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Examples of discontinuous maximal monotone linear operators and the solution to a recent problem posed by B.F. Svaiter

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ABSTRACT

In this paper, we give two explicit examples of unbounded linear maximal monotone operators. The first unbounded linear maximal monotone operator S on ℓ^2 is skew. We show its domain is a proper subset of the domain of its adjoint S^*, and $-S^*$ is not maximal monotone. This gives a negative answer to a recent question posed by Svaiter. The second unbounded linear maximal monotone operator is the inverse Volterra operator T on $L^2[0,1]$. We compare the domain of T with the domain of its adjoint T^* and show that the skew part of T admits two distinct linear maximal monotone skew extensions. These unbounded linear maximal monotone operators show that the constraint qualification for the maximality of the sum of maximal monotone operators cannot be significantly weakened, and they are simpler than the example given by Phelps–Simons. Interesting consequences on Fitzpatrick functions for sums of two maximal monotone operators are also given.

1. Introduction

Linear monotone operators play important roles in modern monotone operator theory and partial differential equations [1,2,8,13,17,21–24], and they are examples that delineate the boundary of the general theory. In this paper, we explicitly construct two unbounded linear monotone operators (not full domain, linear and single-valued on their domains). They answer one of Svaiter's question, have some interesting consequences on Fitzpatrick functions for sums of two maximal monotone operators, and show that the constraint qualification for the maximality of the sum of maximal monotone operators cannot be weaken significantly, see [15], [18, Theorem 5.5] and [21]. Our examples are simpler than the one given by [13].

The paper is organized as follows. Basic facts and auxiliary results are recorded in Section 2. In Section 3, we construct an unbounded maximal monotone skew operator S on ℓ^2. For a maximal monotone skew operator, it is well known that its domain is always a subset of the domain of its adjoint. An interesting question remained is whether or not both of the domains are always same. The maximal monotone skew operator S enjoys the property that the domain of $-S$ is a proper subset of the domain of its adjoint S^*, see Theorem 3.6. Svaiter asked in [20] whether or not $-S^*$ (termed $S^{\vdash}$ in [20]) is maximal monotone provided that S is maximal skew. This operator also answers Svaiter's question in the negative, see Theorem 3.15. In Section 4 we systematically study the inverse Volterra operator T. We show that T is neither skew nor symmetric and compare the domain of T with the domain of its adjoint T^*. It turns out that the skew part of T: $S = \frac{T-T^*}{2}$ admits two distinct linear maximal monotone and skew extensions even when the domain of S is a dense linear subspace in $L^2[0,1]$. It was shown that Fitzpatrick functions satisfy $F_{A+B} = F_A \,\Box_2\, F_B$ when A, B are maximal monotone linear relations

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and $\operatorname{dom} A - \operatorname{dom} B$ is a closed subspace, see [5, Theorem 5.10]. Using these unbounded linear maximal monotone operators in Sections 3 and 4 we also show that the constraint qualification $\operatorname{dom} A - \operatorname{dom} B$ being closed cannot be significantly weakened either.

Throughout this paper, we assume that

$$X \text{ is a real Hilbert space, with inner product } \langle\cdot,\cdot\rangle.$$

Let S be a set-valued operator (also known as multifunction) from X to X. We say that S is *monotone* if

$$\big(\forall(x,x^*)\in \operatorname{gra} S\big)\big(\forall(y,y^*)\in \operatorname{gra} S\big)\quad \langle x-y, x^*-y^*\rangle \geqslant 0,$$

where $\operatorname{gra} S := \{(x,x^*)\in X\times X \mid x^*\in Sx\}$; S is said to be *maximal monotone* if no proper enlargement (in the sense of graph inclusion) of S is monotone. We say T is a maximal monotone extension of S if T is maximal monotone and $\operatorname{gra} T \supseteq \operatorname{gra} S$. The *domain* of S is $\operatorname{dom} S := \{x\in X \mid Sx\neq\varnothing\}$, and its *range* is $\operatorname{ran} S := S(X) = \bigcup_{x\in X} Sx$.

We say S is a *linear relation* if $\operatorname{gra} S$ is linear. The *adjoint* of S, written S^*, is defined by

$$\operatorname{gra} S^* := \big\{(x,x^*)\in X\times X \;\big|\; (x^*,-x)\in(\operatorname{gra} S)^\perp\big\},$$

where, for any subset C of a Hilbert space Z, $C^\perp := \{z\in Z \mid \langle z,c\rangle = 0,\ \forall c\in C\}$. We say a linear relation S is *skew* if $\langle x,x^*\rangle = 0$, $\forall (x,x^*)\in \operatorname{gra} S$, and S is a *maximal monotone skew operator* if S is a maximal monotone operator and S is skew. Svaiter introduced $S^\vdash$ in [20], which is defined by

$$\operatorname{gra} S^\vdash := \big\{(x,x^*)\in X\times X \;\big|\; (x^*,x)\in(\operatorname{gra} S)^\perp\big\}.$$

Hence $S^\vdash = -S^*$. For each function $f: X\to\left]-\infty,+\infty\right]$, f^* stands for the *Fenchel conjugate* given by

$$f^*(x^*) = \sup_{x\in X}\big(\langle x^*,x\rangle - f(x)\big),\quad \forall x^*\in X.$$

2. Auxiliary results and facts

In this section we gather some facts about linear relations, monotone operators, and Fitzpatrick functions. They will be used frequently in sequel.

Fact 2.1 *(Cross). Let $S: X\rightrightarrows X$ be a linear relation. Then the following hold.*

(i) $(S^*)^{-1} = (S^{-1})^*$.
(ii) *If $\operatorname{gra} S$ is closed, then $S^{**} = S$.*
(iii) *If $k\in\mathbb{R}\smallsetminus\{0\}$, then $(kS)^* = kS^*$.*
(iv) $(\forall x\in\operatorname{dom} S^*)(\forall y\in\operatorname{dom} S)$ $\langle S^*x, y\rangle = \langle x, Sy\rangle$ *is a singleton.*

Proof. (i) See [10, Proposition III.1.3(b)]. (ii) See [10, Exercise VIII.1.12]. (iii) See [10, Proposition III.1.3(c)]. (iv) See [10, Proposition III.1.2]. $\square$

If $S: X\rightrightarrows X$ is a linear relation that is at most single-valued, then we will identify S with the corresponding linear operator from $\operatorname{dom} S$ to X and (abusing notation slightly) also write $S: \operatorname{dom} S\to X$. An analogous comment applies conversely to a linear single-valued operator S with domain $\operatorname{dom} S$, which we will identify with the corresponding at most single-valued linear relation from X to X.

Fact 2.2 *(Phelps–Simons). (See [13, Theorem 2.5 and Lemma 4.4].) Let $S: \operatorname{dom} S\to X$ be monotone and linear. The following hold.*

(i) *If S is maximal monotone, then $\operatorname{dom} S$ is dense (and hence S^* is at most single-valued).*
(ii) *Assume that S is a skew operator such that $\operatorname{dom} S$ is dense. Then $\operatorname{dom} S\subseteq\operatorname{dom} S^*$ and $S^*|_{\operatorname{dom} S} = -S$.*

Fact 2.3 *(Brézis–Browder). (See [9, Theorem 2].) Let $S: X\rightrightarrows X$ be a monotone linear relation such that $\operatorname{gra} S$ is closed. Then the following are equivalent.*

(i) *S is maximal monotone.*
(ii) *S^* is maximal monotone.*
(iii) *S^* is monotone.*

For $A: X \rightrightarrows X$, the *Fitzpatrick function* associated with A is defined by

$$F_A: X \times X \to \left]-\infty, +\infty\right]: (x, x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \left(\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\right). \tag{1}$$

Following Penot [14], if $F: X \times X \to \left]-\infty, +\infty\right]$, we set

$$F^{\intercal}: X \times X: (x^*, x) \mapsto F(x, x^*). \tag{2}$$

Fact 2.4 *(Fitzpatrick). (See [11].) Let $A: X \rightrightarrows X$ be monotone. Then $F_A = \langle\cdot,\cdot\rangle$ on $\operatorname{gra} A$ and $F_{A^{-1}} = F_A^{\intercal}$. If A is maximal monotone and $(x, x^*) \in X \times X$, then*

$$F_A(x, x^*) \geqslant \langle x^*, x\rangle,$$

with equality if and only if $(x, x^*) \in \operatorname{gra} A$.

If $A: X \to X$ is a linear operator, we write

$$A_+ = \frac{1}{2}A + \frac{1}{2}A^* \quad \text{and} \quad q_A: X \to \mathbb{R}: x \mapsto \frac{1}{2}\langle x, Ax\rangle. \tag{3}$$

Fact 2.5. *(See [4, Proposition 2.3] and [2, Proposition 2.2(v)].) Let $A: X \to X$ be linear and monotone, and let $(x, x^*) \in X \times X$. Then*

$$F_A(x, x^*) = 2q^*_{A_+}\left(\frac{1}{2}x^* + \frac{1}{2}A^*x\right) = \frac{1}{2}q^*_{A_+}(x^* + A^*x). \tag{4}$$

If $\operatorname{ran} A_+$ is closed, then $\operatorname{dom} q^*_{A_+} = \operatorname{ran} A_+$.

