



Boundary controllability for the semilinear Schrödinger equations on Riemannian manifolds [☆]

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ABSTRACT

We study the boundary exact controllability for the semilinear Schrödinger equation defined on an open, bounded, connected set Ω of a complete, n -dimensional, Riemannian manifold M with metric g . We prove the locally exact controllability around the equilibria under some checkable geometrical conditions. Our results show that exact controllability is geometrical characters of a Riemannian metric, given by the coefficients and equilibria of the semilinear Schrödinger equation. We then establish the globally exact controllability in such a way that the state of the semilinear Schrödinger equation moves from an equilibrium in one location to an equilibrium in another location.

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1. Introduction and the main results

Let M be a complete n -dimensional, Riemannian manifold of class C^3 with C^3 -metric $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ and squared norm $|X|^2 = \langle X, X \rangle$. Let Ω be an open, bounded, connected set of M with smooth boundary $\partial\Omega \equiv \Gamma = \overline{\Gamma_0} \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Here Γ_1 is the uncontrolled or unobserved part of Γ and Γ_0 is the controlled or observed part of Γ , both relatively open in Γ . Let ν be the outward unit normal field along the boundary Γ . We denote by ∇ the gradient, by D the Levi-Civita connection, by D^2 the Hessian, by $\Delta = \text{div}(\nabla)$ the Laplace (Laplace–Beltrami) operator in the metric g .

Let $T > 0$ be given. We consider a controllability problem

$$\begin{cases} iu_t = -a(x, u)\Delta u + i(F(x, u), \nabla u) + b(x, u) & \text{in } (0, T) \times \Omega, \\ u = \varphi & \text{on } (0, T) \times \Gamma_0, \quad u = 0 & \text{on } (0, T) \times \Gamma_1, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $a(x, \xi) = \tilde{a}(x, y, z)$ and $F(x, \xi) = \tilde{F}(x, y, z)$ are real and $b(x, \xi) = \tilde{b}(x, y, z)$ is complex for $\xi = y + iz$, $y, z \in \mathbb{R}$. We assume that $\tilde{F}(x, y, z) : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{X}(M)$ (all vector fields on M), and $\tilde{b}(x, y, z) : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ are smooth with $b(x, 0) = 0$. Moreover, we assume that there is a constant $\lambda > 0$ such that

$$\tilde{a}(x, y, z) \geq \lambda, \quad \forall (x, y, z) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}. \quad (1.2)$$

Let u_0, u_1 be given functions on $\overline{\Omega}$ and let $T > 0$ be given. If there is a boundary function φ on $(0, T) \times \Gamma_0$ such that the solution of the problem (1.1) satisfies $u(T) = u_1$ on Ω , we say that the system (1.1) is exactly controllable from u_0 to u_1 at time T by boundary with the Dirichlet action.

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Controllability of the Schrödinger equation has been an active topic for a long time and a wealth of results on this subject are available in the literature. First, [9] considers the pure Schrödinger equation $i w_t + \Delta w = 0$ in the Euclidean case. Variable coefficients (in space) of the principal part are not permitted, nor are H^1 -energy level terms, by the multiplier methods used in [9]. The same multiplier is just one tool used also in [5], [6, Section 10.9, p. 1042]. Subsequent references, beginning with [11] and continuing with [13,16,7], have greatly generalized the original pure Schrödinger equation in the Euclidean domain. Different approaches have been pursued, all sharing the goal of seeking preliminary Carleman-type estimates: a unifying pseudo-differential approach in [11,12] under a pseudo-convexity assumption; a Riemannian geometric approach yielding Carleman-type estimates [16]. In particular, in [14] and [15] the authors study control problems of the linear Schrödinger equation on Riemannian manifolds.

We study boundary controllability of the semilinear problem (1.1) by some ideas from Yao [18].

Let us choose some Sobolev spaces to formulate our problems. Let

$$2k \geq [n/2] + 4$$

be a given positive integer. We assume initial data $u^0 \in H^{2k-1}(\Omega)$ to study the possibility of moving it to another state in $H^{2k-1}(\Omega)$ at time T via a boundary control $\varphi \in \bigcap_{l=0}^{k-1} C^l([0, T]; H^{2k-3/2-2l}(\Gamma_0))$.

We say $\omega \in H^{2k-1}(\Omega)$ is an equilibrium of the system (1.1) if

$$-a(x, \omega)\Delta\omega + i(F(x, \omega), \nabla\omega) + b(x, \omega) = 0 \quad \text{in } \Omega. \quad (1.3)$$

Since $b(x, 0) = 0$, $\omega = 0$ is an equilibrium of the system (1.1).

Let

$$H_{\Gamma_1}^1(\Omega) = \{v \mid v \in H^1(\Omega), v|_{\Gamma_1} = 0\}.$$

We say that $u_0 \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ and $\varphi \in \bigcap_{l=0}^{k-1} C^l([0, T]; H^{2k-3/2-2l}(\Gamma_0))$ satisfy the compatibility conditions of $2k-1$ order if

$$u_l \in H^{2k-1-2l}(\Omega), \quad u_l|_{\Gamma_1} = 0, \quad \varphi^{(l)}(0) = u_l|_{\Gamma_0}, \quad l = 0, 1, \dots, k-1, \quad (1.4)$$

where for $l \geq 1$,

$$u_l = u^{(l)}(0), \quad (1.5)$$

as computed formally (and recursively) in terms of u_0 , using the equation in (1.1).

Let $\omega \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ be an equilibrium of system (1.1). We define a new metric g_1 on the manifold M

$$g_1 = a^{-1}(x, \omega)g \quad (1.6)$$

as a Riemannian metric on $\overline{\Omega}$ and consider the couple (Ω, g_1) as a Riemannian manifold with the boundary Γ . We denote the inner product induced by g_1 by $\langle \cdot, \cdot \rangle_{g_1}$. Let $x^0 \in \overline{\Omega}$ be given. We denote the distance function from $x \in \overline{\Omega}$ to x^0 under the Riemannian metric g_1 by $\rho(x)$.

Definition. An equilibrium $\omega \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ is called exactly controllable if there is $x^0 \in \overline{\Omega}$ and $\rho_0 > 0$ such that

$$D_{g_1}^2 \rho^2(X, X) \geq \rho_0 |X|_{g_1}^2, \quad \forall X \in M_x, x \in \overline{\Omega}, \quad (1.7)$$

and

$$\inf_{x \in \Omega} |D_{g_1} \rho^2(x)| > 0, \quad (1.8)$$

where $D_{g_1}^2 \rho^2$ denotes the Hessian of the function ρ^2 under the metric g_1 , D_{g_1} denotes the connection under the metric g_1 , and M_x is the tangent space at the point x .

It is well known that the condition (1.7) is locally true. For any $x_0 \in \overline{\Omega}$, there is a neighborhood at x_0 such that the condition (1.7) holds. Whether the condition (1.7) is true depends on the sectional curvature of the metric g_1 . There are some criteria for the condition (1.7) to hold by curvature in Yao [18] and Triggiani [16].

Let

$$h(x) = \rho(x) D_{g_1} \rho, \quad x \in \Omega. \quad (1.9)$$

We have

Theorem 1.1. Let an equilibrium $\omega \in H^{2k-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable. Assume $\langle h(x), \nu_{g_1} \rangle_{g_1} \leq 0$ for $x \in \Gamma_1$, where $h(x)$ is given in (1.9) and ν_{g_1} is the outward normal to Γ in the metric g_1 . Then there is $\epsilon_0 > 0$ and $T_0 > 0$ such that, for any $0 < T < T_0$ given and for any $u_0^i \in H^{2k-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ with

$$\|u_0^i - \omega\|_{2k-1} \leq \epsilon_0, \quad i = 1, 2,$$

we can find $\varphi \in \bigcap_{l=0}^{k-2} C^l([0, T]; H^{2k-2l-3/2}(\Gamma_0))$ with $\varphi^{(k-1)} \in H^1((0, T) \times \Gamma_0)$ which is compatible with the initial data u_0^i of $2k-1$ order, to satisfy

$$u(T) = u_0^2.$$

The above is a local result. However, if we have enough equilibria exactly controllable, we can move one equilibrium to another along an equilibrium curve. This uses the open mapping theorem, locally exact controllability, and a compactness argument. This approach was used by Schmidt [10] for the quasilinear string.

We assume that the system (1.1) further satisfies the condition

$$F(x, \xi) = 0, \quad b(x, \xi) = b_1(x, \xi)\xi, \quad (x, \xi) \in \Omega \times \mathbb{C}, \quad (1.10)$$

where $b_1(x, \xi) = \tilde{b}_1(x, y, z) : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow [0, +\infty)$ is smooth for $\xi = y + iz$. Let $\omega \in H^{2k-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be a given equilibrium. For $\alpha \in [0, 1]$, we assume that $\omega_\alpha \in H^{2k-1}(\Omega)$ are the solutions of the boundary problem

$$\begin{cases} -a(x, \omega_\alpha)\Delta\omega_\alpha + b_1(x, \omega_\alpha)\omega_\alpha = 0, & x \in \Omega, \\ \omega_\alpha|_{\Gamma} = \alpha\omega|_{\Gamma}. \end{cases} \quad (1.11)$$

For the existence of the classical solution to the Dirichlet problem (1.11), for example, see Gilbarg and Trudinger [3].

Theorem 1.2. Suppose (1.10) holds in the system (1.1) and let an equilibrium $\omega \in H^{2k-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable. Let $\omega_\alpha \in H^{2k-1}(\Omega)$, given by (1.11), be also exactly controllable for all $\alpha \in [0, 1]$. Then, for $T > 0$ small, there is

$$\psi \in \bigcap_{l=0}^{k-2} C^l([0, T]; H^{2k-2l-3/2}(\Gamma_0))$$

with $\psi^{(k-1)} \in H^1((0, T) \times \Gamma_0)$ which is compatible with the initial data ω , such that the solution of the problem (1.1) with $u_0 = \omega$ satisfies $u(T) = 0$.

Since the semilinear Schrödinger equation is time-reversible, an equilibrium can be moved to another if they can both be moved to zero.

2. Existence of uniformly small time solutions

There are standard methods to obtain existence of short time solutions, for example, see [1] or [4]. However, intervals of small time for such solutions depend on initial data and boundary values. In order to carry out boundary control, we need a uniform existence interval (maybe small) for all initial data and boundary values around an equilibrium. In this section we establish such results, Theorem 2.1 below.

We consider the problem

$$\begin{cases} iu_t = -a(x, u)\Delta u + i\langle F(x, u), \nabla u \rangle + b(x, u) & \text{in } (0, T) \times \Omega, \\ u = \varphi & \text{on } (0, T) \times \Gamma, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.1)$$

The main results of this section are

Theorem 2.1. Let $k \geq \frac{1}{2}[n/2] + 2$ be given. Let $\omega \in H^{2k-1}(\Omega)$ be an equilibrium of the system (2.1). There exist $\epsilon_0 > 0$ and $T_0 > 0$ such that, for all $u_0 \in H^{2k-1}(\Omega)$ with

$$\|u_0 - \omega\|_{2k-1} + \sum_{l=1}^{k-1} \|u_l\|_{2k-1-2l} \leq \epsilon_0, \quad (2.2)$$

and for all $\varphi \in \bigcap_{l=0}^{k-2} C^l([0, T_0]; H^{2k-2l-3/2}(\Gamma))$ with $\varphi^{(k-1)} \in H^1((0, T_0) \times \Gamma)$ which satisfy the compatibility conditions with u_0 of $2k-1$ order, and such that

$$\sum_{l=0}^{k-2} \sup_{0 \leq t \leq T_0} \|\hat{\varphi}^{(l)}(t)\|_{H^{2k-2l-3/2}(\Gamma)}^2 + \|\hat{\varphi}^{(k-1)}\|_{H^1((0,T_0) \times \Gamma)}^2 \leq \epsilon_0, \quad (2.3)$$

where

$$\hat{\varphi} = \varphi - \omega|_{\Gamma}, \quad (2.4)$$

the system (2.1) has a solution $u \in \bigcap_{l=0}^{k-1} C^l([0, T_0]; H^{2k-1-2l}(\Omega))$.

We collect here a few basic properties of Sobolev spaces on the compact Riemannian manifold (Ω, g) to be involved in the sequel.

Lemma 2.1.

(i) Let $s_1 > s_2 \geq 0$. For any $\epsilon > 0$ there is $c_\epsilon > 0$ such that

$$\|w\|_{s_2}^2 \leq \epsilon \|w\|_{s_1}^2 + c_\epsilon \|w\|^2, \quad \forall w \in H^{s_1}(\Omega). \quad (2.5)$$

(ii) If $s > n/2$, then for each $k = 0, 1, \dots$, we have $H^{s+k}(\Omega) \subset C^k(\overline{\Omega})$ with continuous inclusion.

(iii) Let $s_j \geq 0$, $j = 1, \dots, k$ and $r \triangleq \min_{1 \leq i \leq k} \min_{j_1 \leq \dots \leq j_i} \{s_{j_1} + \dots + s_{j_i} - (i-1)([n/2] + 1)\} \geq 0$. Then there is a constant $c > 0$ such that

$$\|f_1 \cdots f_k\|_r \leq c \|f_1\|_{s_1} \cdots \|f_k\|_{s_k}, \quad \forall f_j \in H^{s_j}(\Omega), \quad 1 \leq j \leq k. \quad (2.6)$$

In addition, we need the following lemma, which is similar to Lemma 2.1 in Yao [17].

Lemma 2.2. Let X be a real vector field on the Riemannian manifold (M, g) . Then, for any $f \in H^1(\overline{\Omega})$ we have

$$2 \operatorname{Re} \langle \nabla \bar{f}, \nabla (X(f)) \rangle(x) = (DX + (DX)^\top)(\nabla \bar{f}, \nabla f)(x) + X(|\nabla f(x)|^2), \quad \forall x \in \overline{\Omega}, \quad (2.7)$$

where $(DX)^\top$ is the transposition of DX .

