du, \tag{3.8}$$

where $\theta(u)=pF(u)-uf(u)$, $\Delta F=F(\alpha)-F(u)$, $\Delta\theta=\theta(\alpha)-\theta(u)$, $\Delta\tilde{f}=\alpha f(\alpha)-uf(u)$, and $\Delta\tilde{\theta}'=\alpha\theta'(\alpha)-u\theta'(u)$. By (3.7) and (3.8), for $\alpha\in(0,\tilde{\beta})$, we obtain that

$$T''(\alpha)+\frac{p}{\alpha}T'(\alpha)=\left(\frac{p-1}{p^{p+1}}\right)^{1/p}\frac{1}{\alpha^2}\int_0^{\alpha}\frac{\Delta F(\phi(\alpha)-\phi(u))+\frac{p+1}{p}(\Delta\theta)^2}{(\Delta F)^{(2p+1)/p}}\,du$$
$$\geqslant\left(\frac{p-1}{p^{p+1}}\right)^{1/p}\frac{1}{\alpha^2}\int_0^{\alpha}\frac{\phi(\alpha)-\phi(u)}{(\Delta F)^{(p+1)/p}}\,du, \tag{3.9}$$

where $\phi(u)=u\theta'(u)-\theta(u)$. Since $\theta''(u)=(p-2)f'(u)-uf''(u)$, $\theta(0)=\theta'(0)=0$, $\theta(\tilde\beta)=pF(\tilde\beta)>0$ and by (3.5), there exist two positive numbers C and D with $\tilde\gamma<C<D<\tilde\beta$ such that

$$\begin{cases}\theta'(u)=(p-1)f(u)-uf'(u)<0 & \text{on } (0,C),\\ \theta'(C)=(p-1)f(C)-Cf'(C)=0, & \\ \theta'(u)=(p-1)f(u)-uf'(u)>0 & \text{on } (C,\tilde\beta),\end{cases} \tag{3.10}$$

and

$$\begin{cases}\theta(u)=pF(u)-uf(u)<0 & \text{on } (0,D),\\ \theta(D)=pF(D)-Df(D)=0, & \\ \theta(u)=pF(u)-uf(u)>0 & \text{on } (D,\tilde\beta).\end{cases} \tag{3.11}$$

By (3.7), (3.10) and (3.11), we obtain that

$$T'(\alpha)<0\quad\text{for } \alpha\in(0,C]\quad\text{and}\quad T'(\alpha)>0\quad\text{for } \alpha\in[D,\tilde\beta). \tag{3.12}$$

Hence $T(\alpha)$ has at least one critical point, a local minimum, on (C,D). We then prove that $T(\alpha)$ has exactly one critical point, a minimum, on (C,D).

It is easy to compute that $\phi(C)=-\theta(C)>0$ and $\phi(D)=D\theta'(D)>0$. In addition, since $\phi'(u)=u\theta''(u)=u[(p-2)f'(u)-uf''(u)]$, $\phi(0)=0$ and by (3.5), then $\phi(C)>\phi(u)$ for $u\in[0,C)$ and $\phi(u)$ is strictly increasing on $[C,D]$. So $\phi(\alpha)>\phi(u)$ for $\alpha\in(C,D)$, $u\in(0,\alpha)$. Hence by (3.9),

$$T''(\alpha)+\frac{p}{\alpha}T'(\alpha)>0\quad\text{for } \alpha\in(C,D).$$

Therefore, if α^* is a critical point of $T(\alpha)$ on (C,D), then $T(\alpha^*)$ must be a minimum. Thus $T(\alpha)$ has exactly one critical point, a minimum, on (C,D). By above analysis, $T(\alpha)$ has exactly one critical point, a minimum, on $(0,\tilde\beta)$.

The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Consider* (3.1) *with $p>2$. Suppose that $f\in C[0,\infty)\cap C^2(0,\infty)$ and there exists a positive number $\tilde\beta$ such that f satisfies* (3.2)*. Assume that $\tilde m_0=\lim_{u\to0^+}f(u)/u^{p-1}\in[0,\infty)$ and*

$$\lim_{u\to\tilde\beta^-}\frac{f(u)}{(\tilde\beta-u)^{p-1}(\ln\frac{1}{\tilde\beta-u})^a}\in(0,\infty] \tag{3.13}$$

for some positive number $a>p$, and there exists a positive number $\tilde\gamma<\tilde\beta$ such that (3.5) *holds. Then*

$$\lim_{\alpha\to0^+}T(\alpha)=\left(\frac{p-1}{\tilde m_0}\right)^{1/p}\frac{\pi}{p}\csc\frac{\pi}{p}\in(0,\infty],\qquad \lim_{\alpha\to\tilde\beta^-}T(\alpha)\in(0,\infty),$$

and $T(\alpha)$ has exactly one critical point, a minimum, on $(0,\tilde\beta)$.

Proof. The result that $\lim_{\alpha\to\tilde\beta^-}T(\alpha)\in(0,\infty)$ follows by (3.13) and by applying [11, Theorem 2.2]. The rest results of Lemma 3.2 follow by the same arguments as those in the proof of Lemma 3.1. □

Remark 6. If $p>2$, then it is easy to check that the condition $-\infty<f'(\tilde\beta)<0$ implies (3.13) by applying L'Hopital's rule.

Define

$$F_\lambda(u)=\int_0^u f_\lambda(t)\,dt,$$

and

$$\theta_{f_\lambda}(u) = pF_\lambda(u) - uf_\lambda(u), \qquad \theta_g(u) = p\int_0^u g(t)\,dt - ug(u), \qquad \theta_h(u) = p\int_0^u h(t)\,dt - uh(u).$$

Then

$$\theta'_{f_\lambda}(u) = (p-1)f_\lambda(u) - uf'_\lambda(u), \qquad \theta'_g(u) = (p-1)g(u) - ug'(u), \qquad \theta'_h(u) = (p-1)h(u) - uh'(u)$$

and

$$\theta''_{f_\lambda}(u) = (p-2)f'_\lambda(u) - uf''_\lambda(u), \qquad \theta''_g(u) = (p-2)g'(u) - ug''(u), \qquad \theta''_h(u) = (p-2)h'(u) - uh''(u).$$

Next, we consider (1.1) where $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1) and (H2). We define

$$T_\lambda(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[F_\lambda(\alpha) - F_\lambda(u)\right]^{-1/p} du \quad \text{for } 0 < \alpha < \beta_\lambda. \tag{3.14}$$

In the next lemma we study some properties of the time map $T_\lambda(\alpha)$.