To study Fitzpatrick functions of sums of maximal monotone operator, one needs the $\Box_2$ operation:

Definition 2.6. Let $F_1, F_2: X \times X \to \left]-\infty, +\infty\right]$. Then the *partial inf-convolution* $F_1 \Box_2 F_2$ is the function defined on $X \times X$ by

$$F_1 \Box_2 F_2: (x, x^*) \mapsto \inf_{y^* \in X} \left(F_1(x, x^* - y^*) + F_2(x, y^*)\right).$$

Fact 2.7. *(See [17, Lemma 23.9] or [3, Proposition 4.2].) Let $A, B: X \rightrightarrows X$ be monotone such that $\operatorname{dom} A \cap \operatorname{dom} B \neq \emptyset$. Then $F_A \Box_2 F_B \geqslant F_{A+B}$.*

Under some constraint qualifications, one has

Fact 2.8.

(i) (*See* [2].) *Let $A, B: X \to X$ be continuous, linear, and monotone operators such that $\operatorname{ran}(A_+ + B_+)$ is closed. Then $F_{A+B} = F_A \Box_2 F_B$.*
(ii) (*See* [5].) *Let $A, B: X \rightrightarrows X$ be maximal monotone linear relations, and suppose that $\operatorname{dom} A - \operatorname{dom} B$ is closed. Then $F_{A+B} = F_A \Box_2 F_B$.*

3. An unbounded skew operator on ℓ^2

In this section, we construct a maximal monotone and skew operator S on ℓ^2 such that $-S^*$ is not maximal monotone. This answers one of Svaiter's question. We explicitly compute the Fitzpatrick functions F_{S+S^*}, F_S, F_{S^*}, and show that $F_{S+S^*} \neq F_S \Box_2 F_{S^*}$ even though S, S^* are linear maximal monotone with $\operatorname{dom} S - \operatorname{dom} S^*$ being a dense linear subspace in ℓ^2.

3.1. The example in ℓ^2

Let ℓ^2 denote the Hilbert space of real square-summable sequences $(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, \ldots)$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Example 3.1. Let $X = \ell^2$, and $S: \operatorname{dom} S \to \ell^2$ be given by

$$Sy := \frac{(\sum_{i<n} y_i - \sum_{i>n} y_i)_{n\in\mathbb{N}}}{2} = \left(\sum_{i<n} y_i + \frac{1}{2}y_n\right)_{n\in\mathbb{N}}, \quad \forall y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S, \tag{5}$$

where $\operatorname{dom} S := \{y = (y_n)_{n\in\mathbb{N}} \in \ell^2 \mid \sum_{i\geqslant 1} y_i = 0,\ (\sum_{i\leqslant n} y_i)_{n\in\mathbb{N}} \in \ell^2\}$ and $\sum_{i<1} y_i = 0$. In matrix form,

$$S = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 1 & 0 & -1 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 1 & 1 & 0 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots & -1 & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \end{pmatrix},$$

or

$$S = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 1 & \frac{1}{2} & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \frac{1}{2} & \cdots & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \end{pmatrix}.$$

Using the second matrix, it is easy to see that S is injective.

Proposition 3.2. *Let S be defined as in Example* 3.1. *Then S is skew.*

Proof. Let $y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S$. Then $(\sum_{i\leqslant n} y_i)_{n\in\mathbb{N}} \in \ell^2$. Thus,

$$\ell^2 \ni \left(\sum_{i\leqslant n} y_i\right)_{n\in\mathbb{N}} - \frac{1}{2}y = \left(\sum_{i\leqslant n} y_i\right)_{n\in\mathbb{N}} - \frac{1}{2}(y_n)_{n\in\mathbb{N}} = \left(\sum_{i<n} y_i + \frac{1}{2}y_n\right)_{n\in\mathbb{N}} = Sy.$$

Hence S is well defined. Clearly, S is linear on $\operatorname{dom} S$. Now we show S is skew.

Let $y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S$, and $s := \sum_{i\geqslant 1} y_i$. Then $(\sum_{i\leqslant n} y_i)_{n\in\mathbb{N}} \in \ell^2$. Hence $(\sum_{i<n} y_i)_{n\in\mathbb{N}} = (\sum_{i\leqslant n} y_i)_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}} \in \ell^2$. By $s = 0$,

$$\ell^2 \ni -\left(\sum_{i<n} y_i\right)_{n\in\mathbb{N}} = 0 - \left(\sum_{i<n} y_i\right)_{n\in\mathbb{N}} = \left(\sum_{i\geqslant 1} y_i - \sum_{i<n} y_i\right)_{n\in\mathbb{N}} = \left(\sum_{i\geqslant n} y_i\right)_{n\in\mathbb{N}},$$

$$\left(\sum_{i\geqslant n+1} y_i\right)_{n\in\mathbb{N}} = 0 - \left(\sum_{i\leqslant n} y_i\right)_{n\in\mathbb{N}} \in \ell^2. \tag{6}$$

Thus, by (6),

$$\begin{aligned} -2\langle Sy, y\rangle &= \left\langle \left(\sum_{i>n} y_i - \sum_{i<n} y_i\right)_{n\in\mathbb{N}}, y\right\rangle = \left\langle \left(\sum_{i\geqslant n+1} y_i + \sum_{i\geqslant n} y_i\right)_{n\in\mathbb{N}}, y\right\rangle \\ &= \left\langle \left(\sum_{i\geqslant 1} y_i, \sum_{i\geqslant 2} y_i, \ldots\right) + \left(\sum_{i\geqslant 2} y_i, \sum_{i\geqslant 3} y_i, \ldots\right), y\right\rangle \\ &= \big\langle \big(s, s - y_1, s - (y_1 + y_2), \ldots\big) + \big(s - y_1, s - (y_1 + y_2), \ldots\big), (y_1, y_2, \ldots)\big\rangle \\ &= \big[sy_1 + (s - y_1)y_2 + \big(s - (y_1 + y_2)\big)y_3 + \cdots\big] \\ &\quad + \big[(s - y_1)y_1 + \big(s - (y_1 + y_2)\big)y_2 + \big(s - (y_1 + y_2 + y_3)\big)y_3 + \cdots\big] \\ &= \lim_n \big[sy_1 + (s - y_1)y_2 + \cdots + \big(s - (y_1 + \cdots + y_{n-1})\big)y_n\big] \\ &\quad + \lim_n \big[(s - y_1)y_1 + \big(s - (y_1 + y_2)\big)y_2 + \cdots + \big(s - (y_1 + \cdots + y_n)\big)y_n\big] \\ &= \lim_n \big[s(y_1 + \cdots + y_n) - y_1y_2 - (y_1 + y_2)y_3 - \cdots - (y_1 + \cdots + y_{n-1})y_n\big] \\ &\quad + \big[s(y_1 + \cdots + y_n) - \big(y_1^2 + \cdots + y_n^2\big) - y_1y_2 - \cdots - (y_1 + \cdots + y_{n-1})y_n\big] \\ &= \lim_n \big[2s(y_1 + \cdots + y_n) - (y_1 + \cdots + y_n)^2\big] = 2s^2 - s^2 = s^2 = 0. \end{aligned} \tag{7}$$

Hence S is skew. $\square$

Remark 3.3. S is unbounded in Example 3.1, since $(1,0,0,\dots,0,\dots)\notin \operatorname{dom} S$.

Fact 3.4 *(Phelps–Simons). (See [13, Proposition 3.2(a)].) Let $S:\operatorname{dom} S\to X$ be linear and monotone. Then $(x,x^*)\in X\times X$ is monotonically related to* $\operatorname{gra} S$ *if and only if*

$$\langle x,x^*\rangle \geqslant 0 \quad \textit{and} \quad \bigl[\langle Sy,x\rangle+\langle x^*,y\rangle\bigr]^2\leqslant 4\langle x^*,x\rangle\langle Sy,y\rangle,\quad \forall y\in\operatorname{dom} S.$$

Proposition 3.5. *Let S be defined as in Example* 3.1. *Then S is a maximal monotone operator. In particular,* $\operatorname{gra} S$ *is closed.*

Proof. By Proposition 3.2, S is skew. Let $(x,x^*)\in X\times X$ be monotonically related to $\operatorname{gra} S$. Write $x=(x_n)_{n\in\mathbb{N}}$ and $x^*=(x_n^*)_{n\in\mathbb{N}}$. By Fact 3.4, we have

$$\langle Sy,x\rangle+\langle x^*,y\rangle=0,\quad \forall y\in\operatorname{dom} S. \tag{8}$$

Let $e_n=(0,\dots,0,1,0,\dots)$: the nth entry is 1 and the others are 0. Then let $y=-e_1+e_n$. Thus $y\in\operatorname{dom} S$ and $Sy=(-\frac{1}{2},-1,\dots,-1,-\frac{1}{2},0,\dots)$. Then by (8),

$$-x_1^*+x_n^*-\frac{1}{2}x_1-\frac{1}{2}x_n-\sum_{i=2}^{n-1}x_i=0 \quad\Rightarrow\quad x_n^*=x_1^*-\frac{1}{2}x_1+\sum_{i=1}^{n-1}x_i+\frac{1}{2}x_n. \tag{9}$$

