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The generalized nonlinear initial–boundary Riemann problem for linearly degenerate quasilinear hyperbolic systems of conservation laws

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ABSTRACT

This work is a continuation of our previous work, in the present paper we study the generalized nonlinear initial–boundary Riemann problem with small BV data for linearly degenerate quasilinear hyperbolic systems of conservation laws with nonlinear boundary conditions in a half space $\{(t,x) \mid t \geqslant 0,\ \ x \geqslant 0\}$. We prove the global existence and uniqueness of piecewise C^1 solution containing only contact discontinuities to a class of the generalized nonlinear initial–boundary Riemann problem, which can be regarded as a small BV perturbation of the corresponding nonlinear initial–boundary Riemann problem, for general $n \times n$ linearly degenerate quasilinear hyperbolic system of conservation laws; moreover, this solution has a global structure similar to the one of the self-similar solution $u = U(\frac{x}{t})$ to the corresponding nonlinear initial–boundary Riemann problem. Some applications to quasilinear hyperbolic systems of conservation laws arising in the string theory and high energy physics are also given.

1. Introduction and main result

Consider the following quasilinear hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(t,x) \in \mathcal{U} \subset \mathbf{R}^n, \tag{1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t,x), $f : \mathcal{U} \to \mathbf{R}^n$ is a given C^3 vector function of u.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given u on the domain under consideration, the Jacobian $A(u) = \nabla f(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \ldots, n)$:

$$l_i(u)A(u) = \lambda_i(u) l_i(u) \quad \big(\text{resp. } A(u) r_i(u) = \lambda_i(u) r_i(u)\big). \tag{1.3}$$

We have

$$\det\big|l_{ij}(u)\big| \neq 0 \quad \big(\text{equivalently, } \det\big|r_{ij}(u)\big| \neq 0\big). \tag{1.4}$$

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Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \dots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker symbol.

Clearly, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \dots, n)$ have the same regularity as $A(u)$, i.e., C^2 regularity.

We assume that on the domain under consideration, each characteristic field is linearly degenerate in the sense of Lax (cf. [1]):

$$\nabla\lambda_i(u)r_i(u) \equiv 0. \tag{1.7}$$

We are interested in solutions taking values in a small neighborhood of a given state in $\mathbf{R}^n$ and, without loss of generality, we can choose this set to be the ball $\mathcal{U} := \mathbf{B}(\eta)$ centered at the origin with suitably small radius η. We first recall that the Riemann problem for system (1.1) is a special Cauchy problem with the piecewise constant initial data

$$t = 0: \quad u = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{1.8}$$

where u_L and u_R are constant states in $\mathcal{U}$. It is well known that the Riemann problem (1.1) and (1.8) has a unique self-similar solution composed of $n+1$ constant states separated by contact discontinuities, provided that the states are in a small neighborhood of a given state (cf. [1]). In the following, the set $\mathcal{U}$ is chosen such that the Riemann problem is always well posed in this sense.

We assume that on the domain under consideration, the eigenvalues of $A(u) = \nabla f(u)$ satisfy the non-characteristic condition

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \dots, m;\ s = m+1, \dots, n). \tag{1.9}$$

We are concerned with the global existence and uniqueness of piecewise C^1 solutions to the generalized nonlinear initial–boundary Riemann problem for system (1.1) in a half space

$$D = \big\{(t, x) \bigm| t \geqslant 0,\ x \geqslant 0\big\} \tag{1.10}$$

with the initial condition

$$t = 0: \quad u = \varphi(x) \quad (x \geqslant 0) \tag{1.11}$$

and the nonlinear boundary condition (cf. [2,3])

$$x = 0: \quad v_s = G_s\big(\alpha(t), v_1, \dots, v_m\big) + h_s(t) \quad (s = m+1, \dots, n), \tag{1.12}$$

where

$$v_i = l_i(u)u \quad (i = 1, \dots, n) \tag{1.13}$$

and

$$\alpha(t) = \big(\alpha_1(t), \dots, \alpha_k(t)\big).$$

Here, $G_s \in C^1$ $(s = m+1, \dots, n)$, $\varphi = (\varphi_1, \dots, \varphi_n)^T$, α and $h(\cdot) = (h_{m+1}(\cdot), \dots, h_n(\cdot)) \in C^1$ with bounded C^1 norm, such that

$$\big\|\varphi(x)\big\|_{C^1}, \big\|\alpha(t)\big\|_{C^1}, \big\|h(t)\big\|_{C^1} \leqslant M, \tag{1.14}$$

for some $M > 0$ bounded but possibly large. Also, we assume that the conditions of C^0 compatibility are not satisfied at the point $(0, 0)$. Without loss of generality, we assume that

$$G_s\big(\alpha(t), 0, \dots, 0\big) \equiv 0 \quad (s = m+1, \dots, n). \tag{1.15}$$