Lemma 3.3. *Let $p > 1$. Consider (1.1) where $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1) and (H2). Then, for each positive number λ_0 and fixed $\alpha \in (0, \beta_{\lambda_0})$, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geqslant \lambda_0$ and $\lim_{\lambda\to\infty} T_\lambda(\alpha) = 0$.*

The proof of Lemma 3.3 is easy but tedious; we omit it. Cf. [20, Lemma 3.2].

The main difference between the bifurcation diagrams in Theorem 2.1 ($1 < p \leqslant 2$) and Theorem 2.3 ($p > 2$) is due to the next key lemma by applying hypothesis (H5). If $1 < p \leqslant 2$, then $\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha) = \infty$ for any $\lambda \in (0,\infty)$; see (4.2) stated below. But if $p > 2$, then $\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha)$ exists and is positive for any $\lambda \in (0,\infty)$; see (4.5) stated below. In the next lemma we study some properties of $\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha)$ for (1.1) and $p > 2$.

Lemma 3.4. *Let $p > 2$. Consider (1.1) where $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1)–(H5). Then $\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha)$ is a strictly decreasing and continuous function of λ on $(0,\infty)$ and $\lim_{\lambda\to\infty}\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha) = 0$.*

Proof. First, we prove that $\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha)$ is a strictly decreasing function of λ on $(0,\infty)$. For $\lambda \in (0,\infty)$, by (3.14), we obtain that

$$\begin{aligned}
\lim_{\alpha\to\beta_\lambda^-} T_\lambda(\alpha) &= \lim_{\alpha\to\beta_\lambda^-} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha f_\lambda(t)\,dt\right]^{-1/p} du \\
&= \lim_{\alpha\to\beta_\lambda^-} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha \frac{h(\beta_\lambda)}{g(\beta_\lambda)} g(t) - h(t)\,dt\right]^{-1/p} du \\
&\qquad \big(\text{since } f_\lambda(u) = \lambda g(u) - h(u) \text{ and by (1.4)}\big) \\
&= \lim_{\alpha\to\beta_\lambda^-} \left(\frac{p-1}{p}\right)^{1/p} \alpha^{(p-1)/p} \int_0^1 \left[\int_v^1 \frac{h(\beta_\lambda)}{g(\beta_\lambda)} g(s\alpha) - h(s\alpha)\,ds\right]^{-1/p} dv \\
&\qquad (\text{let } u = \alpha v \text{ and } t = \alpha s) \\
&= \left(\frac{p-1}{p}\right)^{1/p} \beta_\lambda^{(p-1-p^*)/p} \lim_{\alpha\to\beta_\lambda^-} \int_0^1 \left[\int_v^1 s^{p^*}\left(\frac{h(\beta_\lambda)g(s\alpha)}{g(\beta_\lambda)(s\alpha)^{p^*}} - \frac{h(s\alpha)}{(s\alpha)^{p^*}}\right) ds\right]^{-1/p} dv \\
&= \left(\frac{p-1}{p}\right)^{1/p} \beta_\lambda^{(p-1-p^*)/p} \int_0^1 \left[\int_v^1 \lim_{\alpha\to\beta_\lambda^-} s^{p^*}\left(\frac{h(\beta_\lambda)g(s\alpha)}{g(\beta_\lambda)(s\alpha)^{p^*}} - \frac{h(s\alpha)}{(s\alpha)^{p^*}}\right) ds\right]^{-1/p} dv \\
&\qquad \big(\text{by (H5) and the Monotone Convergence Theorem [21, p. 75]}\big) \\
&= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_\lambda)}{\beta_\lambda^{p-1}} \left(\frac{h(\beta_\lambda)g(s\beta_\lambda)}{g(\beta_\lambda)h(s\beta_\lambda)} - 1\right) ds\right]^{-1/p} dv.
\end{aligned} \tag{3.15}$$

By (H5) and since β_λ is strictly increasing in $\lambda \in (0, \infty)$, we obtain that $\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha)$ is a strictly decreasing function of λ on $(0, \infty)$.

Secondly, we prove that $\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha)$ is a continuous function of λ on $(0, \infty)$. For any number $\lambda_0 \in (0, \infty)$, by (3.15), we obtain that

$$\begin{aligned}
\lim_{\lambda \to \lambda_0} \lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) &= \lim_{\lambda \to \lambda_0} \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_\lambda)}{\beta_\lambda^{p-1}} \left(\frac{h(\beta_\lambda) g(s\beta_\lambda)}{g(\beta_\lambda) h(s\beta_\lambda)} - 1\right) ds\right]^{-1/p} dv \\
&= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \lim_{\lambda \to \lambda_0} \frac{h(s\beta_\lambda)}{\beta_\lambda^{p-1}} \left(\frac{h(\beta_\lambda) g(s\beta_\lambda)}{g(\beta_\lambda) h(s\beta_\lambda)} - 1\right) ds\right]^{-1/p} dv \\
&\qquad \bigl(\text{by (H5) and the Monotone Convergence Theorem [21, p. 75]}\bigr) \\
&= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_{\lambda_0})}{\beta_{\lambda_0}^{p-1}} \left(\frac{h(\beta_{\lambda_0}) g(s\beta_{\lambda_0})}{g(\beta_{\lambda_0}) h(s\beta_{\lambda_0})} - 1\right) ds\right]^{-1/p} dv \\
&= \lim_{\alpha \to \beta_{\lambda_0}^-} T_{\lambda_0}(\alpha).
\end{aligned}$$

So we obtain that $\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha)$ is a continuous function of λ on $(0, \infty)$.