Since $x^*\in\ell^2$ and $x\in\ell^2$, we have $x_n^*\to 0$, $x_n\to 0$. Thus by (9),

$$-\sum_{i\geqslant 1}x_i=x_1^*-\frac{1}{2}x_1. \tag{10}$$

Next we show $-\sum_{i\geqslant 1}x_i=x_1^*-\frac{1}{2}x_1=0$. Let $s=\sum_{i\geqslant 1}x_i$. Then by (9) and (10),

$$\begin{aligned}
2x^*=2\bigl(x_n^*\bigr)_{n\in\mathbb{N}}&=2\Bigl(-\sum_{i\geqslant 1}x_i+\sum_{i<n}x_i+\frac{1}{2}x_n\Bigr)_{n\in\mathbb{N}}=\Bigl(-2\sum_{i\geqslant 1}x_i+2\sum_{i<n}x_i+x_n\Bigr)_{n\in\mathbb{N}}\\
&=\Bigl(-2\sum_{i\geqslant n}x_i+x_n\Bigr)_{n\in\mathbb{N}}=\Bigl(-\sum_{i\geqslant n}x_i-\sum_{i\geqslant n}x_i+x_n\Bigr)_{n\in\mathbb{N}}\\
&=\Bigl(-\sum_{i\geqslant n}x_i-\sum_{i\geqslant n+1}x_i\Bigr)_{n\in\mathbb{N}}.
\end{aligned} \tag{11}$$

On the other hand, by (9),

$$\ell^2\ni x^*-\frac{1}{2}x=\Bigl(-\sum_{i\geqslant 1}x_i+\sum_{i<n}x_i+\frac{1}{2}x_n\Bigr)_{n\in\mathbb{N}}-\Bigl(\frac{1}{2}x_n\Bigr)_{n\in\mathbb{N}}=\Bigl(-\sum_{i\geqslant n}x_i\Bigr)_{n\in\mathbb{N}}.$$

Then by (11),

$$2x^*=\Bigl(-\sum_{i\geqslant n}x_i\Bigr)_{n\in\mathbb{N}}+\Bigl(-\sum_{i\geqslant n+1}x_i\Bigr)_{n\in\mathbb{N}}.$$

Then by Fact 3.4, similar to the proof in (7) in Proposition 3.2, we have

$$\begin{aligned}
0\geqslant -2\langle x^*,x\rangle&=\Bigl\langle\Bigl(\sum_{i\geqslant n}x_i\Bigr)_{n\in\mathbb{N}}+\Bigl(\sum_{i\geqslant n+1}x_i\Bigr)_{n\in\mathbb{N}},x\Bigr\rangle\\
&=\Bigl\langle\Bigl(\sum_{i\geqslant 1}x_i,\sum_{i\geqslant 2}x_i,\dots\Bigr)+\Bigl(\sum_{i\geqslant 2}x_i,\sum_{i\geqslant 3}x_i,\dots\Bigr),x\Bigr\rangle\\
&=2s^2-s^2=s^2.
\end{aligned}$$

Hence $s=0$, i.e., $x_1^*=\frac{1}{2}x_1$. By (9), $x^*=(\sum_{i<n}x_i+\frac{1}{2}x_n)_{n\in\mathbb{N}}$. Thus

$$\ell^2\ni x^*+\frac{1}{2}x=\Bigl(\sum_{i<n}x_i+\frac{1}{2}x_n\Bigr)_{n\in\mathbb{N}}+\Bigl(\frac{1}{2}x_n\Bigr)_{n\in\mathbb{N}}=\Bigl(\sum_{i\leqslant n}x_i\Bigr)_{n\in\mathbb{N}}.$$

Hence $x\in\operatorname{dom} S$ and $x^*=Sx$. Thus, S is maximal monotone. Hence $\operatorname{gra} S$ is closed. □

Proposition 3.6. *Let S be defined as in Example* 3.1. *Then*

$$S^*y=\left(\sum_{i>n} y_i+\frac{1}{2}y_n\right)_{n\in\mathbb{N}},\quad \forall y=(y_n)_{n\in\mathbb{N}}\in\operatorname{dom}S^*, \tag{12}$$

where $\operatorname{dom}S^*=\{y=(y_n)_{n\in\mathbb{N}}\in\ell^2 \mid \sum_{i\geqslant 1} y_i\in\mathbb{R},\ (\sum_{i>n} y_i)_{n\in\mathbb{N}}\in\ell^2\}$. *In matrix form,*

$$S^*:=\begin{pmatrix} \frac{1}{2} & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots\\ 0 & \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots\\ 0 & 0 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & \cdots\\ 0 & 0 & 0 & \frac{1}{2} & 1 & \cdots & 1 & 1 & \cdots\\ 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots & 1 & 1 & \cdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \end{pmatrix}.$$

Moreover, $\operatorname{dom}S\subsetneqq\operatorname{dom}S^*$, $S^*=-S$ *on* $\operatorname{dom}S$, *and* S^* *is not skew.*

Proof. Let $y=(y_n)_{n\in\mathbb{N}}\in\ell^2$ with $(\sum_{i>n} y_i)_{n\in\mathbb{N}}\in\ell^2$, and $y^*=(\sum_{i>n} y_i+\frac{1}{2}y_n)_{n\in\mathbb{N}}$. Now we show $(y,y^*)\in\operatorname{gra}S^*$. Let $s=\sum_{i\geqslant 1} y_i$ and $x\in\operatorname{dom}S$. Then we have

$$\begin{aligned}
\langle y,Sx\rangle+\langle y^*,-x\rangle &= \left\langle y,\frac{1}{2}x+\left(\sum_{i<n}x_i\right)_{n\in\mathbb{N}}\right\rangle+\left\langle \frac{1}{2}y+\left(\sum_{i>n}y_i\right)_{n\in\mathbb{N}},-x\right\rangle\\
&= \left\langle y,\left(\sum_{i<n}x_i\right)_{n\in\mathbb{N}}\right\rangle+\left\langle \left(\sum_{i>n}y_i\right)_{n\in\mathbb{N}},-x\right\rangle\\
&= \lim_n\big[y_2x_1+y_3(x_1+x_2)+\cdots+y_n(x_1+\cdots+x_{n-1})\big]\\
&\quad -\lim_n\big[x_1(s-y_1)+x_2(s-y_1-y_2)+\cdots+x_n(s-y_1-\cdots-y_n)\big]\\
&= \lim_n\big[x_1(y_2+\cdots+y_n)+x_2(y_3+\cdots+y_n)+\cdots+x_{n-1}y_n\big]\\
&\quad -\lim_n\big[x_1(s-y_1)+x_2(s-y_1-y_2)+\cdots+x_n(s-y_1-\cdots-y_n)\big]\\
&= \lim_n\big[x_1(y_1+y_2+\cdots+y_n-s)+x_2(y_1+y_2+\cdots+y_n-s)+\cdots\\
&\quad +x_n(y_1+y_2+\cdots+y_n-s)\big]\\
&= \lim_n\big[(x_1+\cdots+x_n)(y_1+y_2+\cdots+y_n-s)\big]\\
&= 0.
\end{aligned}$$

Hence $(y,y^*)\in\operatorname{gra}S^*$.

On the other hand, let $(a,a^*)\in\operatorname{gra}S^*$ with $a=(a_n)_{n\in\mathbb{N}}$ and $a^*=(a_n^*)_{n\in\mathbb{N}}$. Now we show

$$\left(\sum_{i>n}a_i\right)_{n\in\mathbb{N}}\in\ell^2\quad\text{and}\quad a^*=\left(\sum_{i>n}a_i+\frac{1}{2}a_n\right)_{n\in\mathbb{N}}. \tag{13}$$

Let $e_n=(0,\ldots,0,1,0,\ldots)$: the nth entry is 1 and the others are 0. Then let $y=-e_1+e_n$. Thus $y\in\operatorname{dom}S$ and $Sy=(-\frac{1}{2},-1,\ldots,-1,-\frac{1}{2},0,\ldots)$. Then,

$$0=\langle a^*,y\rangle+\langle -Sy,a\rangle=-a_1^*+a_n^*+\frac{1}{2}a_1+\frac{1}{2}a_n+\sum_{i=2}^{n-1}a_i$$

$$\Rightarrow\quad a_n^*=a_1^*-\frac{1}{2}a_1-\sum_{i=2}^{n-1}a_i-\frac{1}{2}a_n. \tag{14}$$

Since $a^*\in\ell^2$ and $a\in\ell^2$, $a_n^*\to 0$, $a_n\to 0$. Thus by (14),

$$a_1^*=\frac{1}{2}a_1+\sum_{i>1}a_i, \tag{15}$$

from which we see that $\sum_{i\geqslant 1} a_i \in \mathbb{R}$. Combining (14) and (15), we have

$$a_n^* = \sum_{i>n} a_i + \frac{1}{2}a_n.$$

Thus, (13) holds. Hence (12) holds.