Proof. Let $x \in \overline{\Omega}$. Let E_1, \dots, E_n be a frame field normal at x . There are functions h_1, \dots, h_n on some neighborhood of x such that $X = \sum_{l=1}^n h_l E_l$. In addition, we have

$$X(f) = \sum_{l=1}^n h_l E_l(f), \quad \nabla f = \sum_{l=1}^n E_l(f) E_l, \quad \text{and} \quad |\nabla f|^2 = \sum_{l=1}^n (E_l(f))^2. \quad (2.8)$$

We compute

$$\begin{aligned} \langle \nabla \bar{f}, \nabla (X(f)) \rangle(x) &= \sum_{l=1}^n E_l(\bar{f}) E_l(X(f))(x) \\ &= \sum_{l=1}^n E_l(\bar{f})(x) \sum_{s=1}^n (E_l(h_s) E_s(f) + h_s E_l E_s(f))(x) \\ &= \sum_{l,s=1}^n E_l(h_s) E_l(\bar{f}) E_s(f)(x) + \sum_{s=1}^n h_s \left(\sum_{l=1}^n E_l(\bar{f}) E_s E_l(f) \right)(x), \end{aligned} \quad (2.9)$$

where $E_s E_l(f)(x) = E_l E_s f(x)$ is the second covariant differential of f . Then we have

$$\begin{aligned} 2 \operatorname{Re} \langle \nabla \bar{f}, \nabla (X(f)) \rangle(x) &= 2 \operatorname{Re} \sum_{l,s=1}^n E_l(h_s) E_l(\bar{f}) E_s(f)(x) + 2 \operatorname{Re} \sum_{s=1}^n h_s \left(\sum_{l=1}^n E_l(\bar{f}) E_s E_l(f) \right)(x) \\ &= \sum_{l,s=1}^n (E_l(h_s) + E_s(h_l)) E_l(\bar{f}) E_s(f)(x) + \sum_{s=1}^n h_s E_s \left(\sum_{l=1}^n |E_l(f)|^2 \right)(x) \\ &= (DX + (DX)^\top)(\nabla \bar{f}, \nabla f)(x) + X(|\nabla f|^2)(x), \end{aligned} \quad (2.10)$$

where we have used the identity $DX(E_l, E_s)(x) = E_l(h_s)$ established in Yao [17]. \square

Firstly, we need some estimates for the linear, time dependent problem which is introduced below. Given $T > 0$. Let u solve the problem

$$\begin{cases} iu_t = -a(x, t)\Delta u + i\langle F(t, x), \nabla u \rangle + f_1(t, x)u + f_2(t, x)\bar{u} + f(t, x) & \text{in } (0, T) \times \Omega, \\ u = v & \text{on } (0, T) \times \Gamma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (2.11)$$

where $a(t, x) : [0, T] \rightarrow \mathbb{R}$ satisfies

$$\lambda \leq a(t, x) \leq \lambda_1, \quad \forall (t, x) \in [0, T] \times \bar{\Omega}, \quad (2.12)$$

for some constant $\lambda_1 \geq \lambda > 0$. In the system (2.11), $F(t, x)$ is a real vector field and f_1, f_2 are complex-valued functions. They satisfy

$$\begin{aligned} a(t, x) &\in C([0, T]; C^2(\Omega)) \cap W^{1,\infty}((0, T); C(\Omega)), \quad \operatorname{ess\,sup}_{0 \leq t \leq T} |F(t, x)| < \infty, \\ f_1(t, x), f_2(t, x), f(t, x) &\in L^\infty((0, T); C(\Omega)). \end{aligned} \quad (2.13)$$

Lemma 2.3. *Given $T > 0$. Let $u_0 \in H^1(\Omega)$ and $v \in H^1((0, T) \times \Gamma)$. Then the system (2.11) has a unique solution $u \in C([0, T]; H^1(\Omega))$, which satisfies*

$$\|u(t)\|_1^2 \leq C \left(\|u_0\|_1^2 + \|v\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^t \|f\|_1^2 d\tau \right), \quad 0 \leq t \leq T, \quad (2.14)$$

and

$$\int_0^T \int_\Gamma |u_\nu|^2 d\Gamma d\tau \leq C \left(\|u(0)\|_1^2 + \|v\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^T \|f(\tau)\|_1^2 d\tau \right), \quad (2.15)$$

where the positive constant C depends on the coefficients of the system and the time T .

We define

$$g_1(t) = a^{-1}(t, x)g, \quad t \in [0, T]$$

as a new Riemannian metric depending on time in $\bar{\Omega}$ and consider the couple $(\Omega, g_1(t))$ as a Riemannian manifold with a boundary Γ . Let $G = \det(g_{ij})$, $G_1 = \det(g_{1ij})$, $(g^{ij}) = (g_{ij})^{-1}$ and $(g_1^{ij}) = (g_{1ij})^{-1}$. Then we have

$$G_1 = a^{-n}G, \quad \text{and} \quad (g_1^{ij}) = a(x, t)(g^{ij}). \quad (2.16)$$

Furthermore, we have the following lemma:

Lemma 2.4. *Let $x \in \bar{\Omega}$. We denote by M_x the tangent space of x , by ∇_{g_1} the gradient in the metric g_1 , by Δ_{g_1} the Laplace operator in the metric g_1 . Then we have*

$$\nabla_{g_1} u = a \nabla u, \quad \langle X, Y \rangle_{g_1} = \frac{1}{a} \langle X, Y \rangle, \quad \langle \nabla_{g_1} u, \nabla_{g_1} v \rangle_{g_1} = a \langle \nabla u, \nabla v \rangle, \quad (2.17)$$

for all $u, v \in H^1(\Omega)$, $X, Y \in M_x$.

Furthermore,

$$\Delta_{g_1} f = a \Delta f + \left(-\frac{n}{2} + 1 \right) \langle \nabla f, \nabla a \rangle, \quad \forall f \in H^2(\Omega). \quad (2.18)$$

Proof. Let $x \in \bar{\Omega}$. Let $\{x_i\}$ be a local coordinate around x . We have

$$\nabla_{g_1} u = \sum_{ij} g_1^{ij} u_{x_i} \frac{\partial}{\partial x_j} = a \sum_{ij} g^{ij} u_{x_i} \frac{\partial}{\partial x_j} = a \nabla u, \quad (2.19)$$

$$\langle \nabla_{g_1} u, \nabla_{g_1} v \rangle_{g_1} = \sum_{ij} u_{x_i} v_{x_j} g_1^{ij} = a \sum_{ij} u_{x_i} v_{x_j} g^{ij} = a \langle \nabla u, \nabla v \rangle, \quad (2.20)$$

$$\langle X, Y \rangle_{g_1} = \sum_{ij} X_i Y_j g_{1ij} = \frac{1}{a} \sum_{ij} X_i Y_j g_{ij} = \frac{1}{a} \langle X, Y \rangle, \quad (2.21)$$

where $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i}$.

We then show the relation between Δ_{g_1} and Δ .

Let $f \in H^2(\Omega)$. We invoke the relations (2.16) to compute

$$\begin{aligned} \Delta_{g_1} f &= \frac{1}{\sqrt{G_1}} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\sum_{l=1}^n g^{jl} \sqrt{G_1} \frac{\partial f}{\partial x_l} \right) = a^{n/2} \frac{1}{\sqrt{G}} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\sum_{l=1}^n a^{-n/2+1} g^{jl} \sqrt{G} \frac{\partial f}{\partial x_l} \right) \\ &= a \Delta f + \left(-\frac{n}{2} + 1 \right) \sum_{jl} a_{x_j} f_{x_l} g^{jl} = a \Delta f + \left(-\frac{n}{2} + 1 \right) \langle \nabla f, \nabla a \rangle. \quad \square \end{aligned} \quad (2.22)$$

The problem (2.11) is transformed into a new form

$$\begin{cases} i\tilde{u}_t = -\Delta_{g_1} \tilde{u} + i \langle F(t, x), \nabla_{g_1} \tilde{u} \rangle_{g_1} + \tilde{f}_1(t, x) \tilde{u} + f_2(t, x) \tilde{\tilde{u}} + \tilde{f}(t, x) & \text{in } (0, T) \times \Omega, \\ u = \tilde{v} & \text{on } (0, T) \times \Gamma, \\ u(0) = \tilde{u}_0 & \text{in } \Omega, \end{cases} \quad (2.23)$$

by means of the transformation

$$\tilde{u}(t, x) = a^{-\frac{1}{2}(-\frac{n}{2}+1)}(t, x) u(t, x), \quad (t, x) \in [0, T] \times \Omega, \quad (2.24)$$

where

$$\tilde{f}_1 = a^{-\frac{1}{2}(-\frac{n}{2}+1)} \left(\Delta a^{\frac{1}{2}(-\frac{n}{2}+1)} + i \langle F, \nabla a^{\frac{1}{2}(-\frac{n}{2}+1)} \rangle \right) + f_1 + \frac{1}{2} \left(\frac{n}{2} - 1 \right) i a^{-1} a_t, \quad (2.25)$$

$$\tilde{f} = f a^{-\frac{1}{2}(-\frac{n}{2}+1)}, \quad \tilde{v} = a^{-\frac{1}{2}(-\frac{n}{2}+1)} v \quad \text{and} \quad \tilde{u}_0 = a^{-\frac{1}{2}(-\frac{n}{2}+1)}(0, x) u_0. \quad (2.26)$$

Let

$$E(t) = \int_{\Omega} (|\nabla u(t)|^2 + |u(t)|^2) d\Omega \quad \text{and} \quad \tilde{E}(t) = \int_{\Omega} (|\nabla \tilde{u}|^2 + |\tilde{u}|^2) d\Omega. \quad (2.27)$$

Proof of Lemma 2.3. Let ν be the outward unit normal field along the boundary Γ in the metric g . We use Lemma 2.4 to obtain that

$$\nu_{g_1} \equiv \sqrt{a} \nu \quad (2.28)$$

is the outward unit normal field along the boundary Γ in the metric g_1 . Take a real-valued vector field H on Ω such that $H(t, x) = \nu_{g_1}(t, x)$ for $(t, x) \in [0, T] \times \Gamma$.

We denote the metric volume elements in the metrics g and g_1 by $d\Omega$ and $d\Omega_{g_1}$ respectively. We denote the metric surface elements in the induced metric g and g_1 by $d\Gamma$ and $d\Gamma_{g_1}$ respectively. Then we have the relations

$$d\Omega_{g_1} = a^{-\frac{n}{2}}(t, x) d\Omega \quad \text{and} \quad d\Gamma_{g_1} = a^{-\frac{n-1}{2}}(t, x) d\Gamma. \quad (2.29)$$

We give an estimate on $\int_{\Omega} |\nabla \tilde{u}(t)|^2 d\Omega$ for $0 \leq t \leq T$.

We multiply (2.23) by $2H(\tilde{u})$, integrate it by parts over $[0, t] \times \Omega$ for $0 \leq t \leq T$, and obtain

$$\begin{aligned} 2i \int_0^t \int_{\Omega} \tilde{u}_t H(\tilde{u}) d\Omega_{g_1} d\tau &= -2 \int_0^t \int_{\Gamma} \tilde{u} \nu_{g_1} H(\tilde{u}) d\Gamma_{g_1} d\tau + 2 \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} H(\tilde{u}) \rangle_{g_1} d\Omega_{g_1} d\tau \\ &\quad + 2 \int_0^t \int_{\Omega} i \langle F, \nabla_{g_1} \tilde{u} \rangle_{g_1} H(\tilde{u}) d\Omega_{g_1} d\tau \\ &\quad + 2 \int_0^t \int_{\Omega} (\tilde{f}_1 \tilde{u} + f_2 \tilde{\tilde{u}} + \tilde{f}) H(\tilde{u}) d\Omega_{g_1} d\tau. \end{aligned} \quad (2.30)$$

We integrate by parts with respect to t on the left-hand side of (2.30) to obtain

$$\begin{aligned}
 i \int_0^t \int_{\Omega} \tilde{u}_t H(\tilde{u}) d\Omega_{g_1} d\tau &= i \int_0^t \int_{\Omega} \tilde{u}_t H(\tilde{u}) a^{-\frac{n}{2}} d\Omega d\tau \\
 &= i \int_{\Omega} \tilde{u} H(\tilde{u}) a^{-\frac{n}{2}} d\Omega \Big|_0^t - i \int_0^t \int_{\Omega} \tilde{u} (a^{-\frac{n}{2}} H)_t(\tilde{u}) d\Omega d\tau - i \int_0^t \int_{\Omega} \tilde{u} H(\tilde{u}_t) a^{-\frac{n}{2}} d\Omega d\tau \\
 &= i \int_{\Omega} \tilde{u} H(\tilde{u}) a^{-\frac{n}{2}} d\Omega \Big|_0^t - i \int_0^t \int_{\Omega} \tilde{u} (a^{-\frac{n}{2}} H)_t(\tilde{u}) d\Omega d\tau - i \int_0^t \int_{\Omega} \tilde{u} H(\tilde{u}_t) d\Omega_{g_1} d\tau \\
 &= i \int_{\Omega} \tilde{u} H(\tilde{u}) a^{-\frac{n}{2}} d\Omega \Big|_0^t - i \int_0^t \int_{\Omega} \tilde{u} (a^{-\frac{n}{2}} H)_t(\tilde{u}) d\Omega d\tau - i \int_0^t \int_{\Gamma} \tilde{u} \tilde{u}_t \langle H, \nu_{g_1} \rangle_{g_1} d\Gamma_{g_1} d\tau \\
 &\quad + i \int_0^t \int_{\Omega} \tilde{u} \tilde{u}_t \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau + i \int_0^t \int_{\Omega} \tilde{u}_t H(\tilde{u}) d\Omega_{g_1} d\tau, \tag{2.31}
 \end{aligned}$$

which yields

$$\begin{aligned}
 2 \operatorname{Re} i \int_0^t \int_{\Omega} \tilde{u}_t H(\tilde{u}) d\Omega_{g_1} d\tau &= i \int_{\Omega} \tilde{u} H(\tilde{u}) a^{-\frac{n}{2}} d\Omega \Big|_0^t - i \int_0^t \int_{\Omega} \tilde{u} (a^{-\frac{n}{2}} H)_t(\tilde{u}) d\Omega d\tau \\
 &\quad - i \int_0^t \int_{\Gamma} \tilde{u} \tilde{u}_t d\Gamma_{g_1} d\tau + i \int_0^t \int_{\Omega} \tilde{u} \tilde{u}_t \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau, \tag{2.32}
 \end{aligned}$$

where we have used the boundary condition of the system (2.23) and the definition of the vector field H .