Finally, we prove $\lim_{\lambda \to \infty} \lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) = 0$. By (3.15), we obtain that

$$\begin{aligned}
\lim_{\lambda \to \infty} \lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) &= \lim_{\lambda \to \infty} \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_\lambda)}{\beta_\lambda^{p-1}} \left(\frac{h(\beta_\lambda) g(s\beta_\lambda)}{g(\beta_\lambda) h(s\beta_\lambda)} - 1\right) ds\right]^{-1/p} dv \\
&= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \lim_{\lambda \to \infty} \frac{h(s\beta_\lambda)}{\beta_\lambda^{p-1}} \left(\frac{h(\beta_\lambda) g(s\beta_\lambda)}{g(\beta_\lambda) h(s\beta_\lambda)} - 1\right) ds\right]^{-1/p} dv
\end{aligned}$$

by (H5) and the Monotone Convergence Theorem [21, p. 75].

By (H5), we obtain $\lim_{u \to \infty} h(u)/u^{p^*} \in (0, \infty]$ and hence

$$\lim_{u \to \infty} \frac{h(u)}{u^{p-1}} = \lim_{u \to \infty} \frac{h(u)}{u^{p^*}} u^{p^*-p+1} = \infty.$$

Thus, for each fixed $s \in (0, 1)$,

$$\lim_{\lambda \to \infty} \frac{h(s\beta_\lambda)}{\beta_\lambda^{p-1}} \left(\frac{h(\beta_\lambda) g(s\beta_\lambda)}{g(\beta_\lambda) h(s\beta_\lambda)} - 1\right) = s^{p-1} \lim_{u \to \infty} \frac{h(su)}{(su)^{p-1}} \left(\frac{h(u) g(su)}{g(u) h(su)} - 1\right) = \infty$$

by (1.5) and (H5). So $\lim_{\lambda \to \infty} \lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) = 0$.

The proof of Lemma 3.4 is complete. □

4. Proofs of main results

For each fixed $\lambda > 0$, let $u_\lambda(x)$ be a positive solution of (1.1) with $\|u_\lambda\|_\infty = \alpha < \beta_\lambda$. We write

$$f_\lambda(u) = \lambda g(u) - h(u) = \lambda\left[g(u) - \frac{1}{\lambda} h(u)\right],$$

and recall $F_\lambda(u) = \int_0^u f_\lambda(t)\, dt$. Then, by (3.3), it is easy to see that

$$\begin{aligned}
\lambda^{1/p} &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha g(s) - \frac{1}{\lambda} h(s)\, ds\right]^{-1/p} du \\
&= \lambda^{1/p} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha \lambda g(s) - h(s)\, ds\right]^{-1/p} du \\
&= \lambda^{1/p} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \bigl[F_\lambda(\alpha) - F_\lambda(u)\bigr]^{-1/p} du.
\end{aligned}$$

This and (3.14) imply that positive solution $u_\lambda(x)$ of (1.1) satisfying $\|u_\lambda\|_\infty < \beta_\lambda$ corresponds to $\|u_\lambda\|_\infty = \alpha$ and

$$T_\lambda(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[F_\lambda(\alpha) - F_\lambda(u)\right]^{-1/p} du = 1.$$

We first prove Theorems 2.1 and 2.3, then prove Corollaries 2.2 and 2.4.

Proof of Theorem 2.1. Consider (1.1) with $1 < p \leqslant 2$. Suppose that $f_\lambda(u) = \lambda g(u) - h(u)$ and g, h satisfy (H1)–(H4). For each fixed $\lambda > 0$, $f_\lambda(0) = \lambda g(0) - h(0) = 0$. In addition, there exist two positive numbers $\gamma_\lambda < \beta_\lambda$ such that (1.4) and (1.6) hold. First, we study the shape and asymptotic behaviors of the time map $T_\lambda(\alpha)$ on $(0, \beta_\lambda)$.

For each fixed $\lambda > 0$, we obtain that

$$\lim_{u \to 0^+} \frac{f_\lambda(u)}{u^{p-1}} = \lambda \lim_{u \to 0^+} \frac{g(u)}{u^{p-1}} = \lambda m_0^g \in [0, \infty)$$

by (1.2), and

$$\begin{aligned} \lim_{u \to \beta_\lambda^-} \frac{f_\lambda(u)}{(\beta_\lambda - u)^{p-1}} &= \lim_{u \to \beta_\lambda^-} \frac{f_\lambda(u)}{\beta_\lambda - u} (\beta_\lambda - u)^{2-p} \\ &= -f_\lambda'(\beta_\lambda) \lim_{u \to \beta_\lambda^-} (\beta_\lambda - u)^{2-p} \\ &= \begin{cases} 0 & \text{if } 1 < p < 2, \\ -f_\lambda'(\beta_\lambda) \in (0, \infty) & \text{if } p = 2, \end{cases} \end{aligned}$$

since

$$\begin{aligned} f_\lambda'(\beta_\lambda) &= \lambda g'(\beta_\lambda) - h'(\beta_\lambda) \\ &= \frac{h(\beta_\lambda)}{g(\beta_\lambda)} g'(\beta_\lambda) - h'(\beta_\lambda) \quad \bigl(\text{since } \lambda g(\beta_\lambda) - h(\beta_\lambda) = 0 \text{ by } (1.4)\bigr) \\ &= \frac{-1}{g(\beta_\lambda)} \left[g(\beta_\lambda) h'(\beta_\lambda) - g'(\beta_\lambda) h(\beta_\lambda)\right] < 0 \end{aligned} \tag{4.1}$$

by (H1) and (H2). Thus for each fixed $\lambda > 0$, taking $f = f_\lambda$, $\tilde{\beta} = \beta_\lambda$, $\tilde{\gamma} = \gamma_\lambda$, we obtain that $f = f_\lambda(u)$ satisfies all assumptions of Lemma 3.1. So by Lemma 3.1, we obtain that:

(1) For each fixed $\lambda > 0$, $T_\lambda(\alpha)$ has exactly one critical point, a minimum, on $(0, \beta_\lambda)$. In addition, the number

$$\lim_{\alpha \to 0^+} T_\lambda(\alpha) = \left(\frac{p-1}{\lambda m_0^g}\right)^{1/p} \frac{\pi}{p} \csc \frac{\pi}{p} \begin{cases} = \infty & \text{if } m_0^g = 0, \\ < \infty & \text{if } 0 < m_0^g < \infty \end{cases}$$

(note that $\lim_{\alpha \to 0^+} T_{\hat{\lambda}}(\alpha) = 1$ by (2.1)) and it is strictly decreasing in $\lambda > 0$ if $0 < m_0^g < \infty$. Also

$$\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) = \infty \tag{4.2}$$

for $1 < p \leqslant 2$.