Now, consider the nonlinear initial–boundary Riemann problem for system (1.1) in a half space

$$D = \big\{(t, x) \bigm| t \geqslant 0,\ x \geqslant 0\big\}$$

with the constant initial data

$$t = 0: \quad u = \varphi(0) \quad (x \geqslant 0) \tag{1.16}$$

and the nonlinear boundary condition (cf. [2])

$$x=0:\quad v_s=G_s\big(\alpha(0),v_1,\dots,v_m\big)+h_s(0) \\ \triangleq \overline{G}_s(v_1,\dots,v_m)\quad (s=m+1,\dots,n)\ (t\geqslant 0). \tag{1.17}$$

For this problem, Li and Wang [2] obtained the following well-known result.

Theorem 1.1. *Let $u_+\triangleq\varphi(0)$ and $v_i^+\triangleq l_i(u_+)u_+$ $(i=1,\dots,n)$ and suppose that the non-characteristic condition* (1.9) *holds. If $|u_+|$ and $|v_s^+-\overline{G}_s(v_1^+,\dots,v_m^+)|$ $(s=m+1,\dots,n)$ are suitably small, then the nonlinear initial–boundary problem* (1.1)–(1.16)–(1.17) (*known as Riemann problem*) *admits a unique small amplitude self-similar solution $u=U(\frac{x}{t})$ composed of $n-m+1$ constant states $\widehat{u}^{(m)},\widehat{u}^{(m+1)},\dots,\widehat{u}^{(n-1)},\widehat{u}^{(n)}=u_+$ separated by contact discontinuities, i.e.,*

$$u=U\left(\frac{x}{t}\right)=\begin{cases}\widehat{u}^{(m)}, & 0\leqslant x\leqslant\widehat{\lambda}_{m+1}t,\\ \widehat{u}^{(j)}, & \widehat{\lambda}_j t\leqslant x\leqslant\widehat{\lambda}_{j+1}t\ (j=m+1,\dots,n-1),\\ \widehat{u}^{(n)}, & x\geqslant\widehat{\lambda}_n t,\end{cases}$$

where $x=\widehat{\lambda}_j t$ stands for the j-th contact discontinuity $(j=m+1,\dots,n)$.

Remark 1.1. Let $u_-\triangleq(0,\dots,\overset{(m)}{0},h_{m+1}(0),\dots,h_n(0))^T$ and suppose that u_+ and u_- are data in $\mathcal{U}$. Then there exists a suitably small $\eta>0$ such that any Riemann problem (1.1)–(1.16)–(1.17) with data in $\mathcal{U}:=\mathbf{B}(\eta)$ is always well posed in the sense of Theorem 1.1. In the remainder of this paper we will consider the set $\mathcal{U}$ to be the ball $\mathcal{U}:=\mathbf{B}(\eta)$ centered at the origin with suitably small radius η in the sense of Theorem 1.1.

For the self-similar solution of the Riemann problem of general quasilinear hyperbolic systems of conservation laws, the local nonlinear structure stability has been proved by Li and Yu [4] for one-dimensional case, and by Majda [5] for multidimensional case. If each characteristic field with positive velocity is either linearly degenerate or genuinely nonlinear, Li and Wang [2] proved that the self-similar solution with small amplitude to the nonlinear initial–boundary Riemann problem has the global structure stability under perturbation (1.11)–(1.12) satisfying (1.16)–(1.17). Precisely speaking, they obtained the following well-known result.