Now for $x \in \operatorname{dom} S$, since $\sum_{i\geqslant 1} x_i = 0$, we have

$$\begin{aligned} S^*x &= \left(\frac{1}{2}x_n + \sum_{i>n} x_i\right)_{n\in\mathbb{N}} = \left(-\frac{1}{2}x_n + \sum_{i\geqslant n} x_i\right)_{n\in\mathbb{N}} \\ &= \left(-\frac{1}{2}x_n - \sum_{i<n} x_i\right)_{n\in\mathbb{N}} = -Sx. \end{aligned}$$

We note that S^* is not skew since for $e_1 = (1,0,\ldots)$, $\langle S^*e_1, e_1\rangle = \langle 1/2e_1, e_1\rangle = 1/2$. As $e_1 = (1,0,0,\ldots,0,\ldots) \in \operatorname{dom} S^*$ but $e_1 \notin \operatorname{dom} S$, we have $\operatorname{dom} S \subsetneqq \operatorname{dom} S^*$. $\square$

Proposition 3.7. *Let S be defined as in Example* 3.1, *let $y = (y_1, y_2, \ldots) \in \operatorname{dom} S^*$, and set $s = \sum_{i\geqslant 1} y_i$. Then*

$$\langle S^*y, y\rangle = \frac{1}{2}s^2. \tag{16}$$

Proof. By Proposition 3.6, we have $s \in \mathbb{R}$ and

$$\begin{aligned} \langle S^*y, y\rangle &= \left\langle \left(\sum_{i>n} y_i + \frac{1}{2}y_n\right)_{n\in\mathbb{N}}, y\right\rangle = \left\langle \left(\sum_{i\geqslant n} y_i - \frac{1}{2}y_n\right)_{n\in\mathbb{N}}, y\right\rangle \\ &= \lim_n \left[sy_1 + (s-y_1)y_2 + \cdots + (s - y_1 - y_2 - \cdots - y_{n-1})y_n - \frac{1}{2}\left(y_1^2 + y_2^2 + \cdots + y_n^2\right)\right] \\ &= \lim_n \left[s(y_1 + \cdots + y_n) - y_1y_2 - (y_1+y_2)y_3 - \cdots - (y_1 + y_2 + \cdots + y_{n-1})y_n\right] \\ &\quad - \frac{1}{2}\left[y_1^2 + y_2^2 + \cdots + y_n^2\right] \\ &= \lim_n \left[s(y_1 + \cdots + y_n)\right] \\ &\quad - \lim_n \left[y_1y_2 + (y_1+y_2)y_3 + \cdots + (y_1 + y_2 + \cdots + y_{n-1})y_n + \frac{1}{2}\left(y_1^2 + y_2^2 + \cdots + y_n^2\right)\right] \\ &= s^2 - \lim_n \frac{1}{2}[y_1 + y_2 + \cdots + y_n]^2 \\ &= s^2 - \frac{1}{2}s^2 \\ &= \frac{1}{2}s^2. \end{aligned}$$

Hence (16) holds. $\square$

Proposition 3.8. *Let S be defined as in Example* 3.1. *Then $-S$ is not maximal monotone.*

Proof. By Proposition 3.2, $-S$ is skew. Let $e_1 = (1,0,0,\ldots,0,\ldots)$. Then $e_1 \notin \operatorname{dom} S = \operatorname{dom}(-S)$. Thus, $(e_1, \frac{1}{2}e_1) \notin \operatorname{gra}(-S)$. We have for every $y \in \operatorname{dom} S$,

$$\left\langle e_1, \frac{1}{2}e_1\right\rangle \geqslant 0 \quad \text{and} \quad \langle e_1, -Sy\rangle + \left\langle y, \frac{1}{2}e_1\right\rangle = -\frac{1}{2}y_1 + \frac{1}{2}y_1 = 0.$$

By Fact 3.4, $(e_1, \frac{1}{2}e_1)$ is monotonically related to $\operatorname{gra}(-S)$. We deduce that $-S$ is not maximal monotone. $\square$

We proceed to show that for every maximal monotone and skew operator S, the operator $-S$ has a unique maximal monotone extension, namely S^*.

Theorem 3.9. *Let $S : \operatorname{dom} S \to X$ be a maximal monotone skew operator. Then $-S$ has a unique maximal monotone extension*: S^*.

Proof. By Fact 2.2, $\operatorname{gra}(-S) \subseteq \operatorname{gra} S^*$. Assume T is a maximal monotone extension of $-S$. Let $(x, x^*) \in \operatorname{gra} T$. Then (x, x^*) is monotonically related to $\operatorname{gra}(-S)$. By Fact 3.4,

$$\langle x^*, y\rangle + \langle -x, Sy\rangle = \langle x^*, y\rangle + \langle x, -Sy\rangle = 0, \quad \forall y \in \operatorname{dom} S.$$

Thus $(x, x^*) \in \operatorname{gra} S^*$. Since $(x, x^*) \in \operatorname{gra} T$ is arbitrary, we have $\operatorname{gra} T \subseteq \operatorname{gra} S^*$. By Fact 2.3, S^* is maximal monotone. Hence $T = S^*$. □

Remark 3.10. Note that [22, Proposition 17] also implies that $-S$ has a unique maximal monotone extension, where S is as in Theorem 3.9.

Remark 3.11. Define the *right and left shift operators* $R, L : \ell^2 \to \ell^2$ by

$$Rx = (0, x_1, x_2, \ldots), \qquad Lx = (x_2, x_3, \ldots), \quad \forall x = (x_1, x_2, \ldots) \in \ell^2.$$

One can verify that in Example 3.1

$$S = (\mathrm{Id} - R)^{-1} - \frac{\mathrm{Id}}{2}, \qquad S^* = (\mathrm{Id} - L)^{-1} - \frac{\mathrm{Id}}{2}.$$

The maximal monotone operators $(\mathrm{Id} - R)^{-1}$ and $(\mathrm{Id} - L)^{-1}$ have been utilized by Phelps and Simons, see [13, Example 7.4].

3.2. An answer to Svaiter's question

Definition 3.12. Let $S : X \rightrightarrows X$ be skew. We say S is *maximal skew* (termed *"maximal self-canceling" in* [20]) if no proper enlargement (in the sense of graph inclusion) of S is skew. We say T is *a maximal skew extension* of S if T is maximal skew and $\operatorname{gra} T \supseteq \operatorname{gra} S$.

Lemma 3.13. *Let $S : X \rightrightarrows X$ be a maximal monotone skew operator. Then both S and $-S$ are maximal skew.*

Proof. Clearly, S is maximal skew. Now we show $-S$ is maximal skew. Let T be a skew operator such that $\operatorname{gra}(-S) \subseteq \operatorname{gra} T$. Thus, $\operatorname{gra} S \subseteq \operatorname{gra}(-T)$. Since $-T$ is monotone and S is maximal monotone, $\operatorname{gra} S = \operatorname{gra}(-T)$. Then $-S = T$. Hence $-S$ is maximal skew. □

Fact 3.14 *(Svaiter). (See [20].) Let $S : X \rightrightarrows X$ be maximal skew. Then either $-S^*$ (i.e., $S^{\vdash}$) or S^* (i.e., $-S^{\vdash}$) is maximal monotone.*

In [20], Svaiter asked whether or not $-S^*$ (i.e., $S^{\vdash}$) is maximal monotone if S is maximal skew. Now we can give a negative answer, even though S is maximal monotone and skew.

Theorem 3.15. *Let S be defined as in Example* 3.1. *Then S is maximal skew, but $-S^*$ is not monotone, so not maximal monotone.*

Proof. Let $e_1 = (1, 0, 0, \ldots, 0, \ldots)$. By Proposition 3.6, $(e_1, -\frac{1}{2}e_1) \in \operatorname{gra}(-S^*)$, but $\langle e_1, -\frac{1}{2}e_1\rangle = -\frac{1}{2} < 0$. Hence $-S^*$ is not monotone. □

By Theorem 3.15, $-S^*$ (i.e., $S^{\vdash}$) is not always maximal monotone. Can one improve Svaiter's result: "If S is maximal skew, then S^* (i.e., $-S^{\vdash}$) is always maximal monotone"?

Theorem 3.16. *There exists a maximal skew operator T on ℓ^2 such that T^* is not maximal monotone. Consequently, Svaiter's result is optimal.*

Proof. Let $T = -S$, where S is defined as in Example 3.1. By Lemma 3.13, T is maximal skew. Then by Theorem 3.15 and Fact 2.1(iii), $T^* = (-S)^* = -S^*$ is not maximal monotone. Hence Svaiter's result cannot be further improved. □

3.3. The maximal monotonicity and Fitzpatrick functions of a sum

Example 3.17 *($S + S^*$ fails to be maximal monotone).* Let S be defined in Example 3.1. Then neither S nor S^* has full domain. By Fact 2.2, $\forall x \in \operatorname{dom}(S + S^*) = \operatorname{dom} S$, we have

$$(S + S^*)x = 0.$$

Thus $S + S^*$ has a proper monotone extension from $\operatorname{dom}(S + S^*)$ to the 0 map on X. Consequently, $S + S^*$ is not maximal monotone. This supplies a different example for showing that the constraint qualification in the sum problem of maximal monotone operators cannot be substantially weakened, see [13, Example 7.4].

We now compute F_S, F_{S^*}, F_{S+S^*}. As a result, we see that $F_{S+S^*} \neq F_S \,\Box_2\, F_{S^*}$ even though S, S^* are maximal monotone with $\operatorname{dom} S - \operatorname{dom} S^*$ being dense in ℓ^2. Since $\operatorname{ran}(S_+ + (S^*)_+) = \{0\}$ and $F_{S+S^*} \neq F_S \,\Box_2\, F_{S^*}$, this also means that Fact 2.8(i) fails for discontinuous linear maximal monotone operators.