Secondly, for $T_\lambda(\alpha)$ with $\lambda > 0$, we have the following properties:

(2) For $0 < \lambda_1 < \lambda_2$, we obtain $(0 <)\ \beta_{\lambda_1} < \beta_{\lambda_2}$. (Note that $\lim_{\lambda \to 0^+} \beta_\lambda = 0$ and $\lim_{\lambda \to \infty} \beta_\lambda = \infty$.) In addition, $T_{\lambda_1}(\alpha) > T_{\lambda_2}(\alpha)$ for $\alpha \in (0, \beta_{\lambda_1})$ by (3.14) and modifying a comparison theorem of [9, Theorem 2.3] since $f_{\lambda_1}(u) = \lambda_1 g(u) - h(u) < \lambda_2 g(u) - h(u) = f_{\lambda_2}(u)$ for $u \in (0, \beta_{\lambda_1})$.

(3) For each positive number λ_0 and fixed $\alpha \in (0, \beta_{\lambda_0})$, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geqslant \lambda_0$ and $\lim_{\lambda \to \infty} T_\lambda(\alpha) = 0$ by Lemma 3.3.

Define

$$m(\lambda) = \min_{\alpha \in (0, \beta_\lambda)} T_\lambda(\alpha).$$

(Note that $m(\lambda)$ exists for any $\lambda > 0$ by property (1).) By property (2), $m(\lambda)$ is strictly decreasing in $\lambda > 0$.

We then prove that there exist two positive numbers $\lambda_3 < \lambda_4$ such that $m(\lambda_4) < 1 < m(\lambda_3)$ as follows.

We take positive number $\lambda_5 = 1$. It is easy to prove that the number $\sup_{u \in (0, \beta_{\lambda_5})} g(u)/u^{p-1} \in (0, \infty)$ by (H1). Let

$$\lambda_3 \equiv \min\left\{\lambda_5, (p-1)\left(\frac{\pi}{p} \csc \frac{\pi}{p}\right)^p \left(\sup_{u \in (0, \beta_{\lambda_5})} \frac{g(u)}{u^{p-1}}\right)^{-1}\right\}.$$

Then for $0 < u < \beta_{\lambda_3}$ $(\leqslant \beta_{\lambda_5})$,

$$\begin{aligned}
f_{\lambda_3}(u) &= \lambda_3 g(u) - h(u) \\
&< \lambda_3 g(u) \\
&\leqslant \lambda_3 \left(\sup_{u \in (0,\beta_{\lambda_5})} \frac{g(u)}{u^{p-1}} \right) u^{p-1} \\
&\leqslant (p-1)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \left(\sup_{u \in (0,\beta_{\lambda_5})} \frac{g(u)}{u^{p-1}} \right)^{-1} \left(\sup_{u \in (0,\beta_{\lambda_5})} \frac{g(u)}{u^{p-1}} \right) u^{p-1} \\
&= (p-1)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p u^{p-1}.
\end{aligned}$$

So for $0 < \alpha < \beta_{\lambda_3}$, by modifying a comparison theorem of [9, Theorem 2.3], we obtain that

$$\begin{aligned}
T_{\lambda_3}(\alpha) &> \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha (p-1)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p t^{p-1}\,dt \right]^{-1/p} du \\
&= \left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^{-1} \int_0^\alpha \left(\alpha^p - u^p\right)^{-1/p} du \\
&= \left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^{-1} \int_0^1 \left(1 - w^p\right)^{-1/p} dw \quad (\text{let } u = \alpha w) \\
&= \left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^{-1} \left(\frac{\pi}{p}\csc\frac{\pi}{p}\right) \\
&= 1.
\end{aligned}$$

Thus $m(\lambda_3) = \min_{\alpha \in (0,\beta_{\lambda_3})} T_{\lambda_3}(\alpha) > 1$. On the other hand, property (3) simply implies that there exists a number $\lambda_4 > \lambda_3$ such that $m(\lambda_4) < 1$.

Next, we prove that $m(\lambda)$ is continuous on $[\lambda_3, \lambda_4]$. For each fixed number $\lambda_6 \in [\lambda_3, \lambda_4]$, we can choose positive numbers $\tilde{C}$ and $\tilde{D}$ satisfying $\tilde{C} < C_{\lambda_6} < D_{\lambda_6} < \tilde{D} < \beta_{\lambda_6}$, where C_{λ_6} and D_{λ_6} are defined in (3.10) and (3.11) with $f = f_{\lambda_6}$ and $\tilde{\beta} = \beta_{\lambda_6}$, respectively. So

$$\theta_{f_{\lambda_6}}(\tilde{D}) = pF_{\lambda_6}(\tilde{D}) - \tilde{D} f_{\lambda_6}(\tilde{D}) > 0$$

and

$$\theta'_{f_{\lambda_6}}(\tilde{C}) = (p-1) f_{\lambda_6}(\tilde{C}) - \tilde{C} f'_{\lambda_6}(\tilde{C}) < 0.$$

By the continuity of functions θ_{f_λ}, θ'_{f_λ} and β_λ in $\lambda > 0$, there exists a number $\delta > 0$ such that $\beta_\lambda > \tilde{D}$,

$$\theta_{f_\lambda}(\tilde{D}) = pF_\lambda(\tilde{D}) - \tilde{D} f_\lambda(\tilde{D}) > 0$$

and

$$\theta'_{f_\lambda}(\tilde{C}) = (p-1) f_\lambda(\tilde{C}) - \tilde{C} f'_\lambda(\tilde{C}) < 0$$

for $\lambda \in [\lambda_6 - \delta, \lambda_6 + \delta]$. This implies that $\tilde{C} < C_\lambda < D_\lambda < \tilde{D} < \beta_\lambda$ for $\lambda \in [\lambda_6 - \delta, \lambda_6 + \delta]$, where C_λ and D_λ are defined in (3.10) and (3.11) with $f = f_\lambda$ and $\tilde{\beta} = \beta_\lambda$, respectively. Thus

$$m(\lambda) = \min_{\alpha \in (0,\beta_\lambda)} T_\lambda(\alpha) = \min_{\alpha \in [\tilde{C},\tilde{D}]} T_\lambda(\alpha) \quad \text{for } \lambda \in [\lambda_6 - \delta, \lambda_6 + \delta] \tag{4.3}$$

by (3.12). By property (2) and the Dini Theorem [14, p. 195], it is easy to see that

$$\lim_{\lambda \to \lambda_6} \left(\min_{\alpha \in [\tilde{C},\tilde{D}]} T_\lambda(\alpha) \right) = \min_{\alpha \in [\tilde{C},\tilde{D}]} T_{\lambda_6}(\alpha). \tag{4.4}$$