Next, we multiply the complex conjugate of (2.23) by $\tilde{u} \operatorname{div}_{g_1} H$, integrate it by parts over $(0, t) \times \Omega$ to obtain

$$\begin{aligned}
 -i \int_0^t \int_{\Omega} \tilde{u} \tilde{u}_t \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau &= - \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} \tilde{u} \operatorname{div}_{g_1} H d\Gamma_{g_1} d\tau + \int_0^t \int_{\Omega} |\nabla_{g_1} \tilde{u}|_{g_1}^2 \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau \\
 &\quad + \int_0^t \int_{\Omega} \tilde{u} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} (\operatorname{div}_{g_1} H) \rangle_{g_1} d\Omega_{g_1} d\tau \\
 &\quad + \int_0^t \int_{\Omega} (-i \langle F, \nabla_{g_1} \tilde{u} \rangle_{g_1} + \tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) \tilde{u} \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau. \tag{2.33}
 \end{aligned}$$

We use the identity (2.7) and integration by parts over Ω to obtain

$$\begin{aligned}
 2 \operatorname{Re} \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} H(\tilde{u}) \rangle_{g_1} d\Omega_{g_1} d\tau &= \int_0^t \int_{\Omega} (D_{g_1} H + (D_{g_1} H)^\top) (\nabla_{g_1} \tilde{u}, \nabla_{g_1} \tilde{u}) d\Omega_{g_1} d\tau + \int_0^t \int_{\Gamma} |\nabla_{g_1} \tilde{u}|_{g_1}^2 d\Gamma_{g_1} d\tau \\
 &\quad - \int_0^t \int_{\Omega} |\nabla_{g_1} \tilde{u}|_{g_1}^2 \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau. \tag{2.34}
 \end{aligned}$$

We decompose $\nabla_{g_1} \tilde{u}(t, x)$ on Γ as

$$\nabla_{g_1} \tilde{u}(t, x) = \langle \nabla_{g_1} \tilde{u}, \nu_{g_1} \rangle_{g_1} \nu_{g_1} + \nabla_{\Gamma_{g_1}} \tilde{u}(t, x) = \tilde{u}_{\nu_{g_1}} \nu_{g_1} + \nabla_{\Gamma_{g_1}} \tilde{u}(t, x), \tag{2.35}$$

where $\nabla_{\Gamma_{g_1}}$ is the gradient of the induced metric of the boundary Γ from the metric g_1 . Then on Γ we have

$$|\nabla_{g_1} \tilde{u}|_{g_1}^2 = |\tilde{u}_{\nu_{g_1}}|^2 + |\nabla_{\Gamma_{g_1}} \tilde{u}|_{g_1}^2. \tag{2.36}$$

Taking the real part of (2.30), via (2.32), (2.33), (2.34) and (2.36), we have

$$\begin{aligned}
 \int_0^t \int_{\Gamma} |\tilde{u}_{v_{g_1}}|^2 d\Gamma_{g_1} d\tau &= i \int_0^t \int_{\Gamma} \tilde{v} \tilde{v}_t d\Gamma_{g_1} d\tau + \int_0^t \int_{\Gamma} |\nabla_{\Gamma_{g_1}} \tilde{v}|_{g_1}^2 d\Gamma_{g_1} d\tau - \int_0^t \int_{\Gamma} \tilde{u}_{v_{g_1}} \tilde{v} \operatorname{div}_{g_1} H d\Gamma_{g_1} d\tau \\
 &+ \int_0^t \int_{\Omega} (D_{g_1} H + (D_{g_1} H)^\top)(\nabla_{g_1} \tilde{u}, \nabla_{g_1} \tilde{u}) d\Omega_{g_1} d\tau - \int_0^t \int_{\Omega} |\nabla_{g_1} \tilde{u}|_{g_1}^2 \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau \\
 &+ i \int_0^t \int_{\Omega} \tilde{u} (a^{-\frac{n}{2}} H)_t (\tilde{u}) d\Omega_{g_1} d\tau - \int_0^t \int_{\Omega} |\nabla_{g_1} \tilde{u}|_{g_1}^2 \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau \\
 &+ \int_0^t \int_{\Omega} \tilde{u} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} (\operatorname{div}_{g_1} H) \rangle_{g_1} d\Omega_{g_1} d\tau + 2 \operatorname{Re} \int_0^t \int_{\Omega} i \langle F(t, x), \nabla_{g_1} \tilde{u} \rangle_{g_1} H(\tilde{u}) d\Omega_{g_1} d\tau \\
 &+ 2 \operatorname{Re} \int_0^t \int_{\Omega} (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) H(\tilde{u}) d\Omega_{g_1} d\tau + \int_0^t \int_{\Omega} (F(\tilde{u}) + \tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} \\
 &+ \tilde{f}) \tilde{u} \operatorname{div}_{g_1} H d\Omega_{g_1} d\tau - i \int_{\Omega} \tilde{u}(t) H(\tilde{u}(t)) a^{-\frac{n}{2}} d\Omega + i \int_{\Omega} \tilde{u}_0 H(\tilde{u}_0) a^{-\frac{n}{2}} d\Omega. \quad (2.37)
 \end{aligned}$$

We estimate the right-hand side of the identity (2.37), using the inequality

$$ab \leq \epsilon |a|^2 + \frac{1}{4\epsilon} |b|^2, \quad \forall \epsilon > 0, \quad (2.38)$$

to obtain

$$\begin{aligned}
 \int_0^t \int_{\Gamma} |\tilde{u}_{v_{g_1}}|^2 d\Gamma_{g_1} d\tau &\leq C \left(\|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^t \tilde{E}(\tau) d\tau + \tilde{E}(0) + \int_0^t \|\tilde{f}\|^2 d\tau \right) \\
 &+ \epsilon \int_{\Omega} |\nabla \tilde{u}(t)|^2 d\Omega + \frac{B}{4\epsilon} \int_{\Omega} |\tilde{u}(t)|^2 d\Omega, \quad (2.39)
 \end{aligned}$$

where $C > 0$ depends on $a, H, F, \tilde{f}_1, \tilde{f}_2$ and \tilde{f} , and $B = \lambda^{-n} \sup_{(t,x) \in [0,T] \times \Gamma} |H|^2$.

We multiply (2.23) by $2\tilde{u}_t$, integrate it by parts over $(0, T) \times \Omega$ and obtain

$$\begin{aligned}
 2i \int_0^t \int_{\Omega} |u_t|^2 dx d\tau &= -2 \int_0^t \int_{\Gamma} \tilde{u}_{v_{g_1}} \tilde{v}_t d\Gamma_{g_1} d\tau + 2 \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} \tilde{u}_t \rangle_{g_1} d\Omega_{g_1} d\tau \\
 &+ 2 \int_0^t \int_{\Omega} (iF(\tilde{u}) + \tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) \tilde{u}_t d\Omega_{g_1} d\tau. \quad (2.40)
 \end{aligned}$$

We use (2.17) and (2.29) to compute

$$\begin{aligned}
 2 \operatorname{Re} \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} \tilde{u}_t \rangle_{g_1} d\Omega_{g_1} d\tau &= 2 \operatorname{Re} \int_0^t \int_{\Omega} a^{-\frac{n}{2}+1} \langle \nabla \tilde{u}, \nabla \tilde{u}_t \rangle d\Omega d\tau \\
 &= \int_0^t \int_{\Omega} a^{-\frac{n}{2}+1} \langle \nabla \tilde{u}, \nabla \tilde{u}_t \rangle d\Omega d\tau + \int_0^t \int_{\Omega} a^{-\frac{n}{2}+1} \langle \nabla \tilde{u}, \nabla \tilde{u}_t \rangle d\Omega d\tau \\
 &= \int_{\Omega} a^{-\frac{n}{2}+1} (\xi, x) |\nabla \tilde{u}(\xi, x)|^2 d\Omega \Big|_0^t - \int_0^t \int_{\Omega} \frac{\partial}{\partial t} (a^{-\frac{n}{2}+1}) |\nabla \tilde{u}|^2 d\Omega d\tau. \quad (2.41)
 \end{aligned}$$

We multiply the complex conjugate of (2.23) by $F(\tilde{u})$, and integrate it by parts over $(0, t) \times \Omega$ to obtain

$$\begin{aligned} -i \int_0^t \int_{\Omega} F(\tilde{u}) \tilde{u}_t d\Omega_{g_1} d\tau &= - \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} F(\tilde{u}) d\Gamma_{g_1} d\tau + \int_0^t \int_{\Omega} (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) F(\tilde{u}) d\Omega_{g_1} d\tau \\ &\quad - i \int_0^t \int_{\Omega} |F(\tilde{u})|^2 d\Omega_{g_1} d\tau + \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} (F(\tilde{u})) \rangle d\Omega_{g_1} d\tau. \end{aligned} \quad (2.42)$$

On Γ we decompose the vector field F as

$$F = \langle F, \nu_{g_1} \rangle_{g_1} \nu_{g_1} + F_1, \quad (2.43)$$

where F_1 is the projection of F on the submanifold Γ .

We use (2.7), (2.36), (2.43), and integration by parts over $(0, t) \times \Omega$ to obtain

$$\begin{aligned} &-2 \operatorname{Re} i \int_0^t \int_{\Omega} F(\tilde{u}) \tilde{u}_t d\Omega_{g_1} d\tau \\ &= - \int_0^t \int_{\Gamma} |\tilde{u}_{\nu_{g_1}}|^2 \langle F, \nu_{g_1} \rangle_{g_1} d\Gamma_{g_1} d\tau - 2 \operatorname{Re} \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} F_1(\tilde{u}) d\Gamma_{g_1} d\tau \\ &\quad - \int_0^t \int_{\Omega} |\nabla_{g_1} \tilde{u}|_{g_1}^2 \operatorname{div}_{g_1} F d\Omega_{g_1} d\tau + \int_0^t \int_{\Omega} (D_{g_1} F + (D_{g_1} F)^\top) (\nabla_{g_1} \tilde{u}, \nabla_{g_1} \tilde{u}) d\Omega_{g_1} d\tau \\ &\quad + 2 \operatorname{Re} \int_0^t \int_{\Omega} (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) F(\tilde{u}) d\Omega_{g_1} d\tau. \end{aligned} \quad (2.44)$$

We multiply the complex conjugate of (2.23) by $\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}$, and integrate it over $(0, t) \times \Omega$ to obtain

$$\begin{aligned} &\int_0^t \int_{\Omega} \tilde{u}_t (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) d\Omega_{g_1} d\tau \\ &= -i \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) d\Gamma_{g_1} d\tau + i \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) \rangle_{g_1} d\Omega_{g_1} d\tau \\ &\quad + i \int_0^t \int_{\Omega} (-i F(\tilde{u}) + \tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) d\Omega_{g_1} d\tau. \end{aligned} \quad (2.45)$$

We take the real parts of both sides of the identity (2.40), via (2.41), (2.44) and (2.45) to get

$$\begin{aligned} &\int_{\Omega} a^{-\frac{n}{2}+1}(t, x) |\nabla \tilde{u}(t, x)|^2 d\Omega \\ &= \int_{\Omega} a^{-\frac{n}{2}+1}(0, x) |\nabla \tilde{u}_0|^2 d\Omega - \int_0^t \int_{\Gamma} |\tilde{u}_{\nu_{g_1}}|^2 \langle F, \nu_{g_1} \rangle_{g_1} d\Gamma_{g_1} d\tau + 2 \operatorname{Re} \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} \tilde{u}_t d\Gamma_{g_1} d\tau \\ &\quad + 2 \operatorname{Re} \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} (-F_1(\tilde{u}) + i \tilde{f}_1 \tilde{u} + i \tilde{f}_2 \tilde{u} + i \tilde{f}) d\Gamma_{g_1} d\tau - \int_0^t \int_{\Omega} |\nabla_{g_1} \tilde{u}|_{g_1}^2 \operatorname{div}_{g_1} F d\Omega_{g_1} d\tau \\ &\quad + \int_0^t \int_{\Omega} (D_{g_1} F + (D_{g_1} F)^\top) (\nabla_{g_1} \tilde{u}, \nabla_{g_1} \tilde{u}) d\Omega_{g_1} d\tau + 2 \operatorname{Re} \int_0^t \int_{\Omega} (\tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) F(\tilde{u}) d\Omega_{g_1} d\tau \end{aligned}$$

$$\begin{aligned}
& -2 \operatorname{Re} i \int_0^t \int_{\Omega} \langle \nabla_{g_1} \tilde{u}, \nabla_{g_1} (\tilde{f}_1 \tilde{u} + f_2 \tilde{u} + \tilde{f}) \rangle_{g_1} d\Omega_{g_1} d\tau \\
& -2 \operatorname{Re} i \int_0^t \int_{\Omega} (-iF(\tilde{u}) + \tilde{f}_1 \tilde{u} + \tilde{f}_2 \tilde{u} + \tilde{f}) (\tilde{f}_1 \tilde{u} + f_2 \tilde{u} + \tilde{f}) d\Omega_{g_1} d\tau.
\end{aligned} \quad (2.46)$$

From the condition (2.12) we can get

$$\lambda_0 \int_{\Omega} |\nabla \tilde{u}(t, x)|^2 d\Omega \leq B_1 \int_0^t \int_{\Gamma} |\tilde{u}_{\nu_{g_1}}|^2 d\Gamma_{g_1} d\tau + C \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^t \tilde{E}(\tau) d\tau + \int_0^t \|\tilde{f}\|_1^2 d\tau \right), \quad (2.47)$$

where

$$\lambda_0 = \begin{cases} \lambda_1^{-\frac{n}{2}+1}, & n > 2m, \\ 1, & n = 2, \\ \lambda_1^{-\frac{n}{2}+1}, & n = 1, \end{cases} \quad \text{and} \quad B_1 = \sup_{(t,x) \in [0,T] \times \Gamma} |\langle F, \nu_{g_1} \rangle_{g_1}| + 2. \quad (2.48)$$

We then give an estimate on $\int_{\Omega} |\tilde{u}(t)|^2 d\Omega$ for $0 \leq t \leq T$.

Similar to that of $\int_{\Omega} |\nabla \tilde{u}|^2 d\Omega$, we multiply both sides of the system (2.23) by $2\tilde{u}$, integrate it over $(0, T) \times \Omega$ by parts, and take the imaginary parts to obtain

$$\begin{aligned}
\int_{\Omega} |\tilde{u}(t)|^2 a^{-\frac{n}{2}}(t, x) d\Omega &= \int_{\Omega} |\tilde{u}_0|^2 a^{-\frac{n}{2}}(0, x) d\Omega + 2 \operatorname{Im} \int_0^t \int_{\Omega} (iF(\tilde{u}) + \tilde{f}_1 \tilde{u} + f_2 \tilde{u} + \tilde{f}) \tilde{u} d\Omega_{g_1} d\tau \\
&+ \int_0^t \int_{\Omega} |\tilde{u}|^2 \frac{\partial}{\partial t} (a^{-\frac{n}{2}}) d\Omega d\tau - 2 \operatorname{Im} \int_0^t \int_{\Gamma} \tilde{u}_{\nu_{g_1}} \tilde{v} d\Gamma_{g_1} d\tau.
\end{aligned}$$

Then we apply (2.38) and (2.12) to get

$$\begin{aligned}
\lambda_0 \|\tilde{u}(t)\|^2 &\leq \epsilon_1 \int_0^t \int_{\Gamma} |\tilde{u}_{\nu_{g_1}}|^2 d\Gamma_{g_1} d\tau + C_{\epsilon_1} \int_0^t \int_{\Gamma} |\tilde{v}|^2 d\Gamma d\tau \\
&+ C \left(\|\tilde{u}(0)\|^2 + \int_0^t \tilde{E}(\tau) d\tau + \int_0^t \|\tilde{f}(\tau)\|^2 d\tau \right),
\end{aligned} \quad (2.49)$$

where $\epsilon_1 > 0$ and λ_0 is defined in (2.48).