Lemma 3.18. *Let $S : \operatorname{dom} S \to X$ be a maximal monotone skew linear operator. Then*

$$F_S = \iota_{\operatorname{gra}(-S^*)},$$
$$F_{S^*}^{*\intercal} = F_{S^*} = \iota_{\operatorname{gra} S^*} + \langle\cdot,\cdot\rangle.$$

Proof. By [5, Proposition 5.5],

$$F_S^* = (\iota_{\operatorname{gra} S})^\intercal.$$

Then

$$F_S = \big(F_S^{*\intercal}\big)^{*\intercal} = (\iota_{\operatorname{gra} S})^{*\intercal} = \big(\iota_{\operatorname{gra} S}^\intercal\big)^* = (\iota_{\operatorname{gra} S^{-1}})^* = \iota_{(\operatorname{gra} S^{-1})^\perp} = \iota_{\operatorname{gra}(-S^*)}. \tag{17}$$

From Fact 2.2, $\operatorname{gra} -S \subseteq \operatorname{gra} S^*$, we have

$$F_{S^*} \geqslant F_{-S} = \iota_{\operatorname{gra} -(-S)^*} = \iota_{\operatorname{gra} S^*},$$

this shows that $\operatorname{dom} F_{S^*} \subseteq \operatorname{gra} S^*$. By Fact 2.4, $F_{S^*}(x, x^*) = \langle x, x^*\rangle$, $\forall (x, x^*) \in \operatorname{gra} S^*$. Hence $F_{S^*} = \iota_{\operatorname{gra} S^*} + \langle\cdot,\cdot\rangle$. Again by [5, Proposition 5.5], $F_{S^*}^{*\intercal} = \iota_{\operatorname{gra} S^*} + \langle\cdot,\cdot\rangle$. $\square$

Theorem 3.19. *Let S be defined as in Example* 3.1. *Then*

$$F_{S+S^*}\big(x, x^*\big) = \iota_{X\times\{0\}}\big(x, x^*\big),$$
$$F_S \,\Box_2\, F_{S^*}\big(x, x^*\big) = \begin{cases} \frac{1}{2}s^2, & \text{if } (x, x^*) \in \operatorname{dom} S^* \times \{0\} \text{ with } s = \sum_{i\geqslant 1} x_i;\\ \infty & \text{otherwise}.\end{cases} \tag{18}$$

Consequently, $F_S \,\Box_2\, F_{S^*} \neq F_{S+S^*}$.

Proof. By Fact 2.2,

$$\big(S + S^*\big)\big|_{\operatorname{dom} S} = 0. \tag{19}$$

Let $(x, x^*) \in X \times X$. Using (19) and Fact 2.2, we have

$$F_{S+S^*}\big(x, x^*\big) = \sup_{a\in\operatorname{dom} S} \big\langle x^*, a\big\rangle = \iota_{(\operatorname{dom} S)^\perp}\big(x^*\big) = \iota_{\{0\}}\big(x^*\big) = \iota_{X\times\{0\}}\big(x, x^*\big). \tag{20}$$

Then by Fact 2.7, we have

$$F_S \,\Box_2\, F_{S^*}\big(x, x^*\big) = \infty, \quad x^* \neq 0. \tag{21}$$

It follows from Lemma 3.18 that

$$\begin{aligned} F_S \,\Box_2\, F_{S^*}(x, 0) &= \inf_{y^*\in X}\big\{F_S\big(x, y^*\big) + F_{S^*}\big(x, -y^*\big)\big\}\\ &= \inf_{y^*\in X}\big\{\iota_{\operatorname{gra}(-S^*)}\big(x, y^*\big) + \iota_{\operatorname{gra} S^*}\big(x, -y^*\big) + \big\langle x, -y^*\big\rangle\big\}\\ &= \inf_{y^*\in X}\big\{\iota_{\operatorname{gra} S^*}\big(x, -y^*\big) + \big\langle x, -y^*\big\rangle\big\}. \end{aligned} \tag{22}$$

Thus, $F_S \,\Box_2\, F_{S^*}(x, 0) = \infty$ if $x \notin \operatorname{dom} S^*$. Now suppose $x \in \operatorname{dom} S^*$ and $s = \sum_{i\geqslant 1} x_i$. Then by (22) and Proposition 3.7, we have

$$F_S \,\Box_2\, F_{S^*}(x, 0) = \big\langle x, S^* x\big\rangle = \frac{1}{2}s^2.$$

Combining the results above, (18) holds. Since $\operatorname{dom} S^* \neq X$, $F_S \,\Box_2\, F_{S^*} \neq F_{S+S^*}$. $\square$

Remark 3.20. [5, Theorem 7.6] shows that: Let $A : X \rightrightarrows X$ be a maximal monotone linear relation. Then $A^* = -A$ if and only if $\operatorname{dom} A = \operatorname{dom} A^*$ and $F_A = F_A^{*\intercal}$. Let $A = S^*$ with S defined as in Example 3.1. Lemma 3.18 shows that $F_A = F_A^{*\intercal}$, but $A^* = S \neq -S^* = -A$. Hence the requirement $\operatorname{dom} A = \operatorname{dom} A^*$ cannot be omitted.

4. The inverse Volterra operator on $L^2[0,1]$

Let V be the Volterra integral operator. In this section, we systematically study $T = V^{-1}$ and its skew part $S := \frac{1}{2}(T - T^*)$. It turns out that T is neither skew nor symmetric and that its skew part S admits two maximal monotone and skew extensions T_1, T_2 (in fact, anti-self-adjoint) even though $\operatorname{dom} S$ is a dense linear subspace of $L^2[0,1]$. This will give another simpler example of Phelps–Simons' showing that the constraint qualification for the sum of monotone operators cannot be significantly weakened, see [18, Theorem 5.5] or [21]. We compute the Fitzpatrick functions F_T, F_{T^*}, F_{T+T^*}, and we show that $F_T \,\Box_2\, F_{T^*} \neq F_{T+T^*}$. This shows that the constraint qualification for the formula of the Fitzpatrick function of the sum of two maximal monotone operators cannot be significantly weakened either.

Definition 4.1. (See [5].) Let $T : X \rightrightarrows X$ be a linear relation. We say that T is symmetric if $\operatorname{gra} T \subseteq \operatorname{gra} T^*$; T is self-adjoint if $T^* = T$ and anti-self-adjoint if $T^* = -T$.

4.1. Properties of the Volterra operator and its inverse

To study the Volterra operator and its inverse, we shall frequently need the following generalized integration-by-parts formula, see [19, Theorem 6.90].

Fact 4.2 *(Generalized integration by parts). Assume that x, y are absolutely continuous functions on the interval $[a,b]$. Then*

$$\int_a^b xy' + \int_a^b x'y = x(b)y(b) - x(a)y(a).$$

Fact 2.3 allows us to claim that

Proposition 4.3. *Let $A : X \rightrightarrows X$ be a linear relation. If $A^* = -A$, then both A and $-A$ are maximal monotone and skew.*

Proof. Since $A = -A^*$, we have that $\operatorname{dom} A = \operatorname{dom} A^*$ and that A has closed graph. Now $\forall x \in \operatorname{dom} A$, by Fact 2.1(iv),

$$\langle Ax, x\rangle = \big\langle x, A^*x\big\rangle = -\langle x, Ax\rangle \quad\Rightarrow\quad \langle Ax, x\rangle = 0.$$

Hence A and $-A$ are skew. As $A^* = -A$ is monotone, Fact 2.3 shows that A is maximal monotone.

Now $-A = A^* = -(-A)^*$ and $-A$ is a linear relation. Similar arguments show that $-A$ is maximal monotone. $\square$

Example 4.4 *(Volterra operator).* (See [2, Example 3.3].) Set $X = L^2[0,1]$. The *Volterra integration operator* [12, Problem 148] is defined by

$$V : X \to X : x \mapsto Vx, \quad \text{where } Vx : [0,1] \to \mathbb{R} : t \mapsto \int_0^t x, \tag{23}$$

and its adjoint is given by

$$t \mapsto \big(V^*x\big)(t) = \int_t^1 x, \quad \forall x \in X.$$

Then:

(i) Both V and V^* are maximal monotone since they are monotone, continuous and linear.
(ii) Both ranges

$$\operatorname{ran} V = \big\{x \in L^2[0,1]\colon x \text{ is absolutely continuous},\ x(0) = 0,\ x' \in L^2[0,1]\big\}, \tag{24}$$

and

$$\operatorname{ran} V^* = \big\{x \in L^2[0,1]\colon x \text{ is absolutely continuous},\ x(1) = 0,\ x' \in L^2[0,1]\big\}, \tag{25}$$

are dense in X, and both V and V^* are one-to-one.
(iii) $\operatorname{ran} V \cap \operatorname{ran} V^* = \{Vx \mid x \in e^{\perp}\}$, where $e \equiv 1 \in L^2[0,1]$.

(iv) Define $V_+x := \frac{1}{2}(V + V^*)(x) = \frac{1}{2}\langle e, x\rangle e$. Then V_+ is self-adjoint and

$$\operatorname{ran} V_+ = \operatorname{span}\{e\}.$$

(v) Define $V_\circ x := \frac{1}{2}(V - V^*)(x) : t \mapsto \frac{1}{2}[\int_0^t x - \int_t^1 x]$, $\forall x \in L^2[0,1]$, $t \in [0,1]$. Then $V_\circ$ is anti-self-adjoint and

$$\operatorname{ran} V_\circ = \{x \in L^2[0,1]\colon x \text{ is absolutely continuous on } [0,1],\ x' \in L^2[0,1],\ x(0) = -x(1)\}.$$

Proof. (i) By Fact 4.2,

$$\langle x, Vx\rangle = \int_0^1 x(t)\int_0^t x(s)\,ds\,dt = \frac{1}{2}\left(\int_0^1 x(s)\,ds\right)^2 \geqslant 0,$$

so V is monotone.

As $\operatorname{dom} V = L^2[0,1]$ and V is continuous, $\operatorname{dom} V^* = L^2[0,1]$. Let $x, y \in L^2[0,1]$. We have

$$\begin{aligned}\langle Vx, y\rangle &= \int_0^1\int_0^t x(s)\,ds\,y(t)\,dt = \int_0^1 x(t)\,dt\int_0^1 y(s)\,ds - \int_0^1\int_0^t y(s)\,ds\,x(t)\,dt \\ &= \int_0^1\left(\int_0^1 y(s)\,ds - \int_0^t y(s)\,ds\right)x(t)\,dt = \int_0^1\int_t^1 y(s)\,ds\,x(t)\,dt = \langle V^*y, x\rangle,\end{aligned}$$

thus $(V^*y)(t) = \int_t^1 y(s)\,ds$, $\forall t \in [0,1]$.