By (4.3) and (4.4), $\lim_{\lambda \to \lambda_6} m(\lambda) = m(\lambda_6)$. Hence $m(\lambda)$ is continuous on $[\lambda_3, \lambda_4]$.

By above and the Intermediate Value Theorem, there exists a positive number $\lambda^* \in (\lambda_3, \lambda_4)$ such that $m(\lambda^*) = 1$. Since $m(\lambda)$ is strictly decreasing in $\lambda > 0$, λ^* is unique. So we obtain that:

(4) There exists a unique positive number $\lambda^* < \hat{\lambda}$ $(\leqslant \infty)$ such that

$$m(\lambda^*) = \min_{\alpha \in (0, \beta_{\lambda^*})} T_{\lambda^*}(\alpha) = 1$$

by property (1).

So by above, we obtain immediately the exact multiplicity result and ordering results of the solutions in parts (i)–(ii). (Note that the ordering result $u_\lambda < v_\lambda$ can be proved easily.) The proof of part (iii) is easy but tedious; we omit it.

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.3. Consider (1.1) with $p > 2$. Suppose that $f_\lambda(u) = \lambda g(u) - h(u)$ and g, h satisfy (H1)–(H5). For each fixed $\lambda > 0$, $f_\lambda(0) = \lambda g(0) - h(0) = 0$. In addition, there exist two positive numbers $\gamma_\lambda < \beta_\lambda$ such that (1.4) and (1.6) hold. By (4.1) and Remark 6, we obtain that $f = f_\lambda(u) = \lambda g(u) - h(u)$ satisfies (3.13) with $\tilde{\beta} = \beta_\lambda$. In addition, by similar arguments used in the proof of Theorem 2.1, we obtain that $f = f_\lambda(u)$ satisfies all assumptions of Lemma 3.2 with $\tilde{\beta} = \beta_\lambda$ and $\tilde{\gamma} = \gamma_\lambda$. So by Lemma 3.2 and by the same arguments used to prove Theorem 2.1, we obtain that:

(1) For each fixed $\lambda > 0$, $T_\lambda(\alpha)$ has exactly one critical point, a minimum, on $(0, \beta_\lambda)$. In addition, the number

$$\lim_{\alpha \to 0^+} T_\lambda(\alpha) = \left(\frac{p-1}{\lambda m_0^g}\right)^{1/p} \frac{\pi}{p} \csc \frac{\pi}{p} \begin{cases} = \infty & \text{if } m_0^g = 0, \\ < \infty & \text{if } 0 < m_0^g < \infty \end{cases}$$

(note that $\lim_{\alpha \to 0^+} T_{\hat{\lambda}}(\alpha) = 1$) and it is strictly decreasing in $\lambda > 0$ if $0 < m_0^g < \infty$. Also

$$\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) \in (0, \infty) \tag{4.5}$$

for $p > 2$.

(2) For $0 < \lambda_1 < \lambda_2$, we obtain $(0 <)$ $\beta_{\lambda_1} < \beta_{\lambda_2}$ (note that $\lim_{\lambda \to 0^+} \beta_\lambda = 0$ and $\lim_{\lambda \to \infty} \beta_\lambda = \infty$) and $T_{\lambda_1}(\alpha) > T_{\lambda_2}(\alpha)$ for $\alpha \in (0, \beta_{\lambda_1})$.

(3) For each positive number λ_0 and fixed $\alpha \in (0, \beta_{\lambda_0})$, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geqslant \lambda_0$ and $\lim_{\lambda \to \infty} T_\lambda(\alpha) = 0$.

(4) There exists a unique positive number $\lambda^* < \hat{\lambda}$ $(\leqslant \infty)$ such that

$$m(\lambda^*) = \min_{\alpha \in (0, \beta_{\lambda^*})} T_{\lambda^*}(\alpha) = 1.$$

Also, we obtain that:

(5) $\lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha)$ is a strictly decreasing and continuous function of λ on $(0, \infty)$ and $\lim_{\lambda \to \infty} \lim_{\alpha \to \beta_\lambda^-} T_\lambda(\alpha) = 0$ by Lemma 3.4.

By properties (1) and (4), we obtain $\lim_{\alpha \to \beta_{\lambda^*}^-} T_{\lambda^*}(\alpha) \in (1, \infty)$. So by $\lim_{\alpha \to \beta_{\lambda^*}^-} T_{\lambda^*}(\alpha) \in (1, \infty)$, property (5) and the Intermediate Value Theorem, we obtain that:

(6) There exists a unique positive number $\tilde{\lambda} > \lambda^*$ such that $\lim_{\alpha \to \beta_{\tilde{\lambda}}^-} T_{\tilde{\lambda}}(\alpha) = 1$.

So by above, we obtain immediately the exact multiplicity result in parts (i)–(ii) and ordering results of the solutions in parts (iii)–(iv); see, e.g., Figs. 5 and 6. (Note that the ordering result $u_\lambda < v_\lambda$ can be proved easily.) The proof of part (v) is easy but tedious; we omit it.