We insert the inequality (2.39) with $\epsilon = \frac{\lambda_0}{2B_1}$ both into (2.47) and (2.49) to obtain

$$\frac{\lambda_0}{2} \int_{\Omega} |\nabla \tilde{u}(t)|^2 d\Omega - \frac{BB_1^2}{2\lambda_0} \int_{\Omega} |\tilde{u}(t)|^2 d\Omega \leq C \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^t \tilde{E}(\tau) d\tau + \int_0^t \|\tilde{f}\|_1^2 d\tau \right), \quad (2.50)$$

and

$$\begin{aligned}
& \left(\lambda_0 - \frac{\epsilon_1 BB_1}{2\lambda_0} \right) \|\tilde{u}(t)\|^2 - \frac{\epsilon_1 \lambda_0}{2B_1} \int_{\Omega} |\nabla \tilde{u}(t)|^2 d\Omega \\
& \leq C_{\epsilon_1} \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^t \tilde{E}(\tau) d\tau + \int_0^t \|\tilde{f}\|_1^2 d\tau \right).
\end{aligned} \quad (2.51)$$

Let $\epsilon_1 \leq \lambda_0 B_1 (\frac{2BB_1^2}{\lambda_0} + \frac{\lambda_0}{2})^{-1}$ in (2.51). We multiply the inequality (2.51) by $\frac{B_1}{2\epsilon_1}$ and add it to (2.50) to get

$$\frac{\lambda_0}{4} \tilde{E}(t) \leq C \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^t \tilde{E}(\tau) d\tau + \int_0^t \|\tilde{f}\|_1^2 d\tau \right). \quad (2.52)$$

By virtue of Gronwall inequality, we have

$$\tilde{E}(t) \leq C_T \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^T \|\tilde{f}\|_1^2 d\tau \right). \quad (2.53)$$

Insertion of (2.53) into (2.39) with $\epsilon = 1$ yields

$$\int_0^t \int_{\Gamma} |\tilde{u}_{v_{g_1}}|^2 d\Gamma_{g_1} d\tau \leq C_T \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^T \|\tilde{f}\|_1^2 d\tau \right). \quad (2.54)$$

By virtue of (2.17), (2.29) and (2.28), it is easy to check that

$$\int_0^t \int_{\Gamma} |\tilde{u}_{v_{g_1}}|^2 d\Gamma_{g_1} d\tau = \int_0^t \int_{\Gamma} a^{-\frac{n-3}{2}} |\tilde{u}_v|^2 d\Gamma d\tau. \quad (2.55)$$

Then we apply inequality (2.12) to (2.54) and obtain

$$\int_0^t \int_{\Gamma} |\tilde{u}_v|^2 d\Gamma d\tau \leq C_T \left(\tilde{E}(0) + \|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 + \int_0^T \|\tilde{f}\|_1^2 d\tau \right). \quad (2.56)$$

By virtue of the transformation (2.24), the condition (2.12) and the inequality (2.38), we can have

$$C_{1T}E(t) \leq \tilde{E}(t) \leq C_{2T}E(t), \quad 0 \leq t \leq T, \quad (2.57)$$

where C_{1T} and C_{2T} are positive constants depending on T .

Similarly, we have

$$\int_0^t \int_{\Gamma} |u_v|^2 d\Gamma d\tau \leq C_T \left(\int_0^t \int_{\Gamma} |\tilde{u}_{v_{g_1}}|^2 d\Gamma_{g_1} d\tau + \|v\|_{H^1((0,T) \times \Gamma)}^2 \right), \quad (2.58)$$

$$\int_0^t \int_{\Gamma} |\tilde{u}_{v_{g_1}}|^2 d\Gamma_{g_1} d\tau \leq C_T \left(\int_0^t \int_{\Gamma} |u_v|^2 d\Gamma d\tau + \|v\|_{H^1((0,T) \times \Gamma)}^2 \right), \quad (2.59)$$

$$\|\tilde{v}\|_{H^1((0,T) \times \Gamma)}^2 \leq C_T \|v\|_{H^1((0,T) \times \Gamma)}^2, \quad \int_0^T \|\tilde{f}\|_1^2 d\tau \leq C_T \int_0^T \|f\|_1^2 d\tau. \quad (2.60)$$

By virtue of (2.57)–(2.60), we can obtain the inequalities (2.14) and (2.15) from the inequalities (2.53) and (2.56).

From the inequality (2.14) and the fact that $C^\infty([0, T] \times \Gamma)$ is dense in $H^1((0, T) \times \Gamma)$, we obtain that the problem (2.11) has a unique solution $u \in L^\infty((0, T); H^1(\Omega))$. Using the method of Section 8.4 in [8], we obtain the solution $u \in C([0, T]; H^1(\Omega))$. \square

For given $T > 0$ and $r > 0$, let $z(r, T)$ denote the set of all functions w which satisfy

$$\begin{cases} w \in \bigcap_{l=0}^{k-1} W^{l,\infty}((0, T); H^{2k-1-2l}(\Omega)), & \sum_{l=0}^{k-1} \text{ess sup}_{0 \leq t \leq T} \|w^{(l)}(t)\|_{H^{2k-1-2l}(\Omega)}^2 \leq r, \\ w^{(l)}(0) = u_l \quad \text{in } \Omega, \quad l = 0, 1, \dots, k-1, & w = \varphi \quad \text{on } (0, T) \times \Gamma. \end{cases} \quad (2.61)$$

Given $w \in z(r, T)$, we define an elliptic operator B_1 by

$$B_1(t)u = -a(x, w(t))\Delta u + i\langle F(x, w(t)), \nabla u \rangle, \quad u \in H^2(\Omega). \quad (2.62)$$

We need the following lemma:

Lemma 2.5.

(i) Let $h(x, \xi) \triangleq h(x, y, z)$ be a smooth function on $\Omega \times \mathbb{R} \times \mathbb{R}$, where $\xi = y + iz$. Take $w \in z(r, T)$. Then for $l = 0, \dots, k-1$, there is a constant $C_r > 0$ depending only on r such that

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \|h^{(l)}(x, w)\|_{2k-1-2l} \leq C_r, \quad (2.63)$$

where $h^{(l)}(x, w)$ means $[h(x, w(t))]^{(l)}$, the l th order derivative of $h(x, w(t))$ with respect to t .

(ii) Take $w \in z(r, T)$. Let the elliptic operator $B_1(t)$ be defined by (2.62). Then there is $C_r > 0$, which depends on the r , such that

$$\|v\|_l \leq C_r (\|B_1 v\|_{l-2} + \|v\| + \|v\|_{H^{l-1/2}(\Gamma)}), \quad (2.64)$$

for all v such that $B_1 v \in H^{l-2}(\Omega)$, where $l \geq 1$.

Proof. (i) Set $w = w_1 + iw_2$ with $w_1, w_2 \in \mathbb{R}$. We have

$$h^{(l)}(x, w) = \sum_{\substack{p+q=1 \\ p, q \geq 0}}^l \sum_{r_1+\dots+r_p+s_1+\dots+s_q=l} h_{pq}(x, w) w_1^{(r_1)} \dots w_1^{(r_p)} w_2^{(s_1)} \dots w_2^{(s_q)}, \quad (2.65)$$

where $h_{pq}(x, \xi) = h_{pq}(x, y, z)$ for $\xi = y + iz \in \mathbb{C}$ are smooth functions. We use the properties (ii) and (iii) of Lemma 2.1 to obtain

$$\begin{aligned} & \|h_{pq}(x, w) w_1^{(r_1)} \dots w_1^{(r_p)} w_2^{(s_1)} \dots w_2^{(s_q)}\|_{2k-1-2l} \\ & \leq C_r \|w_1^{(r_1)}\|_{2k-1-2r_1} \dots \|w_1^{(r_p)}\|_{2k-1-2r_p} \|w_2^{(s_1)}\|_{2k-1-2s_1} \dots \|w_2^{(s_q)}\|_{2k-1-2s_q} \\ & \leq C_r, \end{aligned} \quad (2.66)$$

where $p+q=1, \dots, l$ and $p, q \geq 0$. Then we have proved (i).

(ii) A standard method as to the linear elliptic problem can give the inequality (2.64), for example see [3] and [2]. \square

Let $u \in \bigcap_{l=0}^{k-1} C^l([0, T]; H^{2k-1-2l}(\Omega))$ be a solution to the system (2.1) for some $T > 0$. We introduce

$$\mathcal{E}(t) = \sum_{l=0}^{k-1} \|u^{(l)}(t)\|_{2k-1-2l}^2, \quad 0 \leq t \leq T. \quad (2.67)$$

Proof of Theorem 2.1. Without loss of generality we suppose that the equilibrium is the zero.

Step 1. Taking $w \in z(r, T)$, we study the following linear system first

$$\begin{cases} iu_t = -a(x, w)\Delta u + i\langle F(x, w), \nabla u \rangle + b(x, w) & \text{in } (0, T) \times \Omega, \\ u = \varphi & \text{on } (0, T) \times \Gamma, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.68)$$

Let $2k \geq [n/2] + 4$ be given. Let $u_0 \in H^{2k-1}(\Omega)$ be given. Let u be a solution to the problem (2.68).

Claim 1. Fix $r > 0$. Let $T > 0$ be small. Then there exists a constant $C_r > 0$ such that

$$\mathcal{E}(t) \leq C_r \left(\mathcal{E}(0) + T + T^2 + \sum_{p=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(p)}(\tau)\|_{H^{2k-2p-3/2}(\Gamma)}^2 + \|\varphi^{(k-1)}\|_{H^1((0, T) \times \Gamma)}^2 \right). \quad (2.69)$$

In addition, the solution to the problem (2.68) satisfies

$$u \in \bigcap_{l=0}^{k-1} C^l([0, T]; H^{2k-1-2l}(\Omega)). \quad (2.70)$$

Proof. For $0 \leq l \leq k-1$, we formally differentiate the system (2.68) l times with respect to t and obtain

$$\begin{cases} iu_t^{(l)} = -a(x, w)\Delta u^{(l)} + i\langle F(x, w), \nabla u^{(l)} \rangle + f_l + b^{(l)}(x, w) & \text{in } (0, T) \times \Omega, \\ u^{(l)} = \varphi^{(l)} & \text{on } (0, T) \times \Gamma, \\ u^{(l)}(0) = u_l & \text{in } \Omega, \end{cases} \quad (2.71)$$

where

$$f_l = \sum_{s=1}^l C_l^s (-a^{(s)}(x, w) \Delta u^{(l-s)} + i \langle F^{(s)}(x, w), \nabla u^{(l-s)} \rangle), \quad 1 \leq l \leq k-1, \quad (2.72)$$

and C_l^s is the binomial coefficient.

We use (2.6) and (2.63) to obtain

$$\begin{aligned} \|f_{k-1}\|_1^2 &\leq C \sum_{s=1}^{k-1} \left(\|a^{(s)}\|_{2k-1-2s}^2 \|\Delta u^{(k-1-s)}\|_{2s-1}^2 + \sum_{s=1}^{k-1} \|F^{(s)}\|_{2k-1-2s}^2 \|\nabla u^{(k-1-s)}\|_{2s}^2 \right) \\ &\leq C_r \mathcal{E}(t), \end{aligned} \quad (2.73)$$

and

$$\|f_l\|_{2k-3-2l}^2 \leq C_r \sum_{s=0}^{l-1} \|u^{(s)}\|_{2k-3-2s}^2, \quad l = 1, \dots, k-2. \quad (2.74)$$

We apply the estimate (2.14) to the problem (2.71) with $l = k-1$, via Lemma 2.5 and the inequality (2.73) to obtain

$$\begin{aligned} \|u^{(k-1)}(t)\|_1^2 &\leq C_r T \left(\|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 + \|u_{k-1}\|_1^2 + \int_0^T \|f_{k-1} + b^{(k-1)}(x, w)\|_1^2 d\tau \right) \\ &\leq C_r T \left(\|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 + \|u_{k-1}\|_1^2 + T + \int_0^T \mathcal{E}(\tau) d\tau \right). \end{aligned} \quad (2.75)$$

Next, we show that the following inequality

$$\begin{aligned} \|u^{(l)}(t)\|_{2k-1-2l}^2 &\leq C_r T \left(\mathcal{E}(0) + \sum_{p=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(p)}(\tau)\|_{H^{2k-2p-3/2}(\Gamma)}^2 + \|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 \right. \\ &\quad \left. + T + T^2 + \int_0^T \mathcal{E}(\tau) d\tau \right), \quad t \in [0, T], \quad l = 0, 1, \dots, k-1, \end{aligned} \quad (2.76)$$

holds by induction in l .

The inequality (2.75) shows that the inequality (2.76) is true for $l = k-1$. Suppose that the inequality (2.76) is true for some $1 \leq l_0 \leq k-1$. We will show that it is true for $l = l_0 - 1$.

Let the elliptic operator B_1 be given by (2.62). We use the ellipticity of the operator B_1 to have, via the problem (2.71) with $l = l_0 - 1$ and (2.74),

$$\begin{aligned} \|u^{(l_0-1)}(t)\|_{2k-2l_0+1}^2 &\leq C_r (\|B_1 u^{(l_0-1)}\|_{2k-1-2l_0}^2 + \|\varphi^{(l_0-1)}(t)\|_{H^{2k-2l_0+1/2}(\Gamma)}^2 + \|u^{(l_0-1)}(t)\|_1^2) \\ &\leq C_r \left(\|u^{(l_0)}\|_{2k-1-2l_0}^2 + \|b^{(l_0-1)}(x, w)\|_{2k-1-2l_0}^2 \right. \\ &\quad \left. + \|\varphi^{(l_0-1)}(t)\|_{H^{2k-2l_0+1/2}(\Gamma)}^2 + \sum_{s=0}^{l_0-1} \|u^{(s)}\|_{2k-3-2s}^2 \right). \end{aligned} \quad (2.77)$$

For $s = 0, \dots, l_0 - 1$, and $0 \leq t \leq T$, we have

$$\|u^{(s)}(t)\|_{2k-3-2s}^2 = \left\| u_s + \int_0^t u^{(1+s)} d\tau \right\|_{2k-3-2s}^2 \leq 2\|u_s\|_{2k-3-2s}^2 + 2t \int_0^t \|u^{(1+s)}\|_{2k-3-2s}^2 d\tau, \quad (2.78)$$

and

$$\begin{aligned}
\|b^{(l_0-1)}(x, w)\|_{2k-1-2l_0}^2 &= \left\| b^{(l_0-1)}(x, w(0)) + \int_0^t b^{(l_0)}(x, w(\tau)) d\tau \right\|_{2k-1-2l_0}^2 \\
&\leq 2\|b^{(l_0-1)}(x, w(0))\|_{2k-1-2l_0}^2 + 2t \int_0^t \|b^{(l_0)}(x, w(\tau))\|_{2k-1-2l_0}^2 d\tau \\
&\leq C_r \mathcal{E}(0) + C_r t^2,
\end{aligned} \tag{2.79}$$

where $w^{(l)}(0) = u_l$ for $l = 0, \dots, k-1$. In the above inequality we have used Lemma 2.1 and the condition $b(x, 0) = 0$. We insert (2.78) and (2.79) into (2.77), and use the induction assumption for l_0 to get

$$\begin{aligned}
\|u^{(l_0-1)}(t)\|_{2k-2l_0+1}^2 &\leq C_{rT} \left(\mathcal{E}(0) + \sum_{p=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(p)}(\tau)\|_{H^{2k-2p-3/2}(\Gamma)}^2 + \|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 \right. \\
&\quad \left. + T + T^2 + \int_0^T \mathcal{E}(\tau) d\tau \right).
\end{aligned} \tag{2.80}$$

Then (2.76) follows by induction.