(ii) To show (24), if $z \in \operatorname{ran} V$, then

$$z(t) = \int_0^t x \quad \text{for some } x \in L^2[0,1],$$

and hence $z(0) = 0$, z is absolutely continuous, and $z' = x \in L^2[0,1]$. On the other hand, if $z(0) = 0$, z is absolutely continuous, $z' \in L^2[0,1]$, then $z = Vz'$.

To show (25), if $z \in \operatorname{ran} V^*$, then

$$z(t) = \int_t^1 x \quad \text{for some } x \in L^2[0,1],$$

and hence $z(1) = 0$, z is absolutely continuous, and $z' = -x \in L^2[0,1]$. On the other hand, if $z(1) = 0$, z is absolutely continuous, $z' \in L^2[0,1]$, then $z = V^*(-z')$.

(iii) follows from (ii) (or see [2]).

(iv) is clear.

(v) If x is absolutely continuous, $x(0) = -x(1)$, $x' \in L^2[0,1]$, we have

$$V_\circ x'(t) = \frac{1}{2}\left(\int_0^t x' - \int_t^1 x'\right) = \frac{1}{2}\big(x(t) - x(0) - x(1) + x(t)\big) = x(t).$$

This shows that $x \in \operatorname{ran} V_\circ$. Conversely, if $x \in \operatorname{ran} V_\circ$, i.e.,

$$x(t) = \frac{1}{2}\int_0^t y - \frac{1}{2}\int_t^1 y \quad \text{for some } y \in L^2[0,1],$$

then x is absolutely continuous, $x' = y \in L^2[0,1]$ and $x(0) = -x(1) = -\frac{1}{2}\int_0^1 y$. $\square$

Theorem 4.5 *(Inverse Volterra operator = Differentiation operator). Let $X = L^2[0,1]$, and V be the Volterra integration operator. We let $T = V^{-1}$ and $D = \operatorname{dom} T \cap \operatorname{dom} T^*$. Then the following hold.*

(i) $T : \operatorname{dom} T \to X$ *is given by* $Tx = x'$ *with*

$$\operatorname{dom} T = \{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous},\ x(0) = 0,\ x' \in L^2[0,1]\},$$

and $T^* : \operatorname{dom} T^* \to X$ *is given by* $T^*x = -x'$ *with*

$$\operatorname{dom} T^* = \{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous},\ x(1) = 0,\ x' \in L^2[0,1]\}.$$

Both T *and* T^* *are maximal monotone linear operators.*

(ii) T *is neither skew nor symmetric.*

(iii) *The linear subspace*

$$D = \{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous},\ x(0) = x(1) = 0,\ x' \in L^2[0,1]\}$$

is dense in X*. Moreover,* T *and* T^* *are skew on* D.

Proof. (i) T and T^* are maximal monotone because $T = V^{-1}$, and $T^* = (V^{-1})^* = (V^*)^{-1}$ and Example 4.4(i). By Example 4.4(ii), $T : L^2[0,1] \to L^2[0,1]$ has

$$\operatorname{dom} T = \{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous},\ x(0) = 0,\ x' \in L^2[0,1]\},$$
$$\operatorname{dom} T^* = \{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous},\ x(1) = 0,\ x' \in L^2[0,1]\},$$
$$Tx = x', \quad \forall x \in \operatorname{dom} T, \quad \text{and} \quad T^*y = -y', \quad \forall y \in \operatorname{dom} T^*.$$

Note that by Fact 4.2,

$$\langle Tx, x\rangle = \int_0^1 x'x = \frac{1}{2}x^2(1) - \frac{1}{2}x^2(0) = \frac{1}{2}x(1)^2, \quad \forall x \in \operatorname{dom} T, \tag{26}$$

$$\langle T^*x, x\rangle = \int_0^1 -x'x = -\left(\frac{1}{2}x(1)^2 - \frac{1}{2}x(0)^2\right) = \frac{1}{2}x(0)^2, \quad \forall x \in \operatorname{dom} T^*. \tag{27}$$

(ii) Letting $x(t) = t,\ y(t) = t^2$ we have

$$\langle Tx, x\rangle = \int_0^1 t = \frac{1}{2}, \qquad \langle x, Ty\rangle = \int_0^1 2t^2 = \frac{2}{3} \neq \frac{1}{3} = \int_0^1 t^2 = \langle Tx, y\rangle \quad \Rightarrow \quad \langle Tx, x\rangle \neq 0, \qquad \langle Tx, y\rangle \neq \langle x, Ty\rangle.$$

(iii) By (i), $D = \operatorname{dom} T \cap \operatorname{dom} T^*$ is clearly a linear subspace. For $x \in D$, $x(0) = x(1) = 0$, from (26) and (27),

$$\langle Tx, x\rangle = \frac{1}{2}x(1)^2 = 0, \qquad \langle T^*x, x\rangle = \frac{1}{2}x(0)^2 = 0.$$

Hence both T and T^* are skew on D. The fact that D is dense in $L^2[0,1]$ follows from [19, Theorem 6.111]. □

Our proof of (ii), (iii) in the following theorem follows the ideas of [16, Example 13.4].

Theorem 4.6 *(The skew part of inverse Volterra operator). Let* $X = L^2[0,1]$*, and* T *be defined as in Theorem* 4.5*. Let* $S := \frac{T-T^*}{2}$.

(i) $Sx = x'$ $(\forall x \in \operatorname{dom} S)$ *and* $\operatorname{gra} S = \{(Vx, x) \mid x \in e^{\perp}\}$*, where* $e \equiv 1 \in L^2[0,1]$*. In particular,*

$$\operatorname{dom} S = \{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous},\ x(0) = x(1) = 0,\ x' \in L^2[0,1]\},$$
$$\operatorname{ran} S = \{y \in L^2[0,1]\colon\ \langle e, y\rangle = 0\} = e^{\perp}.$$

Moreover, $\operatorname{dom} S$ *is dense, and*

$$S^{-1} = V|_{e^{\perp}}, \qquad (-S)^{-1} = V^*|_{e^{\perp}}, \tag{28}$$

consequently, S *is skew, and neither* S *nor* $-S$ *is maximal monotone.*

(ii) *The adjoint of S has* $\operatorname{gra} S^* = \{(V^*x^* + le, x^*) \mid x^* \in X,\ l \in \mathbb{R}\}$. *More precisely,*

$$S^*x = -x', \quad \forall x \in \operatorname{dom} S^*, \quad \textit{with}$$
$$\operatorname{dom} S^* = \big\{x \in L^2[0,1]\colon\ x \textit{ is absolutely continuous on } [0,1],\ x' \in L^2[0,1]\big\},$$
$$\operatorname{ran} S^* = L^2[0,1].$$

Neither S^* nor $-S^*$ is monotone. Moreover, $S^{**} = S$.

(iii) *Let $T_1 : \operatorname{dom} T_1 \to X$ be defined by*

$$T_1 x = x', \quad \forall x \in \operatorname{dom} T_1 := \big\{x \in L^2[0,1]\colon\ x \textit{ is absolutely continuous},\ x(0) = x(1),\ x' \in L^2[0,1]\big\}.$$

Then $T_1^* = -T_1$,

$$\operatorname{ran} T_1 = e^{\perp}. \tag{29}$$

Hence T_1 is skew, and a maximal monotone extension of S; and $-T_1$ is skew and a maximal monotone extension of $-S$.

Proof. (i) By Theorem 4.5(iii), we directly get $\operatorname{dom} S$. Now $(\forall x \in \operatorname{dom} S = \operatorname{dom} T \cap \operatorname{dom} T^*)$ $Tx = x'$ and $T^*x = -x'$, so $Sx = x'$. Then Example 4.4(iii) implies $\operatorname{gra} S = \{(Vx, x) \mid x \in e^{\perp}\}$. Hence

$$\operatorname{gra} S^{-1} = \big\{(x, Vx)\colon\ x \in e^{\perp}\big\}. \tag{30}$$

Theorem 4.5(iii) implies $\operatorname{dom} S$ is dense. Furthermore, $\operatorname{gra}(-S) = \{(Vx, -x)\colon\ x \in e^{\perp}\}$, so

$$\operatorname{gra}(-S)^{-1} = \big\{(x, -Vx)\colon\ x \in e^{\perp}\big\}.$$

Since

$$V^*x(t) = \int_t^1 x - 0 = \int_t^1 x - \int_0^1 x = -\int_0^t x = -Vx(t), \quad \forall t \in [0,1],\ \forall x \in e^{\perp},$$

we have $-Vx = V^*x$, $\forall x \in e^{\perp}$. Then

$$\operatorname{gra}(-S)^{-1} = \big\{\big(x, V^*x\big)\colon\ x \in e^{\perp}\big\}. \tag{31}$$

Hence, (30) and (31) together establish (28). As both V, V^* are maximal monotone with full domain, we conclude that $S^{-1}, (-S)^{-1}$ are not maximal monotone, thus $S, -S$ are not maximal monotone.