The proof of Theorem 2.3 is complete. □

Proof of Corollary 2.2. For part (i) where $g(u) = u^q$ and $h(u) = u^r$ with $1 < p \leqslant 2$ and $r > q > p - 1$, it is easy to see that $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy (H1) with $m_0^g = 0$, and hence $\hat{\lambda} = \infty$. We then compute that $h(u)/g(u) = u^{r-q}$ which is positive and strictly increasing on $(0, \infty)$ and satisfies

$$\lim_{u \to 0^+} \frac{h(u)}{g(u)} = 0, \qquad \lim_{u \to \infty} \frac{h(u)}{g(u)} = \infty.$$

Thus g, h satisfy (H2). It is clear that $(p-2)g'(u) - ug''(u) = (p-1-q)qu^{q-1} < 0$ on $(0, \infty)$ and $(p-2)h'(u) - uh''(u) = (p-1-r)ru^{r-1} < 0$ on $(0, \infty)$. Thus g, h satisfy (H3). Finally, we compute that

$$\frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \frac{(r-p+1)r}{(q-p+1)q} u^{r-q}$$

which is positive and strictly increasing on $(0, \infty)$ and satisfies

$$\lim_{u \to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0, \qquad \lim_{u \to \infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

So g, h satisfy (H4). We conclude that g, h satisfy (H1)–(H4). So part (i) follows.

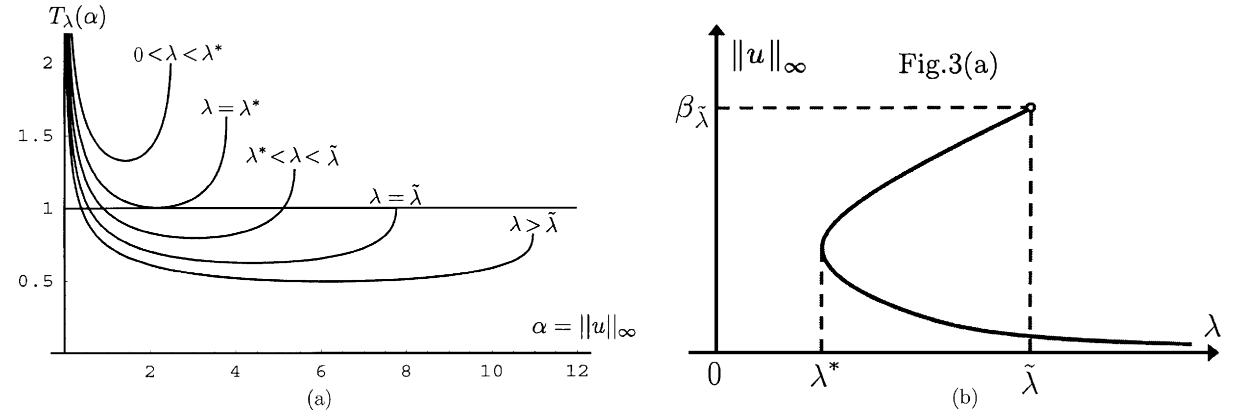

Fig. 5. (a) Graphs of $T_\lambda(\alpha)$ for $\alpha \in (0, \beta_\lambda)$ with varying $\lambda > 0$ in the case $\lim_{\alpha\to 0^+} T_{\tilde{\lambda}}(\alpha) = \infty$. (b) The corresponding bifurcation diagram of (1.1) with $p > 2$.

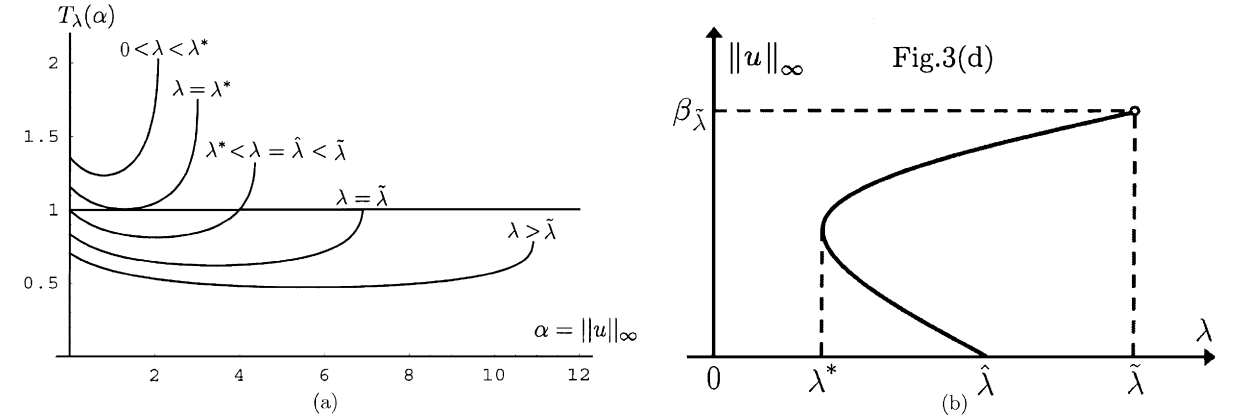

Fig. 6. (a) Graphs of $T_\lambda(\alpha)$ for $\alpha \in (0, \beta_\lambda)$ with varying $\lambda > 0$ in the case $0 < \lim_{\alpha\to 0^+} T_{\tilde{\lambda}}(\alpha) < 1$. (b) The corresponding bifurcation diagram of (1.1) with $p > 2$.

For part (ii) where $g(u) = ku^{p-1} + u^q$ and $h(u) = u^r$ with $1 < p \leqslant 2$, $r > q > p-1$ and $k > 0$, it is easy to see that $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1) with $m_0^g = k$, and hence $\hat{\lambda} = ((p-1)/k)((\pi/p)\csc(\pi/p))^p \in (0,\infty)$. We then compute that $h(u)/g(u) = u^{r-q}/(ku^{p-1-q}+1)$ which is positive and strictly increasing on $(0,\infty)$ and satisfies

$$\lim_{u\to 0^+} \frac{h(u)}{g(u)} = 0, \qquad \lim_{u\to\infty} \frac{h(u)}{g(u)} = \infty.$$

Thus g, h satisfy (H2). It is clear that $(p-2)g'(u) - ug''(u) = (p-1-q)qu^{q-1} < 0$ on $(0,\infty)$ and $(p-2)h'(u) - uh''(u) = (p-1-r)ru^{r-1} < 0$ on $(0,\infty)$. Thus g, h satisfy (H3). Finally, we compute that

$$\frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \frac{(r-p+1)r}{(q-p+1)q} u^{r-q}$$

which is positive and strictly increasing on $(0,\infty)$ and satisfies

$$\lim_{u\to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0, \qquad \lim_{u\to\infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

So g, h satisfy (H4). We conclude that g, h satisfy (H1)–(H4). So part (ii) follows.