Theorem 1.2. *Suppose that φ,α,G_s,h_s $(s=m+1,\dots,n)$ are all C^1 functions with respect to their arguments, satisfying that there exists a constant $\mu>0$ such that*

$$\theta\triangleq\sup_{x\geqslant 0}(1+x)^{1+\mu}\big(\big|\varphi(x)\big|+\big|\varphi'(x)\big|\big)+\sup_{t\geqslant 0}(1+t)^{1+\mu}\big(\big|\alpha(t)\big|+\big|h(t)\big|+\big|\alpha'(t)\big|+\big|h'(t)\big|\big)<\infty. \tag{1.18}$$

Suppose furthermore that the conditions of C^0 compatibility are not satisfied at the point (0, 0). *Suppose finally that the corresponding Riemann problem* (1.1)–(1.16)–(1.17) *admits a unique small amplitude self-similar solution $u=U(\frac{x}{t})$ composed of $n-m+1$ constant states $\widehat{u}^{(m)},\widehat{u}^{(m+1)},\dots,\widehat{u}^{(n-1)},\widehat{u}^{(n)}=u_+$ and $n-m$ small amplitude elementary waves $x=\widehat{\lambda}_k t$ $(k=m+1,\dots,n)$* (*shocks corresponding to the genuinely nonlinear characteristics and contact discontinuities corresponding to the linearly degenerate characteristics*):

$$u=U\left(\frac{x}{t}\right)=\begin{cases}\widehat{u}^{(m)}, & 0\leqslant x\leqslant\widehat{\lambda}_{m+1}t,\\ \widehat{u}^{(l)}, & \widehat{\lambda}_l t\leqslant x\leqslant\widehat{\lambda}_{l+1}t\ (l=m+1,\dots,n-1),\\ \widehat{u}^{(n)}, & x\geqslant\widehat{\lambda}_n t.\end{cases} \tag{1.19}$$

Then there exists $\theta_0>0$ so small that for any given $\theta\in(0,\theta_0]$, the generalized nonlinear initial–boundary Riemann problem (1.1)–(1.11)–(1.12) *admits a unique global piecewise C^1 solution*

$$u=u(t,x)=\begin{cases}u^{(m)}(t,x), & 0\leqslant x\leqslant x_{m+1}(t),\\ u^{(l)}(t,x), & x_l(t)\leqslant x\leqslant x_{l+1}(t)\ (l=m+1,\dots,n-1),\\ u^{(n)}(t,x), & x\geqslant x_n(t),\end{cases} \tag{1.20}$$

where, for $l=m,\dots,n$, $u^{(l)}(t,x)\in C^1$ satisfies system (1.1) *in the classical sense on the corresponding angular domain. Moreover, for $k=m+1,\dots,n$, $u^{(k-1)}(t,x)$ and $u^{(k)}(t,x)$ are connected to each other by the k-th small amplitude wave $x=x_k(t)$ with $x_k(0)=0$* (*the k-th shock corresponding to the genuinely nonlinear characteristic or the k-th contact discontinuity corresponding to the linearly degenerate characteristic*). *This solution possesses a global structure similar to that of the self-similar solution* (1.19) *to the nonlinear initial–boundary Riemann problem* (1.1)–(1.16)–(1.17).

Remark 1.2. Recently, under the assumption that Lax's Riemann solution of the system (1.1) only contains non-degenerate shocks and contact discontinuities but no centered rarefaction waves and other weak discontinuities, Kong [6,7] proved the global structure stability of this kind of Lax's Riemann solution with small amplitude, Shao [8,9] also studied that the global structure stability of this kind of Lax's Riemann solution with small amplitude in a half space.

However, it is well known that the BV space is a suitable framework for one-dimensional Cauchy problem for the hyperbolic systems of conservation laws (see Bressan [10], Glimm [11]), the result in Bressan [12] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large. So the following question arises naturally: if the BV norm of the initial and boundary data is suitably small, then can we obtain the global existence and uniqueness of piecewise C^1 solution containing only contact discontinuities to a class of the generalized nonlinear initial–boundary Riemann problem, which can be regarded as a small BV perturbation of the corresponding Riemann problem, for general $n \times n$ linearly degenerate quasilinear hyperbolic system of conservation laws? Here, it is important to mention that the global existence of weak solutions to a strictly hyperbolic system of conservation laws in one space dimension when the initial data is a small BV perturbation of a solvable Riemann problem has been proved by Schochet [13], unfortunately his method is not useful to show that the solutions are still contact discontinuities. An analogous result on stability of contact discontinuities under perturbations of small bounded variation is stated by Corli and Sable-Tougeron [14]. In this paper we exploit to some extent the ideas of Bressan [12], we will develop the method of using continuous Glimm's functional to solve this problem globally and to provide a new, concise proof of the above mentioned problem of the stability of contact discontinuities. The basic idea we will use here is to combine the techniques employed by Li and Kong [15], especially both the decomposition of waves and the global behavior of waves on the contact discontinuity curves, with the method of using continuous Glimm's functional. However, we must modify Glimm's functional in order to take care of the presence of contact discontinuities. This makes our new analysis more complicated than those for the C^1 solutions of the Cauchy problem for linearly degenerate quasilinear hyperbolic systems in Bressan [12], Dai and Kong [16], Zhou [17]. Moreover, due to the presence of a boundary, any waves with negative speed are expected to be reflected at the boundary, some additional difficulties appear. Therefore new proofs are required to overcome them. This also makes our new analysis more complicated than that for the Cauchy problem case in Shao [18]. The present paper can be viewed as a development of [12,16,17].