(ii) By (i), we have

$$\begin{aligned}\big(x, x^*\big) \in \operatorname{gra} S^* \quad &\Leftrightarrow \quad \langle -x, y\rangle + \big\langle x^*, Vy\big\rangle = 0, \quad \forall y \in e^{\perp}\\ &\Leftrightarrow \quad \big\langle -x + V^*x^*, y\big\rangle = 0, \quad \forall y \in e^{\perp} \quad \Leftrightarrow \quad x - V^*x^* \in \operatorname{span}\{e\}.\end{aligned}$$

Equivalently, $x = V^*x^* + ke$ for some $k \in \mathbb{R}$. This means that x is absolutely continuous, $x^* = -x' \in L^2[0,1]$. On the other hand, if x is absolutely continuous and $x' \in L^2[0,1]$, observe that

$$x(t) = \int_t^1 -x' + x(1)e,$$

so that $x - V^*(-x') \in \operatorname{span}\{e\}$ and $(x, -x') \in \operatorname{gra} S^*$. It follows that

$$\operatorname{dom} S^* = \big\{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous on } [0,1],\ x' \in L^2[0,1]\big\},$$
$$\operatorname{ran} S^* = L^2[0,1], \quad \text{and}$$
$$S^*x = -x', \quad \forall x \in \operatorname{dom} S^*.$$

Since

$$\big\langle S^*x, x\big\rangle = -\int_0^1 x'x = -\left(\frac{1}{2}x(1)^2 - \frac{1}{2}x(0)^2\right),$$

we conclude that neither S^* nor $-S^*$ is monotone.

We proceed to show that $S^{**}=S$. Note that $\forall x\in\operatorname{dom}S^*$, $z\in\operatorname{dom}S$, we have $z(0)=z(1)=0$ and

$$\langle S^*x,z\rangle=\int_0^1 -x'z=-\left(x(1)z(1)-x(0)z(0)-\int_0^1 xz'\right)=\int_0^1 xz'=\langle x,Sz\rangle,$$

this implies that $S^{**}z=Sz$, $\forall z\in\operatorname{dom}S$, i.e., $S^{**}|_{\operatorname{dom}S}=S$. Suppose now that $x\in\operatorname{dom}S^{**}$, $\varphi=S^{**}x$. Put $\Phi=V$. Then $\forall z\in\operatorname{dom}S^*$,

$$\begin{aligned}\langle S^*z,x\rangle&=\int_0^1 -z'x=\langle z,S^{**}x\rangle\\&=\langle z,\varphi\rangle=\int_0^1 z\varphi=\bigl[z(1)\Phi(1)-z(0)\Phi(0)\bigr]-\int_0^1 \Phi z'\\&=z(1)\Phi(1)-\int_0^1\Phi z'.\end{aligned}$$

Using $z=e\in\operatorname{dom}S^*$ gives $\Phi(1)=0$. It follows that

$$\int_0^1[\Phi-x]z'=0,\quad \forall z\in\operatorname{dom}S^*\quad\Rightarrow\quad \Phi-x\in\left(\operatorname{ran}S^*\right)^{\perp},$$

then $\Phi=x$ since $\operatorname{ran}S^*=L^2[0,1]$. As $\Phi(1)=\Phi(0)=0$ and Φ is absolutely continuous, we have $x\in\operatorname{dom}S$. Since $x\in\operatorname{dom}S^{**}$ was arbitrary, we conclude that $\operatorname{dom}S^{**}\subseteq\operatorname{dom}S$. Hence $S^{**}=S$. (Alternatively, V is continuous $\Rightarrow$ $V|_{e^\perp}$ has closed graph $\Rightarrow$ S^{-1} has closed graph $\Rightarrow$ S has closed graph $\Rightarrow$ $\operatorname{gra}S=\operatorname{gra}S^{**}$ $\Rightarrow$ $S^{**}=S$.)

(iii) To show (29), suppose that x is absolutely continuous and that $x(0)=x(1)$. Then

$$\int_0^1 x'=x(1)-x(0)=0\quad\Rightarrow\quad T_1x=x'\in e^{\perp}.$$

Conversely, if $x\in L^2[0,1]$ satisfies $\langle e,x\rangle=0$, we define $z=Vx$, then z is absolutely continuous, $z(0)=z(1)$, $T_1z=x$. Hence $\operatorname{ran}T_1=e^{\perp}$.

T_1 is skew, because for every $x\in\operatorname{dom}T_1$, we have

$$\langle T_1x,x\rangle=\int_0^1 x'x=\frac12x(1)^2-\frac12x(0)^2=0.$$

Moreover, $T_1^*=-T_1$: indeed, as T_1 is skew, by Fact 2.2, $\operatorname{gra}(-T_1)\subseteq\operatorname{gra}T_1^*$. To show that $T_1^*=-T_1$, take $z\in\operatorname{dom}T_1^*$, $\varphi=T_1^*z$. Put $\Phi=V\varphi$. We have $\forall y\in\operatorname{dom}T_1$,

$$\int_0^1 y'z=\langle T_1y,z\rangle=\langle T_1^*z,y\rangle=\langle\varphi,y\rangle=\int_0^1 y\varphi=\int_0^1 y\Phi' \tag{32}$$

$$=\bigl[\Phi(1)y(1)-\Phi(0)y(0)\bigr]-\int_0^1\Phi y'. \tag{33}$$

Using $y=e\in\operatorname{dom}T_1$ gives $\Phi(1)-\Phi(0)=0$, from which $\Phi(1)=\Phi(0)=0$. It follows from (32)–(33) that $\int_0^1 y'(z+\Phi)=0$, $\forall y\in\operatorname{dom}T_1$. Since $\operatorname{ran}T_1=e^{\perp}$, $z+\Phi\in\operatorname{span}\{e\}$, say $z+\Phi=ke$ for some constant $k\in\mathbb{R}$. Then z is absolutely continuous, $z(0)=z(1)$ since $\Phi(0)=\Phi(1)=0$, and $T_1^*z=\varphi=\Phi'=-z'$. This implies that $\operatorname{dom}T_1^*\subseteq\operatorname{dom}T_1$. Then by Fact 2.2, $T_1^*=-T_1$. It remains to apply Proposition 4.3. □

Fact 4.7. *Let $A:X\rightrightarrows X$ be a multifunction. Then $(-A)^{-1}=A^{-1}\circ(-\mathrm{Id})$. If A is a linear relation, then*

$$(-A)^{-1}=-A^{-1}.$$

Proof. This follows from the set-valued inverse definition. Indeed, $x \in (-A)^{-1}(x^*) \Leftrightarrow (x, x^*) \in \operatorname{gra}(-A) \Leftrightarrow (x, -x^*) \in \operatorname{gra} A \Leftrightarrow x \in A^{-1}(-x^*)$. When A is a linear relation, $x \in (-A)^{-1}(x^*) \Leftrightarrow (x, -x^*) \in \operatorname{gra} A \Leftrightarrow (-x, x^*) \in \operatorname{gra} A \Leftrightarrow -x \in A^{-1}x^* \Leftrightarrow x \in -A^{-1}(x^*)$. $\square$

Theorem 4.8 *(The inverse of the skew part of Volterra operator). Let $X = L^2[0,1]$, and V be the Volterra integration operator, and $V_\circ : L^2[0,1] \to L^2[0,1]$ be given by*

$$V_\circ = \frac{V - V^*}{2}.$$

Define $T_2 : \operatorname{dom} T_2 \to L^2[0,1]$ by $T_2 = V_\circ^{-1}$. Then

(i) *$T_2 x = x'$, $\forall x \in \operatorname{dom} T_2$ where*

$$\operatorname{dom} T_2 = \bigl\{x \in L^2[0,1]\colon\ x \text{ is absolutely continuous on } [0,1],\ x' \in L^2[0,1],\ x(0) = -x(1)\bigr\}. \tag{34}$$

(ii) *$T_2^* = -T_2$, and both $T_2, -T_2$ are maximal monotone and skew.*

Proof. (i) Since

$$V_\circ x(t) = \frac{1}{2}\left(\int_0^t x - \int_t^1 x\right),$$

$V_\circ$ is a one-to-one map. Then

$$V_\circ^{-1}\left(\frac{1}{2}\left(\int_0^t x - \int_t^1 x\right)\right) = x(t) = \left(\frac{1}{2}\left(\int_0^t x - \int_t^1 x\right)\right)',$$

which implies $T_2 x = V_\circ^{-1} x = x'$ for $x \in \operatorname{ran} V_\circ$. As $\operatorname{dom} T_2 = \operatorname{ran} V_\circ$, by Example 4.4(v), $\operatorname{ran} V_\circ$ can be written as (34).

(ii) Since $\operatorname{dom} V = \operatorname{dom} V^* = L^2[0,1]$, $V_\circ$ is skew on $L^2[0,1]$, so maximal monotone. Then $T_2 = V_\circ^{-1}$ is maximal monotone.

Since $V_\circ$ is skew and $\operatorname{dom} V_\circ = L^2[0,1]$, we have $V_\circ^* = -V_\circ$, by Fact 4.7,

$$T_2^* = \left(V_\circ^{-1}\right)^* = \left(V_\circ^*\right)^{-1} = (-V_\circ)^{-1} = -V_\circ^{-1} = -T_2.$$

By Proposition 4.3, both T_2 and $-T_2$ are maximal monotone and skew. $\square$

Remark 4.9. Note that while $V_\circ$ is continuous on $L^2[0,1]$, the operator S given in Example 3.1 is discontinuous.

Combining Theorem 4.5, Theorem 4.6 and Theorem 4.8, we can summarize the nice relationships among the differentiation operators encountered in this section.