The proof of Corollary 2.2 is complete. □

Proof of Corollary 2.4. For part (i) where $g(u) = u^q$ and $h(u) = u^r$ with $p > 2$ and $r > q > p-1$, it was proved that $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1)–(H4) with $m_0^g = 0$, and hence $\hat{\lambda} = \infty$ in the proof of Corollary 2.2(i). So, to complete the proof of part (i), it suffices to prove that g, h satisfy (H5). Taking $p^* = (r+q)/2 \in (q, r)$, we compute that $g(u)/u^{p^*} = u^{q-p^*}$ which is strictly decreasing on $(0,\infty)$ and $h(u)/u^{p^*} = u^{r-p^*}$ which is strictly increasing on $(0,\infty)$. In addition, for each fixed $s \in (0,1)$, we compute that $h(u)g(su)/(g(u)h(su)) = s^{q-r} \in (1,\infty)$ which is a positive constant function of u on $(0,\infty)$ and hence (1.3) holds. Also,

$$\frac{h(su)}{u^{p-1}}\left[\frac{h(u)g(su)}{g(u)h(su)}-1\right]=\left(s^q-s^r\right)u^{r-p+1}$$

is a strictly increasing function of u on $(0,\infty)$. So g,h satisfy (H5). We conclude that g,h satisfy (H1)–(H5). So part (i) follows.

For part (ii) where $g(u)=ku^{p-1}+u^q$ and $h(u)=u^r$ with $p>2$, $r>q>p-1$ and $k>0$, it was proved that $g,h\in C[0,\infty)\cap C^2(0,\infty)$ satisfy (H1)–(H4) with $m_0^g=k$, and hence $\hat{\lambda}=((p-1)/k)((\pi/p)\csc(\pi/p))^p\in(0,\infty)$ in the proof of Corollary 2.2(ii). We then prove that g,h satisfy (H5). Taking $p^*=(r+q)/2\in(q,r)$, we compute that $g(u)/u^{p^*}=ku^{p-1-p^*}+u^{q-p^*}$ which is strictly decreasing on $(0,\infty)$ and $h(u)/u^{p^*}=u^{r-p^*}$ which is strictly increasing on $(0,\infty)$. In addition, for each fixed $s\in(0,1)$, we compute and see that

$$\frac{h(su)}{u^{p-1}}\left[\frac{h(u)g(su)}{g(u)h(su)}-1\right]=\frac{k(s^{p-1}-s^r)u^{r-q}+(s^q-s^r)u^{r-p+1}}{1+ku^{p-1-q}} \tag{4.6}$$

is a strictly increasing function of u on $(0,\infty)$, and

$$\lim_{u\to\infty}\frac{h(u)g(su)}{g(u)h(su)}=\lim_{u\to\infty}\frac{ks^{p-1}u^{p-1}+s^qu^q}{s^r(ku^{p-1}+u^q)}=s^{q-r}\in(1,\infty).$$

So g,h satisfy (H5). We conclude that g,h satisfy (H1)–(H5).

Finally, we prove that there exists a unique positive number $k^*=k^*(p,q,r)$ such that:

(a) If $0<k<k^*$, then $0<\tilde{\lambda}(k)<\hat{\lambda}(k)<\infty$.
(b) If $k=k^*$, then $0<\hat{\lambda}(k)=\tilde{\lambda}(k)<\infty$.
(c) If $k>k^*$, then $0<\hat{\lambda}(k)<\tilde{\lambda}(k)<\infty$.

For each fixed $k>0$, we denote that $f_\lambda=f_{\lambda,k}$, $\beta_\lambda=\beta_{\lambda,k}$, $T_\lambda=T_{\lambda,k}$, $\hat{\lambda}=\hat{\lambda}(k)$, and $\tilde{\lambda}=\tilde{\lambda}(k)$. By Remark 4, we obtain that

$$\hat{\lambda}(k)=\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p.$$

We define $C_p=(p-1)(\frac{\pi}{p}\csc\frac{\pi}{p})^p$ and we obtain that

$$f_{\hat{\lambda}(k),k}(u)=C_pu^{p-1}+\frac{C_p}{k}u^q-u^r$$

for $u\in(0,\infty)$. Thus, for each fixed $u\in(0,\infty)$, $f_{\hat{\lambda}(k),k}(u)$ is strictly decreasing in $k\in(0,\infty)$. So by (1.4), $\beta_{\hat{\lambda}(k),k}$ is a strictly decreasing function of k on $(0,\infty)$. In addition,

$$\lim_{k\to0^+}\beta_{\hat{\lambda}(k),k}=\infty\quad\text{and}\quad\lim_{k\to\infty}\beta_{\hat{\lambda}(k),k}=(C_p)^{1/(r-p+1)}.$$

For each fixed $s\in(0,1)$, by (4.6), we obtain that

$$\begin{aligned}\frac{h(s\beta_{\hat{\lambda}(k),k})}{\beta^{p-1}_{\hat{\lambda}(k),k}}\left[\frac{h(\beta_{\hat{\lambda}(k),k})g(s\beta_{\hat{\lambda}(k),k})}{g(\beta_{\hat{\lambda}(k),k})h(s\beta_{\hat{\lambda}(k),k})}-1\right]&=\frac{k(s^{p-1}-s^r)\beta^{r-q}_{\hat{\lambda}(k),k}+(s^q-s^r)\beta^{r-p+1}_{\hat{\lambda}(k),k}}{1+k\beta^{p-1-q}_{\hat{\lambda}(k),k}}\\&=\frac{k(s^{p-1}-s^r)\beta^{r}_{\hat{\lambda}(k),k}+(s^q-s^r)\beta^{r+q-p+1}_{\hat{\lambda}(k),k}}{k\beta^{p-1}_{\hat{\lambda}(k),k}+\beta^{q}_{\hat{\lambda}(k),k}}\\&=\frac{k(s^{p-1}-s^r)\beta^{r}_{\hat{\lambda}(k),k}+(s^q-s^r)\beta^{r+q-p+1}_{\hat{\lambda}(k),k}}{\frac{k}{C_p}\beta^{r}_{\hat{\lambda}(k),k}}\quad\big(\text{by (1.4)}\big)\\&=C_p\left[\left(s^{p-1}-s^r\right)+\frac{(s^q-s^r)}{k}\beta^{q-p+1}_{\hat{\lambda}(k),k}\right]\end{aligned}$$