Adding up the inequality (2.76) from $l = 0$ to $l = k-1$ yields

$$\begin{aligned}
\mathcal{E}(t) &\leq C_{rT} \left(\mathcal{E}(0) + \sum_{p=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(p)}(\tau)\|_{H^{2k-2p-3/2}(\Gamma)}^2 + \|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 \right. \\
&\quad \left. + T + T^2 + \int_0^T \mathcal{E}(\tau) d\tau \right), \quad 0 \leq t \leq T.
\end{aligned} \tag{2.81}$$

Integration of the inequality (2.81) over $(0, T)$ yields

$$\begin{aligned}
(1 - C_{rT}T) \int_0^T \mathcal{E}(t) dt &\leq C_{rT}T \left(\mathcal{E}(0) + \sum_{p=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(p)}(\tau)\|_{H^{2k-2p-3/2}(\Gamma)}^2 \right. \\
&\quad \left. + \|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 + T + T^2 \right).
\end{aligned} \tag{2.82}$$

Insertion of the inequality (2.82) with $C_{rT}T \leq \frac{1}{2}$ into (2.81) yields the inequality (2.69).

To complete the proof of Claim 1, it remains to show that the relation (2.70) holds true. The inequality (2.69) implies $u \in \bigcap_{l=0}^{k-1} W^{l,\infty}(0, T; H^{2k-1-2l}(\Omega))$. Applying Lemma 2.3, we have $u^{(k-1)} \in C([0, T]; H^1(\Omega))$. A similar argument as in Theorem 3.1 in [1] shows that the relation (2.70) holds true. \square

Step 2. Let $r > 0$ be given. Let $C_{r1} > 0$ be such that $C_{rT} \leq C_{r1}$ for all $T > 0$ small, where C_{rT} is the constant in the inequality (2.69).

Given $u_0 \in H^{2k-1}(\Omega)$ and boundary function φ such that

$$\mathcal{E}(0) + \sum_{p=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(p)}(\tau)\|_{H^{2k-2p-3/2}(\Gamma)}^2 + \|\varphi^{(k-1)}\|_{H^1((0,T) \times \Gamma)}^2 \leq \frac{r}{2C_{r1}}. \tag{2.83}$$

We denote by \mathcal{Y} the map which carries $w \in z(r, T)$ into the solution u of the problem (2.68). Now we fix T_0 small such that $0 < T_0 + T_0^2 \leq \frac{r}{2C_{r1}}$ and $T_0 \leq \frac{1}{2C_{r1}}$. By (2.69) we have $\mathcal{E}(t) \leq r$ for $0 \leq t \leq T \leq T_0$. Then

$$\mathcal{Y} : z(r, T) \rightarrow z(r, T), \quad T \leq T_0. \tag{2.84}$$

To complete the proof of Theorem 2.1, we next show that \mathcal{Y} is a strictly contractive map.

Let $w, \hat{w} \in z(r, T)$. Let $u = \mathcal{Y}(w)$ and $\hat{u} = \mathcal{Y}(\hat{w})$. We equip $z(r, T)$ with the complete metric ϱ defined by

$$\varrho(w, \hat{w}) = \operatorname{ess\,sup}_{0 \leq t \leq T} \|w(t) - \hat{w}(t)\|_1. \tag{2.85}$$

Set $U = u - \hat{u}$. Then U is the solution to the following system

$$\begin{cases} iU_t = -a(x, w)\Delta U + i\langle F(x, w), \nabla U \rangle + Q & \text{in } (0, T) \times \Omega, \\ U = 0 & \text{on } (0, T) \times \Gamma, \\ U(0) = 0 & \text{in } \Omega, \end{cases} \quad (2.86)$$

where

$$Q = (a(x, \hat{w}) - a(x, w))\Delta \hat{u} + i\langle F(x, w) - F(x, \hat{w}), \nabla \hat{u} \rangle + b(x, w) - b(x, \hat{w}). \quad (2.87)$$

Applying the mean value theorem and the inequality (2.14) to (2.86), we have, via the boundary and initial condition of U ,

$$\|U(t)\|_1^2 \leq C_{rT} \int_0^T \|Q(\tau)\|_1^2 d\tau \leq C_{r1} T Q(w, \hat{w})^2. \quad (2.88)$$

Choosing $T_0 > 0$ so small such that $C_{r1}T_0 < 1$, we obtain the map Υ is strictly contractive for $0 < T \leq T_0$. \square

3. Locally exact controllability

The proof for the locally exact controllability depends on the following fact: Let \mathcal{X} and \mathcal{Y} be Banach spaces and let $\Phi : \mathcal{O} \rightarrow \mathcal{Y}$, where \mathcal{O} is an open subset of \mathcal{X} , be Frechét differentiable. Let $X_0 \in \mathcal{O}$. If $\Phi'(X_0) : \mathcal{X} \rightarrow \mathcal{Y}$ is surjective, then there is an open neighborhood of $Y_0 = \Phi(X_0)$ contained in the image $\Phi(\mathcal{O})$.

Given $0 < T \leq T_0$, where T_0 is given in Theorem 2.1, we introduce two Banach spaces:

$$\mathcal{X}_0^{2k-1}(T) \triangleq \left\{ \varphi \left| \begin{array}{l} \varphi \in \bigcap_{l=0}^{k-2} C^l([0, T]; H^{2k-3/2-2l}(\Gamma_0)) \\ \varphi^{(l)}(0) = 0, \quad 0 \leq l \leq k-1, \quad \varphi^{(k-1)} \in H^1((0, T) \times \Gamma_0) \end{array} \right. \right\} \quad (3.1)$$

and

$$\tilde{\mathcal{X}}_0^{2k-1}(T) \triangleq \left\{ \varphi \left| \begin{array}{l} \varphi \in \bigcap_{l=0}^{k-2} C^l([0, T]; H^{2k-3/2-2l}(\Gamma_0)) \\ \varphi^{(l)}(T) = 0, \quad 0 \leq l \leq k-1, \quad \varphi^{(k-1)} \in H^1((0, T) \times \Gamma_0) \end{array} \right. \right\}, \quad (3.2)$$

with the same norm $\|\varphi\|_{\mathcal{X}_0}^2 = \sum_{l=0}^{k-2} \sup_{0 \leq \tau \leq T} \|\varphi^{(l)}(\tau)\|_{H^{2k-2l-3/2}(\Gamma_0)}^2 + \|\varphi^{(k-1)}\|_{H^1((0, T) \times \Gamma_0)}^2$.

Let $\omega \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ be exactly controllable. We invoke Theorem 2.1 to define a map for $\varphi \in \mathcal{X}_0^{2k-1}(T)$ by setting

$$\Phi(\varphi) = u(T), \quad (3.3)$$

where u is the solution of the following problem

$$\begin{cases} iu_t = -a(x, u)\Delta u + i\langle F(x, u), \nabla u \rangle + b(x, u) & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \Gamma_1, \quad u = \omega|_{\Gamma_0} + \varphi & \text{on } (0, T) \times \Gamma_0, \\ u(0) = \omega & \text{in } \Omega. \end{cases} \quad (3.4)$$

Let $\epsilon_0 > 0$ be given in Theorem 2.1. Then

$$\Phi : B_{\mathcal{X}_0^{2k-1}(T)}(0, \epsilon_0) \rightarrow H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega), \quad (3.5)$$

where $B_{\mathcal{X}_0^{2k-1}(T)}(0, \epsilon_0)$ is the ball in the space $\mathcal{X}_0^{2k-1}(T)$ with the radius ϵ_0 centered at 0. We observe that $\Phi(0) = \omega$.

We need to compute

$$\Phi'(0)\varphi = \frac{\partial}{\partial \sigma} \Phi(\sigma\varphi) \Big|_{\sigma=0}, \quad \varphi \in \mathcal{X}_0^{2k-1}(T). \quad (3.6)$$

It is easy to check that

$$\Phi'(0)\varphi = v(T), \quad (3.7)$$

where $v(t, x)$ is the solution of the linear system

$$\begin{cases} iv_t = -a(x, \omega)\Delta v + i\langle F(x, \omega), \nabla v \rangle + \frac{r - i\hat{r}}{2}v + \frac{r + i\hat{r}}{2}\bar{v} & \text{in } (0, T) \times \Omega, \\ v = 0 & \text{on } (0, T) \times \Gamma_1, \quad v = \varphi & \text{on } (0, T) \times \Gamma_0, \\ v(0) = 0 & \text{in } \Omega, \end{cases} \quad (3.8)$$

where

$$r = -\frac{\partial}{\partial y}a(x, \omega)\Delta\omega + i\left\langle \frac{\partial}{\partial y}F(x, \omega), \nabla\omega \right\rangle + \frac{\partial}{\partial y}b(x, \omega), \quad (3.9)$$

and

$$\hat{r} = -\frac{\partial}{\partial z}a(x, \omega)\Delta\omega + i\left\langle \frac{\partial}{\partial z}F(x, \omega), \nabla\omega \right\rangle + \frac{\partial}{\partial z}b(x, \omega). \quad (3.10)$$

We now verify that $\Phi'(0) : \varphi \in \mathcal{X}_0^{2k-1}(T) \rightarrow H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ is surjective. In the language of control theory the surjection is just exact controllability, which for a reversible system such as (3.8) is equivalent to null controllability.

Explicitly, one has to show that, for specified $T > 0$, given $v_0 \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$, one can find $\varphi \in \tilde{\mathcal{X}}_0^{2k-1}(T)$ such that the solution to

$$\begin{cases} iv_t = -a(x, \omega)\Delta v + i\langle F(x, \omega), \nabla v \rangle + \frac{r - i\hat{r}}{2}v + \frac{r + i\hat{r}}{2}\bar{v} & \text{in } (0, T) \times \Omega, \\ v = 0 & \text{on } (0, T) \times \Gamma_1, \quad v = \varphi & \text{on } (0, T) \times \Gamma_0, \\ v(0) = v_0 & \text{in } \Omega, \end{cases} \quad (3.11)$$

satisfies

$$v(T) = 0. \quad (3.12)$$

Since the second-order term in the problem (3.11) is not a Laplace operator, the direct study of the null controllability of the problem (3.11) is not an easy job. Then we give a transformation similar to that of the problem (2.23).

We define

$$g_1 = a^{-1}(x, \omega)g$$

as a new Riemannian metric on $\bar{\Omega}$ and consider the couple (Ω, g_1) as a Riemannian manifold with a boundary Γ .

The problem (3.11) is transformed into a new form

$$\begin{cases} i\tilde{v}_t = \mathcal{B}\tilde{v} & \text{in } (0, T) \times \Omega, \\ \tilde{v} = 0 & \text{on } (0, T) \times \Gamma_1, \quad \tilde{v} = \tilde{\varphi} & \text{on } (0, T) \times \Gamma_0, \\ \tilde{v}(0) = \tilde{v}_0 & \text{in } \Omega, \end{cases} \quad (3.13)$$

by means of the transformation

$$\tilde{v}(t, x) = a^{-\frac{1}{2}(-\frac{n}{2}+1)}(x, \omega)v(t, x), \quad (t, x) \in [0, T] \times \Omega, \quad (3.14)$$

where

$$\begin{aligned} \mathcal{B}\tilde{v} = & -\Delta_{g_1}\tilde{v} + i\langle F, \nabla_{g_1}\tilde{v} \rangle_{g_1} + \left(a^{-\frac{1}{2}(-\frac{n}{2}+1)}\Delta a^{\frac{1}{2}(-\frac{n}{2}+1)} + \frac{1}{2}\left(-\frac{n}{2}+1\right)a^{-1}i\langle F, \nabla_{g_1}a \rangle_{g_1} \right. \\ & \left. + \frac{r - i\hat{r}}{2}\right)\tilde{v} + \frac{r + i\hat{r}}{2}\bar{\tilde{v}}, \end{aligned} \quad (3.15)$$

and

$$\tilde{\varphi} = a^{-\frac{1}{2}(-\frac{n}{2}+1)}(x, \omega)\varphi, \quad \tilde{v}_0 = a^{-\frac{1}{2}(-\frac{n}{2}+1)}(x, \omega)v_0. \quad (3.16)$$

Then the null controllability of the problem (3.11) is equivalent to that of the problem (3.13).

In order to obtain the null controllability of the system (3.13), we have to study observability estimates of an adjoint system of the system (3.13). However, since \tilde{v} and $\bar{\tilde{v}}$ appear in the formula (3.15) at the same time, a formal adjoint operator of \mathcal{B} does not exist. To overcome this difficulty, we consider an equivalent real system instead of the problem (3.13).

Let $\tilde{v} = \tilde{v}_1 + i\tilde{v}_2$. We consider a real system with $(\tilde{v}_1, \tilde{v}_2)$ being unknown

$$\begin{cases} \begin{pmatrix} \tilde{v}_{1t} \\ \tilde{v}_{2t} \end{pmatrix} = B \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} & \text{in } (0, T) \times \Omega, \\ \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = 0 & \text{on } (0, T) \times \Gamma_1, \quad \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} & \text{on } (0, T) \times \Gamma_0, \\ \begin{pmatrix} \tilde{v}_1(0) \\ \tilde{v}_2(0) \end{pmatrix} = \begin{pmatrix} \tilde{v}_{10} \\ \tilde{v}_{20} \end{pmatrix} & \text{in } \Omega, \end{cases} \quad (3.17)$$

where

$$B \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} -\Delta_{g_1} \tilde{v}_2 + \langle F, \nabla_{g_1} \tilde{v}_1 \rangle_{g_1} + h_1 \tilde{v}_1 + h_2 \tilde{v}_2 \\ \Delta_{g_1} \tilde{v}_1 + \langle F, \nabla_{g_1} \tilde{v}_2 \rangle_{g_1} + h_3 \tilde{v}_1 + h_4 \tilde{v}_2 \end{pmatrix}, \quad (3.18)$$

$$r = r_1 + ir_2, \quad \hat{r} = \hat{r}_1 + i\hat{r}_2,$$

$$h_1 = \frac{1}{2} \left(-\frac{n}{2} + 1 \right) a^{-1} \langle F, \nabla_{g_1} a \rangle_{g_1} + r_2, \quad h_2 = a^{-\frac{1}{2}(-\frac{n}{2}+1)} \Delta a^{\frac{1}{2}(-\frac{n}{2}+1)} + \hat{r}_2, \quad (3.19)$$

and

$$h_3 = -a^{-\frac{1}{2}(-\frac{n}{2}+1)} \Delta a^{\frac{1}{2}(-\frac{n}{2}+1)} - r_1, \quad h_4 = \frac{1}{2} \left(-\frac{n}{2} + 1 \right) a^{-1} \langle F, \nabla_{g_1} a \rangle_{g_1} - \hat{r}_1. \quad (3.20)$$

Then the null controllability of the problem (3.13) is equivalent to that of the problem (3.17). Moreover, the dual system of (3.17) is

$$\begin{cases} \begin{pmatrix} \phi_{1t} \\ \phi_{2t} \end{pmatrix} = B^* \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} & \text{in } (0, T) \times \Omega, \\ \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 & \text{on } (0, T) \times \Gamma, \\ \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}, \end{cases} \quad (3.21)$$

where

$$B^* \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\Delta_{g_1} \phi_2 + \langle F, \nabla_{g_1} \phi_1 \rangle_{g_1} + (\operatorname{div} F - h_1) \phi_1 - h_3 \phi_2 \\ \Delta_{g_1} \phi_1 + \langle F, \nabla_{g_1} \phi_2 \rangle_{g_1} - h_2 \phi_1 + (\operatorname{div} F - h_4) \phi_2 \end{pmatrix}. \quad (3.22)$$

Next, we study observability estimates of the adjoint system (3.21) to obtain the null controllability of the system (3.17) or equivalently that of the system (3.13). Fortunately, the real system (3.21) is equivalent to the following complex-valued system where $\phi = \phi_1 + i\phi_2$

$$\begin{cases} i\phi_t = \hat{\mathcal{B}}\phi & \text{in } (0, T) \times \Omega, \\ \phi = 0 & \text{on } (0, T) \times \Gamma, \\ \phi(0) = \phi_0 & \text{in } \Omega, \end{cases} \quad (3.23)$$

where

$$\hat{\mathcal{B}} = -\Delta_{g_1} \phi + i \langle F, \nabla_{g_1} \phi \rangle_{g_1} + R\phi + \hat{R}\bar{\phi}, \quad (3.24)$$

$$R = \frac{1}{2} (2i \operatorname{div} F - ih_1 + h_2 - h_3 - ih_4), \quad \text{and} \quad \hat{R} = \frac{1}{2} (-ih_1 + h_2 + h_3 + ih_4). \quad (3.25)$$

To obtain the null controllability of the system (3.13), therefore we need to study the observability estimates of the system (3.23). Note that the systems (3.13) and (3.23) are not dual with each other in the traditional sense.