Corollary 4.10. *The domain of the skew operator S is dense in $L^2[0,1]$. Neither S nor $-S$ is maximal monotone. Neither S^* nor $-S^*$ is monotone.*

The linear operators S, T, T_1, T_2 satisfy:

$$\operatorname{gra} S \subsetneqq \operatorname{gra} T \subsetneqq \operatorname{gra}\bigl(-S^*\bigr),$$
$$\operatorname{gra} S \subsetneqq \operatorname{gra} T_1 \subsetneqq \operatorname{gra}\bigl(-S^*\bigr),$$
$$\operatorname{gra} S \subsetneqq \operatorname{gra} T_2 \subsetneqq \operatorname{gra}\bigl(-S^*\bigr).$$

While S is skew, T, T_1, T_2 are maximal monotone and T_1, T_2 are skew. Also,

$$\operatorname{gra}(-S) \subsetneqq \operatorname{gra}\bigl(T^*\bigr) \subsetneqq \operatorname{gra} S^*,$$
$$\operatorname{gra}(-S) \subsetneqq \operatorname{gra}(-T_1) \subsetneqq \operatorname{gra} S^*,$$
$$\operatorname{gra}(-S) \subsetneqq \operatorname{gra}(-T_2) \subsetneqq \operatorname{gra} S^*.$$

While $-S$ is skew, $T^*, -T_1, -T_2$ are maximal monotone and $-T_1, -T_2$ are skew.

Remark 4.11. (i) Note that while T_1, T_2 are maximal monotone, $-T_1, -T_2$ are also maximal monotone. This is in stark contrast with the maximal monotone skew operator given in Proposition 3.5 and Proposition 3.8 such that its negative is not maximal monotone.

(ii) Even though the skew operator S in Theorem 4.6 has $\operatorname{dom} S$ dense in $L^2[0,1]$, it still admits two distinct maximal monotone and skew extensions T_1, T_2.

4.2. Consequences on sum of maximal monotone operators and Fitzpatrick functions of a sum

Example 4.12 *($T + T^*$ fails to be maximal monotone).* Let T be defined as in Theorem 4.5. Now $\forall x \in \operatorname{dom} T \cap \operatorname{dom} T^*$, we have

$$Tx + T^*x = x' - x' = 0.$$

Thus $T + T^*$ has a proper monotone extension from $\operatorname{dom} T \cap \operatorname{dom} T^* \subsetneqq X$ to the 0 map on X. Consequently, $T + T^*$ is not maximal monotone. Note that $\operatorname{dom} T \cap \operatorname{dom} T^*$ is dense in X and that $\operatorname{dom} T - \operatorname{dom} T^*$ is a dense subspace of X. This supplies a simpler example for showing that the constraint qualification in the sum problem of maximal monotone operators cannot be substantially weakened, see [13, Example 7.4]. Similarly, by Theorems 4.6 and 4.8, $T_i^* = -T_i$, we conclude that $T_i + T_i^* = 0$ on $\operatorname{dom} T_i$, a dense subset of $L^2[0,1]$; thus, $T_i + T_i^*$ fails to be maximal monotone while both T_i, T_i^* are maximal monotone.

To study Fitzpatrick functions of sums of maximal monotone operators, we need

Lemma 4.13. *Let V be the Volterra integration operator (see Example 4.4). Then*

$$q_{V_+}^*(z) = \iota_{\operatorname{span}\{e\}}(z) + \langle z, e\rangle^2, \quad \forall z \in X,$$

where $e \equiv 1 \in L^2[0,1]$.

Proof. Let $z \in X$. By Example 4.4(iv) and Fact 2.5, we have

$$q_{V_+}^*(z) = \infty, \quad \text{if } z \notin \operatorname{span}\{e\}.$$

Now suppose that $z = le$ for some $l \in \mathbb{R}$. By Example 4.4(iv),

$$\begin{aligned} q_{V_+}^*(z) &= \sup_{x\in X}\bigl\{\langle x, z\rangle - q_{V_+}(x)\bigr\} = \sup_{x\in X}\left\{\langle x, le\rangle - \frac{1}{4}\langle x, e\rangle^2\right\} \\ &= l^2 = \langle le, e\rangle^2 = \langle z, e\rangle^2. \end{aligned}$$

Hence $q_{V_+}^*(z) = \iota_{\operatorname{span}\{e\}}(z) + \langle z, e\rangle^2$. $\square$

Lemma 4.14. *Let T be defined as in Theorem 4.5. We have*

$$\begin{aligned} F_T(x, y^*) &= F_V(y^*, x) = \iota_{\operatorname{span}\{e\}}(x + V^*y^*) + \frac{1}{2}\langle x + V^*y^*, e\rangle^2, \\ F_{T^*}(x, y^*) &= F_{V^*}(y^*, x) = \iota_{\operatorname{span}\{e\}}(x + Vy^*) + \frac{1}{2}\langle x + Vy^*, e\rangle^2, \quad \forall (x, y^*) \in X \times X. \end{aligned} \tag{35}$$

Proof. Apply Fact 2.4, Fact 2.5 and Lemma 4.13. $\square$

Remark 4.15. Theorem 4.16 below gives another example showing that $F_{T+T^*} \neq F_T \,\square_2\, F_{T^*}$ while T, T^* are maximal monotone, and $\operatorname{dom} T - \operatorname{dom} T^*$ is a dense subspace in $L^2[0,1]$. Moreover, $\operatorname{ran}(T_+ + (T^*)_+) = \{0\}$. This again shows that the assumption that $\operatorname{dom} A - \operatorname{dom} B$ is closed in Fact 2.8(ii) cannot be weakened substantially, and that Fact 2.8(i) fails for discontinuous linear monotone operators.

Theorem 4.16. *Let T be defined as in Theorem 4.5, and set*

$$H := \bigl\{x \in L^2[0,1]\colon \text{ x is absolutely continuous, and } x' \in L^2[0,1]\bigr\}.$$

Then

$$\begin{aligned} F_{T+T^*}(x, x^*) &= \iota_{X\times\{0\}}(x, x^*), \quad \forall (x, x^*) \in X \times X, \\ F_T \,\square_2\, F_{T^*}(x, x^*) &= \begin{cases} \frac{1}{2}[x(1)^2 + x(0)^2], & \text{if } (x, x^*) \in H \times \{0\}; \\ \infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{36}$$

Consequently, $F_T \,\square_2\, F_{T^*} \neq F_{T+T^*}$.

Proof. By Theorem 4.5(i) and Example 4.4(iii),

$$\big(T+T^*\big)y=0,\quad \forall y\in \operatorname{dom}T\cap\operatorname{dom}T^*=\big\{Vx \,\big|\, x\in e^{\perp}\big\},\tag{37}$$

where $e\equiv 1\in L^2[0,1]$. Let $(x,x^*)\in X\times X$. Using Theorem 4.5(i), we see that

$$F_{T+T^*}\big(x,x^*\big)=\sup_{y\in\operatorname{dom}T\cap\operatorname{dom}T^*}\big\langle x^*,y\big\rangle=\sup_{y\in X}\big\langle x^*,y\big\rangle=\iota_{\{0\}}\big(x^*\big)=\iota_{X\times\{0\}}\big(x,x^*\big).\tag{38}$$

By Fact 2.7, we have

$$(F_T\,\Box_2\,F_{T^*})\big(x,x^*\big)=\infty,\quad \forall x^*\neq 0.\tag{39}$$

When $x^*=0$, by (35),

$$\begin{aligned}(F_T\,\Box_2\,F_{T^*})(x,0)&=\inf_{y^*\in X}\big\{F_T\big(x,y^*\big)+F_{T^*}\big(x,-y^*\big)\big\}\\&=\inf_{y^*\in X}\bigg\{\iota_{\operatorname{span}\{e\}}\big(x+V^*y^*\big)+\frac{1}{2}\big\langle x+V^*y^*,e\big\rangle^2+\iota_{\operatorname{span}\{e\}}\big(x-Vy^*\big)+\frac{1}{2}\big\langle x-Vy^*,e\big\rangle^2\bigg\}.\end{aligned}\tag{40}$$

Observe that

$$\begin{aligned}&x+V^*y^*\in\operatorname{span}\{e\},\qquad x-Vy^*\in\operatorname{span}\{e\}\\&\quad\Leftrightarrow\quad x-Vy^*+Vy^*+V^*y^*\in\operatorname{span}\{e\},\qquad x-Vy^*\in\operatorname{span}\{e\}\\&\quad\Leftrightarrow\quad x-Vy^*\in\operatorname{span}\{e\}\quad\big(\text{by Example 4.4(iv)}\big)\\&\quad\Leftrightarrow\quad x\in Vy^*+\operatorname{span}\{e\}\quad\Leftrightarrow\quad x\text{ is absolutely continuous and }y^*=x'.\end{aligned}$$

Therefore, $(F_T\,\Box_2\,F_{T^*})(x,0)=\infty$ if $x\notin H$. For $x\in H$, using (40) and the fact that $x-Vx'=x(0)e$ and $x+V^*x'=x(1)e$, we obtain

$$\begin{aligned}(F_T\,\Box_2\,F_{T^*})(x,0)&=\frac{1}{2}\big\langle x+V^*x',e\big\rangle^2+\frac{1}{2}\big\langle x-Vx',e\big\rangle^2\\&=\frac{1}{2}x(1)^2+\frac{1}{2}x(0)^2=\frac{1}{2}\big[x(1)^2+x(0)^2\big].\end{aligned}$$

Thus, (36) holds. Consequently, $F_T\,\Box_2\,F_{T^*}\neq F_{T+T^*}$. $\Box$

Finally, we remark that the examples given in Sections 3 and 4 have important consequences on *decompositions of monotone operator*, namely Borwein–Wiersma decomposition and Asplund decomposition [7]. This will be addressed in the forthcoming paper [6].

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