is a strictly decreasing function of k on $(0,\infty)$. So by (3.15), $\lim_{\alpha\to\beta^-_{\hat{\lambda}(k),k}}T_{\hat{\lambda}(k),k}(\alpha)$ is a strictly increasing function of k on $(0,\infty)$. In addition, for any number $k_0\in(0,\infty)$, by (3.15), we obtain that

$$\lim_{k\to k_0}\lim_{\alpha\to\beta^-_{\hat{\lambda}(k),k}}T_{\hat{\lambda}(k),k}(\alpha)=\lim_{k\to k_0}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1\frac{h(s\beta_{\hat{\lambda}(k),k})}{\beta^{p-1}_{\hat{\lambda}(k),k}}\left(\frac{h(\beta_{\hat{\lambda}(k),k})g(s\beta_{\hat{\lambda}(k),k})}{g(\beta_{\hat{\lambda}(k),k})h(s\beta_{\hat{\lambda}(k),k})}-1\right)ds\right]^{-1/p}dv$$

$$= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \lim_{k\to k_0} \frac{h(s\beta_{\hat\lambda(k),k})}{\beta^{p-1}_{\hat\lambda(k),k}} \left(\frac{h(\beta_{\hat\lambda(k),k})g(s\beta_{\hat\lambda(k),k})}{g(\beta_{\hat\lambda(k),k})h(s\beta_{\hat\lambda(k),k})} - 1\right) ds\right]^{-1/p} dv$$

(by the Monotone Convergence Theorem [21, p. 75])

$$= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_{\hat\lambda(k_0),k_0})}{\beta^{p-1}_{\hat\lambda(k_0),k_0}} \left(\frac{h(\beta_{\hat\lambda(k_0),k_0})g(s\beta_{\hat\lambda(k_0),k_0})}{g(\beta_{\hat\lambda(k_0),k_0})h(s\beta_{\hat\lambda(k_0),k_0})} - 1\right) ds\right]^{-1/p} dv$$

$$= \lim_{\alpha\to\beta^-_{\hat\lambda(k_0),k_0}} T_{\hat\lambda(k_0),k_0}(\alpha).$$

So $\lim_{\alpha\to\beta^-_{\hat\lambda(k),k}} T_{\hat\lambda(k),k}(\alpha)$ is a continuous function of k on $(0,\infty)$. Also, for each fixed $s\in(0,1)$, we obtain that

$$\lim_{k\to 0^+} \frac{h(s\beta_{\hat\lambda(k),k})}{\beta^{p-1}_{\hat\lambda(k),k}} \left[\frac{h(\beta_{\hat\lambda(k),k})g(s\beta_{\hat\lambda(k),k})}{g(\beta_{\hat\lambda(k),k})h(s\beta_{\hat\lambda(k),k})} - 1\right] = \lim_{k\to 0^+} C_p\left[\left(s^{p-1}-s^r\right) + \frac{(s^q-s^r)}{k}\beta^{q-p+1}_{\hat\lambda(k),k}\right] = \infty$$

and

$$\begin{aligned}\lim_{k\to \infty} \frac{h(s\beta_{\hat\lambda(k),k})}{\beta^{p-1}_{\hat\lambda(k),k}} \left[\frac{h(\beta_{\hat\lambda(k),k})g(s\beta_{\hat\lambda(k),k})}{g(\beta_{\hat\lambda(k),k})h(s\beta_{\hat\lambda(k),k})} - 1\right] &= \lim_{k\to \infty} C_p\left[\left(s^{p-1}-s^r\right) + \frac{(s^q-s^r)}{k}\beta^{q-p+1}_{\hat\lambda(k),k}\right] \\ &= C_p\left(s^{p-1}-s^r\right) \\ &< C_p s^{p-1} = (p-1)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p s^{p-1}.\end{aligned}$$

So by (3.15), $\lim_{k\to 0^+}\lim_{\alpha\to\beta^-_{\hat\lambda(k),k}} T_{\hat\lambda(k),k}(\alpha) = 0$, and

$$\begin{aligned}\lim_{k\to\infty}\lim_{\alpha\to\beta^-_{\hat\lambda(k),k}} T_{\hat\lambda(k),k}(\alpha) &> \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 (p-1)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p s^{p-1}\, ds\right]^{-1/p} dv \\ &= \left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^{-1} \int_0^1 \left(1-v^p\right)^{-1/p} dv \\ &= \left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^{-1}\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right) = 1.\end{aligned}$$

By above results and the Intermediate Value Theorem, we obtain that there exists a unique positive number $k^* = k^*(p,q,r)$ such that:

(1) If $0<k<k^*$, then $\lim_{\alpha\to\beta^-_{\hat\lambda(k),k}} T_{\hat\lambda(k),k}(\alpha) < 1$.
(2) If $k=k^*$, then $\lim_{\alpha\to\beta^-_{\hat\lambda(k^*),k^*}} T_{\hat\lambda(k^*),k^*}(\alpha) = 1$.
(3) If $k>k^*$, then $\lim_{\alpha\to\beta^-_{\hat\lambda(k),k}} T_{\hat\lambda(k),k}(\alpha) > 1$.

Since for each fixed $k>0$, $\lim_{\alpha\to\beta^-_{\tilde\lambda(k),k}} T_{\tilde\lambda(k),k}(\alpha) = 1$ and $\lim_{\alpha\to\beta^-_{\lambda,k}} T_{\lambda,k}(\alpha)$ is a strictly decreasing function of λ on $(0,\infty)$. Thus for such positive $k^* = k^*(p,q,r)$, we obtain that properties (a)–(c) hold. So part (ii) follows.

The proof of Corollary 2.4 is complete. □

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