Now we write out the controlled systems (3.13) and (3.17) as follows. Let $\phi = \phi_1 + i\phi_2$ solve the problem (3.23) (or equivalently (ϕ_1, ϕ_2) solve the problem (3.21)). Let ν_{g_1} be the outward unit normal field along the boundary Γ in the metric g_1 . We take a boundary control $\varphi = -i\phi_{\nu_{g_1}}$ in (3.13) (or equivalently $(\varphi_1, \varphi_2) = (\phi_{2\nu_{g_1}}, -\phi_{1\nu_{g_1}})$ in (3.17)) and obtain the controlled system in a complex-valued form

$$\begin{cases} i\psi_t = \mathcal{B}\psi & \text{in } (0, T) \times \Omega, \\ \psi = 0 & \text{on } (0, T) \times \Gamma_1, \quad \psi = -i\phi_{\nu_{g_1}} & \text{on } (0, T) \times \Gamma_0, \\ \psi(T) = 0 & \text{in } \Omega, \end{cases} \quad (3.26)$$

or equivalently a real-valued form

$$\begin{cases} \begin{pmatrix} \psi_{1t} \\ \psi_{2t} \end{pmatrix} = B \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & \text{in } (0, T) \times \Omega, \\ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 & \text{on } (0, T) \times \Gamma_1, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \phi_{2\nu_{g_1}} \\ -\phi_{1\nu_{g_1}} \end{pmatrix} & \text{on } (0, T) \times \Gamma_0, \\ \begin{pmatrix} \psi_1(T) \\ \psi_2(T) \end{pmatrix} = 0 & \text{in } \Omega. \end{cases} \quad (3.27)$$

We then have the following Green formula

$$\left(B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, -B^* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) + \int_{\Gamma_0} (v_2 u_{1\nu_{g_1}} - v_1 u_{2\nu_{g_1}}) d\Gamma_{g_1}, \quad (3.28)$$

for all $v = v_1 + iv_2 \in H^2(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ and $u = u_1 + iu_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, where (\cdot, \cdot) denotes the inner product in $L^2(\Omega) \times L^2(\Omega)$.

Given $\phi_0 = \phi_{10} + i\phi_{20} \in H_0^1(\Omega)$ (or $(\phi_{10}, \phi_{20}) \in H_0^1(\Omega) \times H_0^1(\Omega)$). We solve the problem (3.23) (or (3.21)). Then we solve the controlled system (3.26) (or (3.27)). We define linear maps $\Lambda : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ or $\tilde{\Lambda} : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow H^{-1}(\Omega) \times H^{-1}(\Omega)$ by

$$\Lambda(\phi_0) = \psi(0), \quad \text{or} \quad \tilde{\Lambda}(\phi_{10}, \phi_{20}) = (\psi_1(0), \psi_2(0)). \quad (3.29)$$

Let $\varphi_0 = \varphi_{10} + i\varphi_{20} \in H_0^1(\Omega)$ be given. Let φ (or (φ_1, φ_2)) be the solution to the problem (3.23) with the initial data φ_0 (or the problem (3.21) with initial data $(\varphi_{10}, \varphi_{20})$). We multiply both sides of Eq. (3.27) by (φ_1, φ_2) , and integrate it by parts over $(0, T) \times \Omega$, via the identity (3.28), to obtain

$$\left(\tilde{\Lambda} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}, \begin{pmatrix} \varphi_{10} \\ \varphi_{20} \end{pmatrix} \right) = \int_0^T \int_{\Gamma_0} (\phi_{1\nu_{g_1}} \varphi_{1\nu_{g_1}} + \phi_{2\nu_{g_1}} \varphi_{2\nu_{g_1}}) d\Gamma_{g_1} dt. \quad (3.30)$$

In particular, if $\varphi_0 = \phi_0$, then

$$\left(\tilde{\Lambda} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}, \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) = \int_0^T \int_{\Gamma_0} |\phi_{\nu_{g_1}}|^2 d\Gamma_{g_1} dt. \quad (3.31)$$

3.1. Distributed control

Fortunately, the observability inequality of the system (3.23) (or equivalently the system (3.21)) has been studied by [16] at the norm of the space $H_0^1(\Omega)$.

Lemma 3.1. Let $\omega \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ be an exactly controllable equilibrium, and $T > 0$ be arbitrarily given. Assume $\langle h(x), \nu_{g_1}(x) \rangle_{g_1} \leq 0$ for $x \in \Gamma_1$, where $h(x)$ is given in (1.9), ν_{g_1} is the unit outward normal vector to Γ in the Riemannian metric g_1 . Then, there exist two constants $C_{1T}, C_{2T} > 0$ such that

$$C_{1T} \|\phi_0\|_1^2 \leq \int_0^T \int_{\Gamma_0} |\phi_{\nu_{g_1}}|^2 d\Gamma_{g_1} d\tau \leq C_{2T} \|\phi_0\|_1^2. \quad (3.32)$$

Remark 3.1. Theorem 2.3.1 in [16] gives the observability inequality in $H^1(\Omega)$ level to problems of the form:

$$\begin{cases} i\phi_t = -\Delta_{g_1} \phi + i\langle F, \nabla_{g_1} \phi \rangle_{g_1} + l(x)\phi & \text{in } (0, T) \times \Omega, \\ \phi = 0 & \text{on } (0, T) \times \Gamma, \\ \phi(0) = \phi_0 & \text{in } \Omega, \end{cases} \quad (3.33)$$

where $l(x) \in L^\infty(\Omega)$. It seems that we can't use Theorem 2.3.1 in [16] to the problem (3.23). However, since $\bar{\phi}$ is zero-order term which does not affect the energy estimates in the proof of Theorem 2.3.1 in [16], the problem (3.23) still has the observability inequality in $H^1(\Omega)$ level. The right-hand side of the inequality (3.32) is easy to be obtained.

Remark 3.2. Let $\phi = \phi_1 + i\phi_2$. It is easy to check that the norm $\|\phi\|_s$ is equivalent to the norm $\|(\phi_1, \phi_2)\|_s$ for all $\phi \in H^s(\Omega)$ and any integer $s \geq -1$, where $\|(\cdot, \cdot)\|_s$ is the norm of the product space $H^s(\Omega) \times H^s(\Omega)$. Thus by the formula (3.31), the inequality (3.32) is equivalent to the inequality

$$C_{1T} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_1^2 \leq \left(\tilde{\Lambda} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}, \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) \leq C_{2T} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_1^2. \quad (3.34)$$

Then the map $\tilde{\Lambda}$ is an isomorphism from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $H^{-1}(\Omega) \times H^{-1}(\Omega)$ by Lax–Milgram theorem (or equivalently, Λ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$). By Lemma 3.1, we have constructed a control $-i\phi_{v_{g_1}} \in L^2((0, T) \times \Gamma_0)$ which moves the initial state $\psi(0) \in H^{-1}(\Omega)$ of the system (3.26) to rest at the time T .

However, the above control strategy only gives distributed control function because solution $\psi(t, x)$ of the controlled system (3.26) is only in $H^{-1}(\Omega)$ no matter ϕ_0 is smooth or not. Indeed, since $-i\phi_{v_{g_1}} \neq 0$ for any $x \in \Gamma_0$, the compatible condition $\psi(T) = -i\phi_{v_{g_1}}(T)$ for $x \in \Gamma_0$ is never true.

3.2. Smooth control

We shall modify the above control strategy to obtain smooth control as in [18]. Our main results of this section are

Theorem 3.1. *Let k be an integer with $2k \geq [n/2] + 4$. Assume the same hypothesis as in Lemma 3.1. Then, for any $\tilde{v}_0 \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$, there is a control function $\tilde{\varphi} \in \tilde{\mathcal{X}}_0^{2k-1}(T)$ such that the solution $\tilde{v} \in \bigcap_{l=0}^{k-1} C^l([0, T]; H^{2k-1-2l}(\Omega))$ of the problem (3.13) satisfies $\tilde{v}(T) = 0$.*

In the following we shall show that Theorem 3.1 is equivalent to observability estimates. Let $T > T_1 > 0$. We assume that $z \in C^\infty(-\infty, +\infty)$ is such that $0 \leq z(t) \leq 1$ with

$$z(t) = \begin{cases} 0, & t \geq T, \\ 1, & t \leq T_1. \end{cases} \quad (3.35)$$

Let l be a positive integer. Let $\mathcal{E}_0^{2l+1}(\Omega)$ consist of all functions $u \in H^{2l+1}(\Omega)$ with the boundary conditions

$$B^{*s} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Big|_{\Gamma} = 0, \quad \text{or equivalently,} \quad \hat{B}^s u|_{\Gamma} = 0, \quad 0 \leq s \leq l, \quad (3.36)$$

with the norm of $H^{2l+1}(\Omega)$, where B^* and \hat{B} are given by (3.22) and (3.24) respectively.

Given $\phi_0 = \phi_{10} + i\phi_{20} \in \mathcal{E}_0^{2k+1}(\Omega)$, we solve the real problem (3.21) to have a complex-valued solution $\phi = \phi_1 + i\phi_2$. Then instead of (3.26), we solve the following problem

$$\begin{cases} i\psi_t = \mathcal{B}\psi & \text{in } (0, T) \times \Omega, \\ \psi = 0 & \text{on } (0, T) \times \Gamma_1, \quad \psi = -iz(t)\phi_{v_{g_1}} & \text{on } (0, T) \times \Gamma_0, \\ \psi(T) = 0 & \text{in } \Omega. \end{cases} \quad (3.37)$$

Let Λ be given in (3.29) where ψ in (3.29) is the solution to the problem (3.37) this time. By the identity (3.30), we have

$$\left(\tilde{\Lambda} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}, \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) = \int_0^T \int_{\Gamma_0} z(t) (\phi_{1v_{g_1}} \varphi_{1v_{g_1}} + \phi_{2v_{g_1}} \varphi_{2v_{g_1}}) d\Gamma_{g_1} dt. \quad (3.38)$$

We need the following boundary regularity.

Lemma 3.2. *Let ϕ solve the problem (3.23) with the initial data $\phi_0 \in \mathcal{E}_0^3(\Omega)$. Then $\phi_{v_{g_1}} \in H^1((0, T) \times \Gamma_0)$ with $\|\phi_{v_{g_1}}\|_{H^1((0, T) \times \Gamma_0)} \leq C_T \|\phi_0\|_3$ for some constant $C_T > 0$.*

Proof. Formal differentiation of the problem (3.23) with respect to t yields

$$\begin{cases} i\phi_{tt} = \hat{\mathcal{B}}\phi_t & \text{in } (0, T) \times \Omega, \\ \phi_t = 0 & \text{on } (0, T) \times \Gamma, \\ \phi_t(0) = -i\hat{\mathcal{B}}\phi_0 & \text{in } \Omega. \end{cases} \quad (3.39)$$

The condition $\phi_0 \in \mathcal{E}_0^3(\Omega)$ implies $\hat{\mathcal{B}}\phi_0 \in H_0^1(\Omega)$. By Lemma 3.1 we have

$$\int_0^T \int_{\Gamma_0} |\phi_{tv_{g_1}}|^2 d\Gamma_{g_1} dt \leq C \|\hat{\mathcal{B}}\phi_0\|_1^2 \leq C \|\phi_0\|_3^2. \quad (3.40)$$

We next prove that $\phi_{v_{g_1}} \in L^2((0, T); H^1(\Gamma))$. Let X be a vector field on the manifold Γ , that is, $X(x) \in \Gamma_x$ for each $x \in \Gamma$. We extend X to the whole $\bar{\Omega}$ to be a vector field on the manifold $(\bar{\Omega}, g_1)$. Let

$$v = X(\phi), \quad (t, x) \in (0, T) \times \Omega. \quad (3.41)$$

Then v solves

$$\begin{cases} iv_t = \hat{B}v + [X, \hat{B}]\phi & \text{in } (0, T) \times \Omega, \\ v = 0 & \text{on } (0, T) \times \Gamma, \\ v(0) = X(\phi_0) & \text{in } \Omega, \end{cases} \quad (3.42)$$

where $[X, \hat{B}]\phi = X(\hat{B}\phi) - \hat{B}X(\phi)$. Note that $[X, \hat{B}]\phi$ is a second-order term.

We use the inequality (2.53) to the problem (3.39) and the ellipticity of the operator \hat{B} to obtain

$$\|\phi(t)\|_3 \leq C\|\phi_0\|_3, \quad 0 \leq t \leq T. \quad (3.43)$$

Applying the inequalities (2.54) and (3.43), we have

$$\int_0^T \int_{\Gamma} |v_{v_{g_1}}|^2 d\Gamma_{g_1} dt \leq C \left(\|X(\phi_0)\|_1^2 + \int_0^T \| [X, \hat{B}]\phi(t) \|_1^2 dt \right) \leq C_T \|\phi_0\|_3. \quad (3.44)$$

Since

$$X(\phi_{v_{g_1}}) = v_{g_1}(X(\phi)) + [X, v_{g_1}]\phi,$$

by virtue of the inequalities (3.44), (3.43) and trace theorem, we have

$$\int_0^T \int_{\Gamma} |X(\phi_{v_{g_1}})|^2 d\Gamma_{g_1} dt \leq C_T \|\phi_0\|_3^2,$$

for any vector field X on the submanifold Γ , that is, $\phi_{v_{g_1}} \in L^2((0, T); H^1(\Gamma))$. \square

We need the following ellipticity of the operator B which is defined in (3.18).

Lemma 3.3. *Let the operator B be defined in (3.18). Then for any integer $s \geq 1$, there is a constant $C > 0$ such that*

$$\left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_s \leq C \left(\left\| B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{s-2} + \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{H^{s-1/2}(\Gamma)} + \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| \right), \quad (3.45)$$

for all $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that $B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H^{s-2}(\Omega)$.

Proof. We use the inequality (2.5), the formula (3.18) and Lemma 2.5, to obtain

$$\begin{aligned} \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_s^2 &= \|v_1\|_s^2 + \|v_2\|_s^2 \leq C (\|\Delta_{g_1} v_1 + h_3 v_1\|_{s-2}^2 + \|\Delta_{g_1} v_2 + h_2 v_2\|_{s-2}^2 \\ &\quad + \|v_1\|_{H^{s-1/2}(\Gamma)}^2 + \|v_2\|_{H^{s-1/2}(\Gamma)}^2 + \|v_1\|^2 + \|v_2\|^2) \\ &\leq C (\|\Delta_{g_1} v_1 + \langle F, \nabla_{g_1} v_2 \rangle_{g_1} + h_3 v_1 + h_4 v_2\|_{s-2}^2 + \|\nabla_{g_1} v_2\|_{s-2}^2 \\ &\quad + \|\Delta_{g_1} v_2 + \langle F, \nabla_{g_1} v_1 \rangle_{g_1} + h_1 v_1 + h_2 v_2\|_{s-2}^2 + \|\nabla_{g_1} v_1\|_{s-2}^2 + \|v_2\|_{s-2}^2 \\ &\quad + \|v_1\|^2 + \|v_2\|^2 + \|v_1\|_{H^{s-1/2}(\Gamma)}^2 + \|v_2\|_{H^{s-1/2}(\Gamma)}^2 + \|v_1\|_{s-2}^2) \\ &\leq C \left(\left\| B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{s-2}^2 + \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{H^{s-1/2}(\Gamma)}^2 + \epsilon \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_s^2 + C_\epsilon \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|^2 \right), \end{aligned} \quad (3.46)$$

$0 < \epsilon < 1$, which completes the proof. \square

Remark 3.3. It is easy to check that Lemma 3.3 is also true if the operator B is replaced by B^* .

Lemma 3.4. Let k be an integer with $2k \geq [n/2] + 4$. Let l be an integer with $k \geq l \geq 0$. Assume the same condition as in Lemma 3.1. Then there are constants $C_{1T}, C_{2T} > 0$ such that

$$C_{1T} \|\phi_0\|_{2l+1} \leq \|\Lambda \phi_0\|_{2l-1} \leq C_{2T} \|\phi_0\|_{2l+1}. \quad (3.47)$$

In particular, Λ are isomorphisms from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and from $\Xi_0^{2l+1}(\Omega)$ to $H^{2l-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ for $l \geq 1$, respectively.

Proof. According to Remark 3.2 and the definitions of Λ and $\tilde{\Lambda}$, the inequality (3.47) is equivalent to the following inequality

$$C_{1T} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+1} \leq \left\| \tilde{\Lambda} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l-1} \leq C_{2T} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+1}. \quad (3.48)$$

We then prove the inequality (3.48) instead of the inequality (3.47).

Step 1. Let $\phi_0 = \phi_{10} + i\phi_{20} \in \Xi_0^{2l+1}(\Omega)$. Let (ψ_1, ψ_2) solve the problem (3.27) with the boundary control $(z\phi_{2\nu_{g_1}}, -z\phi_{1\nu_{g_1}})$. In the sequel, C, C_T are general constants, and C_T depends on T as well. We use Lemma 3.3 on the operator B to obtain

$$\begin{aligned} \left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{2l-1} &\leq C \left(\left\| B \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{2l-3} + \left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{H^{2l-3/2}(\Gamma_0)} + \left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\| \right) \\ &\leq C \left(\left\| B^2 \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{2l-5} + \left\| B \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{H^{2l-7/2}(\Gamma)} + \left\| B^2 \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\| \right. \\ &\quad \left. + \left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{H^{2l-3/2}(\Gamma_0)} + \left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\| \right) \\ &\leq \dots \\ &\leq C \left(\left\| B^{l-1} \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_1 + \sum_{s=1}^{l-2} \left\| B^s \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{H^{2l-2s-3/2}(\Gamma)} \right. \\ &\quad \left. + \sum_{s=0}^{l-2} \left\| B^s \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{H^{2l-3/2}(\Gamma_0)} \right). \end{aligned} \quad (3.49)$$

We firstly give some estimates on $\|B^{l-1}(\frac{\psi_1(0)}{\psi_2(0)})\|_1$ and $\sum_{s=0}^{l-2} \|B^s(\frac{\psi_1(0)}{\psi_2(0)})\|$.

Denote $B^s(\frac{\psi_1(0)}{\psi_2(0)})$ by $(\frac{\psi_1^s}{\psi_2^s})$ for $0 \leq s \leq k-1$. Then $(\frac{\psi_1^{(s)}}{\psi_2^{(s)}})$ solves

$$\begin{cases} \begin{pmatrix} \psi_1^{(s+1)} \\ \psi_2^{(s+1)} \end{pmatrix} = B \begin{pmatrix} \psi_1^{(s)} \\ \psi_2^{(s)} \end{pmatrix} & \text{in } (0, T) \times \Omega, \\ \begin{pmatrix} \psi_1^{(s)} \\ \psi_2^{(s)} \end{pmatrix} = 0 & \text{on } (0, T) \times \Gamma_1, \quad \begin{pmatrix} \psi_1^{(s)} \\ \psi_2^{(s)} \end{pmatrix} = \sum_{q=0}^s C_s^q z^{(q)} \begin{pmatrix} \phi_{2\nu_{g_1}}^{(s-q)} \\ -\phi_{1\nu_{g_1}}^{(s-q)} \end{pmatrix} & \text{on } (0, T) \times \Gamma_0, \\ \begin{pmatrix} \psi_1^{(s)}(T) \\ \psi_2^{(s)}(T) \end{pmatrix} = 0, \quad \begin{pmatrix} \psi_1^{(s)}(0) \\ \psi_2^{(s)}(0) \end{pmatrix} = \begin{pmatrix} \Psi_1^s \\ \Psi_2^s \end{pmatrix} & \text{in } \Omega, \end{cases} \quad (3.50)$$

where C_s^q is the binomial coefficient, or equivalently, $\psi^{(s)}$ solves

$$\begin{cases} i\psi^{(s+1)} = \mathcal{B}\psi^{(s)} & \text{in } (0, T) \times \Omega, \\ \psi^{(s)} = 0 & \text{on } (0, T) \times \Gamma_1, \quad \psi^{(s)} = -i \sum_{q=0}^s C_s^q z^{(q)} \phi_{\nu_{g_1}}^{(s-q)} & \text{on } (0, T) \times \Gamma_0, \\ \psi^{(s)}(T) = 0, \quad \psi^{(s)}(0) = \Psi_1^s + i\Psi_2^s & \text{in } \Omega. \end{cases} \quad (3.51)$$

Using the inequality (2.14) to the system (3.51) with $\psi^{(s)}(T)$ viewed as the initial data, we have

$$\left\| \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_1 \leq C \|\psi^{(s)}(0)\|_1 \leq C_T \sum_{q=0}^s \|\phi_{\nu_{g_1}}^{(s-q)}\|_{H^1((0,T) \times \Gamma_0)}. \quad (3.52)$$

For $0 \leq s \leq l-1$, $(\phi_1^{(s)}, \phi_2^{(s)})$ is the solution to the system (3.21) with the initial data $\begin{pmatrix} \phi_1^s \\ \phi_2^s \end{pmatrix} \triangleq B^{*s} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}$ (or equivalently, $\phi^{(s)}$ is the solution to the system (3.23) with the initial data $\Phi_1^s + i\Phi_2^s$), and $\Phi_1^s + i\Phi_2^s \in \Xi_0^3(\Omega)$. We then use Lemma 3.2 to have

$$\|\phi_{v_{g_1}}^{(s)}\|_{H^1((0,T) \times \Gamma_0)} \leq C \|\Phi_1^s + i\Phi_2^s\|_3 = C \|B^{*s} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}\|_3 \leq C \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2s+3}. \quad (3.53)$$

Moreover, for $0 \leq s \leq l-2$, invoking trace theorem we have

$$\begin{aligned} \left\| B^s \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} \right\|_{H^{2l-2s-3/2}(\Gamma)} &= \left\| \begin{pmatrix} \psi_1^{(s)}(0) \\ \psi_2^{(s)}(0) \end{pmatrix} \right\|_{H^{2l-2s-3/2}(\Gamma)} \\ &= \left\| \begin{pmatrix} \phi_{1v_{g_1}}^{(s)}(0) \\ \phi_{2v_{g_1}}^{(s)}(0) \end{pmatrix} \right\|_{H^{2l-2s-3/2}(\Gamma_0)} = \left\| \begin{pmatrix} B^{*s} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \end{pmatrix} \right\|_{v_{g_1}} \Big|_{H^{2l-2s-3/2}(\Gamma_0)} \\ &\leq C \|(\phi_{10}, \phi_{20})\|_{2l} \leq C \|(\phi_{10}, \phi_{20})\|_{2l+1}. \end{aligned} \quad (3.54)$$

We insert (3.52) and (3.53) with $s \leq l-1$ and (3.54) with $s \leq l-2$ into (3.49) to obtain the right-hand side of the inequality (3.48).

Step 2. Next, we proceed to prove the left-hand side of the inequality (3.48) by induction in l . Lemma 3.1 and Remark 3.2 show that the inequality (3.48) is true for $l=0$. Let the left-hand side of the inequality (3.48) be true for some integer $k-1 \geq l \geq 0$. We will show that it is also true with l replaced by $l+1$.

Take $\phi_0 \in \Xi_0^{4l+5}(\Omega)$. Then $(\phi_1^{(2l+2)}, \phi_2^{(2l+2)})$ is the solution to the system (3.21) with the initial data $B^{*2l+2} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}$. According to the identity (3.38), we have

$$\left(\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}, B^{*2l+2} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) = \int_0^T \int_{\Gamma_0} z(t) (\phi_{1v_{g_1}} \phi_{1v_{g_1}}^{(2l+2)} + \phi_{2v_{g_1}} \phi_{2v_{g_1}}^{(2l+2)}) d\Gamma_{g_1} dt. \quad (3.55)$$

On one hand, we use the Green formula (3.28) to the left-hand side of (3.55), to obtain

$$\begin{aligned} \text{LHS of (3.55)} &= - \left(B \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}, B^{*2l+1} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) + \int_{\Gamma_0} (\psi_2(0) \phi_{1v_{g_1}}^{2l+1} - \psi_1(0) \phi_{2v_{g_1}}^{2l+1}) d\Gamma_{g_1} \\ &= \left(B^2 \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}, B^{*2l} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) \\ &\quad - \int_{\Gamma_0} (\psi_2^1 \phi_{1v_{g_1}}^{2l} - \psi_1^1 \phi_{2v_{g_1}}^{2l}) d\Gamma_{g_1} + \int_{\Gamma_0} (\psi_2(0) \phi_{1v_{g_1}}^{2l+1} - \psi_1(0) \phi_{2v_{g_1}}^{2l+1}) d\Gamma_{g_1} \\ &= \dots \\ &= (-1)^{l+1} \left(B^{l+1} \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}, B^{*l+1} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right) \\ &\quad + \sum_{s=0}^l (-1)^s \int_{\Gamma_0} (\psi_2^s \phi_{1v_{g_1}}^{2l+1-s} - \psi_1^s \phi_{2v_{g_1}}^{2l+1-s}) d\Gamma_{g_1}. \end{aligned} \quad (3.56)$$

On the other hand, integration by parts with respect to t on the right-hand side of the identity (3.55) gives

$$\begin{aligned} \text{RHS of (3.55)} &= - \int_{\Gamma_0} (\phi_{1v_{g_1}}(0) \phi_{1v_{g_1}}^{(2l+1)}(0) + \phi_{2v_{g_1}}(0) \phi_{2v_{g_1}}^{(2l+1)}(0)) d\Gamma_{g_1} \\ &\quad - \int_0^T \int_{\Gamma_0} [(z(t) \phi_{1v_{g_1}})_t \phi_{1v_{g_1}}^{(2l+1)} + (z(t) \phi_{2v_{g_1}})_t \phi_{2v_{g_1}}^{(2l+1)}] d\Gamma_{g_1} dt \\ &= - \int_{\Gamma_0} (\phi_{1v_{g_1}}(0) \phi_{1v_{g_1}}^{(2l+1)}(0) + \phi_{2v_{g_1}}(0) \phi_{2v_{g_1}}^{(2l+1)}(0)) d\Gamma_{g_1} \\ &\quad + \int_{\Gamma_0} ((\phi_{1v_{g_1}})_t(0) \phi_{1v_{g_1}}^{(2l)}(0) + (\phi_{2v_{g_1}})_t(0) \phi_{2v_{g_1}}^{(2l)}(0)) d\Gamma_{g_1} \end{aligned}$$

$$\begin{aligned}
& + \int_0^T \int_{\Gamma_0} [(z(t)\phi_{1\nu_{g_1}})_{tt} \phi_{1\nu_{g_1}}^{(2l)} + (z(t)\phi_{2\nu_{g_1}})_{tt} \phi_{2\nu_{g_1}}^{(2l)}] d\Gamma_{g_1} dt \\
& = \dots \\
& = (-1)^{l+1} \int_0^T \int_{\Gamma_0} [(z(t)\phi_{1\nu_{g_1}})^{(l+1)} \phi_{1\nu_{g_1}}^{(l+1)} + (z(t)\phi_{2\nu_{g_1}})^{(l+1)} \phi_{2\nu_{g_1}}^{(l+1)}] d\Gamma_{g_1} dt \\
& \quad + \sum_{s=0}^l (-1)^{s+1} \int_{\Gamma_0} (\phi_{1\nu_{g_1}}^{(s)}(0) \phi_{1\nu_{g_1}}^{(2l+1-s)}(0) + \phi_{2\nu_{g_1}}^{(s)}(0) \phi_{2\nu_{g_1}}^{(2l+1-s)}(0)) d\Gamma_{g_1}. \tag{3.57}
\end{aligned}$$

On Γ_0 , we have

$$\begin{pmatrix} \psi_1^s \\ \psi_2^s \end{pmatrix} = B^s \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} \psi_1^{(s)}(0) \\ \psi_2^{(s)}(0) \end{pmatrix} = \begin{pmatrix} \phi_{2\nu_{g_1}}^{(s)}(0) \\ -\phi_{1\nu_{g_1}}^{(s)}(0) \end{pmatrix}, \tag{3.58}$$

and in Ω , we have

$$\begin{pmatrix} \phi_1^{2l+1-s} \\ \phi_2^{2l+1-s} \end{pmatrix} = B^{*2l+1-s} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_1^{(2l+1-s)}(0) \\ \phi_2^{(2l+1-s)}(0) \end{pmatrix}, \quad s = 0, \dots, l. \tag{3.59}$$

Insertion of (3.56)–(3.59) into (3.55) yields

$$\begin{pmatrix} B^{l+1} \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}, B^{*l+1} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \end{pmatrix} = \int_0^T \int_{\Gamma_0} [(z(t)\phi_{1\nu_{g_1}})^{(l+1)} \phi_{1\nu_{g_1}}^{(l+1)} + (z(t)\phi_{2\nu_{g_1}})^{(l+1)} \phi_{2\nu_{g_1}}^{(l+1)}] d\Gamma_{g_1} dt. \tag{3.60}$$

We use the relation (3.18) and integration by parts on the left-hand side of the identity (3.60) to obtain

$$\begin{aligned}
\text{LHS of (3.60)} & = \left(B \begin{pmatrix} \psi_1^l \\ \psi_2^l \end{pmatrix}, \begin{pmatrix} \phi_1^{l+1} \\ \phi_2^{l+1} \end{pmatrix} \right) \\
& = \int_{\Omega} (\langle \nabla_{g_1} \psi_2^l, \nabla_{g_1} \phi_1^{l+1} \rangle_{g_1} - \langle \nabla_{g_1} \psi_1^l, \nabla_{g_1} \phi_2^{l+1} \rangle_{g_1}) d\Omega_{g_1} \\
& \quad + \left(\left(\langle F, \nabla_{g_1} \psi_1^l \rangle_{g_1} + h_1 \psi_1^l + h_2 \psi_2^l \right), \begin{pmatrix} \phi_1^{l+1} \\ \phi_2^{l+1} \end{pmatrix} \right) \\
& \quad + \left(\left(\langle F, \nabla_{g_1} \psi_2^l \rangle_{g_1} + h_3 \psi_1^l + h_4 \psi_2^l \right), \begin{pmatrix} \phi_1^{l+1} \\ \phi_2^{l+1} \end{pmatrix} \right) \\
& \leq C (\|\psi_1^l\|_1 + \|\psi_2^l\|_1) (\|\phi_1^{l+1}\|_1 + \|\phi_2^{l+1}\|_1) \\
& \leq C (\|\psi_1(0)\|_{2l+1} + \|\psi_2(0)\|_{2l+1}) (\|\phi_{10}\|_{2l+3} + \|\phi_{20}\|_{2l+3}). \tag{3.61}
\end{aligned}$$

We use the formula (3.35) and the property (2.38) on the right-hand side of (3.60) to obtain

$$\begin{aligned}
\text{RHS of (3.60)} & = \int_0^T \int_{\Gamma_0} z(t) (|\phi_{1\nu_{g_1}}^{(l+1)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1)}|^2) d\Gamma_{g_1} dt \\
& \quad + \sum_{s=1}^{l+1} \int_0^T \int_{\Gamma_0} C_{l+1}^s z^{(s)}(t) (\phi_{1\nu_{g_1}}^{(l+1-s)} \phi_{1\nu_{g_1}}^{(l+1)} + \phi_{2\nu_{g_1}}^{(l+1-s)} \phi_{2\nu_{g_1}}^{(l+1)}) d\Gamma_{g_1} dt \\
& \geq \int_0^{T_1} \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1)}|^2) d\Gamma_{g_1} dt - \epsilon \int_0^T \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1)}|^2) d\Gamma_{g_1} dt \\
& \quad - C_{\epsilon} \sum_{s=1}^{l+1} \int_0^T \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1-s)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1-s)}|^2) d\Gamma_{g_1} dt. \tag{3.62}
\end{aligned}$$

Since $\phi^{(l+1-s)}$ is the solution to the system (3.23) with the initial data $(-i)^{l+1-s} \hat{\mathcal{B}}^{l+1-s} \phi_0$ which belongs to $H_0^1(\Omega)$ for $s = 1, \dots, l+1$, by virtue of Lemma 3.1, we have

$$\int_0^T \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1-s)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1-s)}|^2) d\Gamma_{g_1} dt \leq C_T \|\hat{B}^{l+1-s} \phi_0\|_1^2 \leq C \|\phi_0\|_{2l+1}^2, \quad s = 1, \dots, l+1, \quad (3.63)$$

and

$$\int_0^T \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1)}|^2) d\Gamma dt \leq C_T \|\phi_0\|_{2l+3}^2. \quad (3.64)$$

On the other hand, we use Lemma 3.1 again to get

$$\begin{aligned} \int_0^{T_1} \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1)}|^2) d\Gamma_{g_1} dt &\geq C_{T_1} \|\hat{B}^{l+1} \phi_0\|_1^2 = C_{T_1} \|\Phi_1^{l+1} + i\Phi_2^{l+1}\|_1^2 \\ &\geq C_{T_1} \|(\Phi_1^{l+1}, \Phi_2^{l+1})\|_1^2 = C_{T_1} \|B^{*l+1} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix}\|_1^2. \end{aligned} \quad (3.65)$$

We use Lemma 3.3 with the operator B replaced by the operator B^* , to get

$$\begin{aligned} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3} &\leq C \left(\left\| B^* \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+1} + \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\| \right) \\ &\leq \dots \\ &\leq C \left(\left\| B^{*l+1} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_1 + \sum_{s=0}^l \left\| B^{*s} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\| \right) \\ &\leq C \left(\left\| B^{*l+1} \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_1 + \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l} \right), \end{aligned} \quad (3.66)$$

and consequently,

$$\int_0^{T_1} \int_{\Gamma_0} (|\phi_{1\nu_{g_1}}^{(l+1)}|^2 + |\phi_{2\nu_{g_1}}^{(l+1)}|^2) d\Gamma dt \geq C_{T_1} \left(\left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3}^2 - \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l}^2 \right). \quad (3.67)$$

We insert the inequalities (3.63), (3.64) and (3.67) into (3.62), and choose $\epsilon > 0$ so small, to obtain

$$\text{RHS of (3.60)} \geq C_T \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3}^2 - C_T \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+1}^2. \quad (3.68)$$

We combine (3.61) with (3.68), and use the induction to get

$$\begin{aligned} C_T \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3}^2 &\leq C \left\| (\psi_1(0), \psi_2(0)) \right\|_{2l+1} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3} + C_T \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+1}^2 \\ &\leq C \left\| (\psi_1(0), \psi_2(0)) \right\|_{2l+1} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3} \\ &\quad + C_T \left\| (\psi_1(0), \psi_2(0)) \right\|_{2l-1} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+1} \\ &\leq C_T \left\| (\psi_1(0), \psi_2(0)) \right\|_{2l+1} \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3}, \end{aligned} \quad (3.69)$$

and hence

$$C_T \left\| \begin{pmatrix} \phi_{10} \\ \phi_{20} \end{pmatrix} \right\|_{2l+3, 2l+3} \leq \left\| (\psi_1(0), \psi_2(0)) \right\|_{2l+1, 2l+1}. \quad (3.70)$$

As $\mathcal{E}_0^{4l+5}(\Omega)$ is dense in $\mathcal{E}_0^{2l+3}(\Omega)$, the inequality (3.70) holds with $\phi_0 \in \mathcal{E}_0^{2l+3}(\Omega)$. Then the left-hand side of the inequality (3.48) follows by induction.

From Remark 3.2, we know that $\Lambda : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isomorphism. Then for any $\psi \in H^{2l-1}(\Omega)$, there is a $\phi_0 \in H_0^1(\Omega)$ such that $\psi = \Lambda(\phi_0)$. In addition the inequality (3.47) shows that $\psi \in H^{2l-1}(\Omega)$ if and only if $\phi_0 \in H^{2l+1}(\Omega)$. Finally, we obtain $H^{2l-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega) = \{\psi \in H^{2l-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega); \exists \phi \in H_0^1(\Omega) \text{ such that } \Lambda(\phi) = \psi, \text{ and } \|\phi\|_{2l+1} < \infty\}$. Then Λ is an isomorphism from $\mathcal{E}_0^{2l+1}(\Omega)$ to $H^{2l-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$. \square

Proof of Theorem 3.1. Let $\tilde{v}_0 \in H^{2k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)$ be given. By Lemma 3.4, there is $\phi_0 \in \mathcal{E}_0^{2k+1}(\Omega)$ such that the control $-iz\phi_{\nu_{g_1}}$ on $(0, T) \times \Gamma_0$ drives the system (3.13) to rest at time T , where ϕ is the solution to the problem (3.23) with the initial data ϕ_0 .

Since $\phi^{(k-1)}$ is the solution to the problem (3.23) with the initial data $(-i)^{k-1} \hat{B}^{k-1} \phi_0 \in \mathcal{E}_0^3(\Omega)$, Lemma 3.2 implies $-iz\phi_{v_{g_1}} \in \tilde{\mathcal{X}}_0^{2k-1}(T)$. \square

4. Globally exact controllability

Proof of Theorem 1.2. By Theorem 1.1 and the compactness principle it will suffice to prove that the map $\Pi : [0, 1] \rightarrow H^{2k-1}(\Omega)$ with $\Pi(\alpha) = \omega_\alpha$ is continuous in $\alpha \in [0, 1]$.

We show that $\{\omega_\alpha; \alpha \in [0, 1]\}$ is compact in $H^{2k-1}(\Omega)$. For fixed $\alpha \in [0, 1]$, by virtue of the condition (1.2) the problem (1.11) is equivalent to the following problem

$$\begin{cases} -\Delta \omega_\alpha + b_2(x, \omega_\alpha) \omega_\alpha = 0, & x \in \Omega, \\ \omega_\alpha|_\Gamma = \alpha \omega|_\Gamma, \end{cases} \quad (4.1)$$

where $b_2(x, \xi) \equiv a^{-1}(x, \xi) b_1(x, \xi) \geq 0$ for $\xi \in \mathbb{C}$.

Applying the maximum principle to the problem (4.1), gives

$$\sup_{x \in \Omega} |\omega_\alpha| \leq \alpha \sup_{x \in \Gamma} |\omega|. \quad (4.2)$$

Let

$$B_1(\alpha)u = -\Delta u + b_2(x, \omega_\alpha)u, \quad u \in H^2(\Omega), \quad \alpha \in [0, 1]. \quad (4.3)$$

Then $B_1(\alpha)\omega_\alpha = 0$. We use the ellipticity of the operator $B_1(\alpha)$, the property (ii) of Lemma 2.1 and the inequality (4.2) to have

$$\begin{aligned} \|\omega_\alpha\|_{2k-1} &\leq C_\alpha (\|B_1(\alpha)\omega_\alpha\|_{2k-3} + \|\omega_\alpha\|_{H^{2k-3/2}(\Gamma)} + \|\omega_\alpha\|) \\ &\leq C_\alpha \left(\alpha \|\omega\|_{H^{2k-3/2}(\Gamma)} + \sup_{x \in \Gamma} |\omega| \right). \end{aligned} \quad (4.4)$$

Then there is a sequence $\{\omega_{\alpha_l}\}_{l=1}^{+\infty}$ converging weakly in $H^{2k-1}(\Omega)$. According to the compact imbedding: $H^{2k-1}(\Omega) \rightarrow L^2(\Omega)$, there is a subsequence, still denoted by $\{\omega_{\alpha_l}\}_{l=1}^{+\infty}$, converging strongly in $L^2(\Omega)$. For any integer $m, l > 0$, using the ellipticity of the operator $B_1(\alpha_m)$, we get

$$\begin{aligned} \|\omega_{\alpha_m} - \omega_{\alpha_l}\|_{2k-1} &\leq C_{\alpha_m} (\|B_1(\alpha_m)(\omega_{\alpha_m} - \omega_{\alpha_l})\|_{2k-3} + \|\omega_{\alpha_m} - \omega_{\alpha_l}\|_{H^{2k-3/2}(\Gamma)} + \|\omega_{\alpha_m} - \omega_{\alpha_l}\|) \\ &= C_{\alpha_m} (\|B_1(\alpha_m)(\omega_{\alpha_m} - \omega_{\alpha_l})\|_{2k-3} + |\alpha_m - \alpha_l| \|\omega\|_{H^{2k-3/2}(\Gamma_0)} + \|\omega_{\alpha_m} - \omega_{\alpha_l}\|). \end{aligned} \quad (4.5)$$

Next, let us estimate $\|B_1(\alpha_m)(\omega_{\alpha_m} - \omega_{\alpha_l})\|_{2k-3}$.

The term $(B_1(\alpha_m) - B_1(\alpha_l))\omega_{\alpha_l}$ can be written as a sum of some terms of the form

$$f(x, \omega_{1\alpha_m}, \omega_{2\alpha_m}, \omega_{1\alpha_l}, \omega_{2\alpha_l}) \omega_{\alpha_l} (\omega_{s\alpha_m} - \omega_{s\alpha_l}), \quad s = 1, 2,$$

where the subscripts “1” and “2” denote the real and imaginary parts of a function. We apply the properties (i) and (ii) of Lemma 2.1 to the above products, via the bound (4.2), to obtain

$$\begin{aligned} \|B_1(\alpha_m)(\omega_{\alpha_m} - \omega_{\alpha_l})\|_{2k-3} &= \|(B_1(\alpha_m) - B_1(\alpha_l))\omega_{\alpha_l}\|_{2k-3} \leq C \|\omega_{\alpha_m} - \omega_{\alpha_l}\|_{2k-3} \\ &\leq \epsilon \|\omega_{\alpha_m} - \omega_{\alpha_l}\|_{2k-1} + C_\epsilon \|\omega_{\alpha_m} - \omega_{\alpha_l}\|. \end{aligned} \quad (4.6)$$

We insert (4.6) into (4.5) and choose $\epsilon > 0$ so small to get

$$\|\omega_{\alpha_m} - \omega_{\alpha_l}\|_{2k-1} \leq C (\|\omega_{\alpha_m} - \omega_{\alpha_l}\| + |\alpha_m - \alpha_l| \|\omega\|_{H^{2k-3/2}(\Gamma_0)}). \quad (4.7)$$

As $\{\alpha_m\}$ is bounded, there is a converging subsequence, which we still denote by $\{\alpha_m\}$. Then from the strong convergence of $\{\omega_{\alpha_l}\}$ in $L^2(\Omega)$ and the inequality (4.7) we know that $\{\omega_{\alpha_m}\}$ converges in $H^{2k-1}(\Omega)$. Consequently, $\{\omega_\alpha; \alpha \in [0, 1]\}$ is relatively compact in $H^{2k-1}(\Omega)$.

We extend the map Π to the whole line \mathbb{R} by

$$\tilde{\Pi}(\alpha) \equiv \begin{cases} 0, & \alpha < 0, \\ \omega_\alpha, & 0 \leq \alpha \leq 1, \\ \omega, & \alpha > 1. \end{cases} \quad (4.8)$$

We use the same method as above to show that $\{\tilde{\Pi}(\alpha); \alpha \in A\}$ is relatively compact in $H^{2k-1}(\Omega)$ for any bounded set $A \subset \mathbb{R}$. Then the map $\tilde{\Pi} : (-\infty, +\infty) \rightarrow H^{2k-1}(\Omega)$ is compact, and consequently continuous. \square

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