Our main results can be summarized as follows:

Theorem 1.3. *Suppose that system* (1.1) *is strictly hyperbolic and linearly degenerate. Suppose furthermore that φ, α, G_s and h_s ($s = m+1, \ldots, n$) are all C^1 functions with respect to their arguments, satisfying that the conditions of C^0 compatibility are not satisfied at the point* (0, 0)*, and the non-characteristic condition* (1.9) *holds. Suppose finally that u_+ and u_- are data in $\mathcal{U} := \mathbf{B}(\eta)$ and $\eta > 0$ is suitably small. Then for any constant $M > 0$, there exists a positive constant ε so small that if* (1.14) *holds together with*

$$\int_0^{+\infty} \left|\varphi'(x)\right| dx, \ \int_0^{+\infty} \left|\alpha'(t)\right| dt, \ \int_0^{+\infty} \left|h'(t)\right| dt \leqslant \varepsilon, \tag{1.21}$$

then the generalized nonlinear initial–boundary Riemann problem (1.1), (1.11) *and* (1.12) *admits a unique global piecewise C^1 solution $u = u(t,x)$ only containing $n-m$ contact discontinuities $x = x_i(t)$ ($x_i(0) = 0$) ($i = m+1, \ldots, n$) in a half space $\{(t,x) \mid t \geqslant 0,\ x \geqslant 0\}$. This solution has a global structure similar to the one of the self-similar solution $u = U(\frac{x}{t})$ of the corresponding Riemann problem* (1.1), (1.16) *and* (1.17)*. Precisely speaking,*

$$u = u(t,x) = \begin{cases} u^{(m)}(t,x), & 0 \leqslant x \leqslant x_{m+1}(t), \\ u^{(l)}(t,x), & x_l(t) \leqslant x \leqslant x_{l+1}(t)\ (l = m+1, \ldots, n-1), \\ u^{(n)}(t,x), & x \geqslant x_n(t), \end{cases}$$

where $u^{(l)}(t,x)$ ($l = m, \ldots, n$) are all C^1 solutions to system (1.1) *on each corresponding domain respectively and for $i = m+1, \ldots, n$, $u^{(i-1)}(t,x)$ and $u^{(i)}(t,x)$ are connected to each other by the i-th contact discontinuity $x = x_i(t)$ with $x_i(0) = 0$.*

Remark 1.3. Our result indicates that the self-similar solution $u = U(\frac{x}{t})$ to the nonlinear initial–boundary Riemann problem (1.1), (1.17) and (1.18) possesses a global nonlinear structure stability under a small BV perturbation of the initial and boundary data.

Remark 1.4. Suppose that system (1.1) is non-strictly hyperbolic but each characteristic has a constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) < \cdots < \lambda_m(u) < 0 < \lambda_m(u) < \cdots < \lambda_{p+1}(u) \equiv \cdots \equiv \lambda_n(u) \quad (m \leqslant p \leqslant n). \tag{1.22}$$

Then the conclusion of Theorem 1.3 still holds (cf. [16]).

Some of the results related to these topics are listed below. Chen et al. [19–21] investigated the asymptotic stability of Riemann waves for hyperbolic conservation laws. Hsiao and Tang [22] investigated the construction and qualitative behavior of the solution of the perturbated Riemann problem for the system of one-dimensional isentropic flow with damping. Xin et al. [23–25] proved the nonlinear stability of contact discontinuities in systems of conservation laws. Smoller et al. [26] investigated the instability of rarefaction shocks in systems of conservation laws. For the overcompressive shock waves, Liu [27] proved the nonlinear stability and instability. Bressan and LeFloch [28] investigated the structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws. Lions et al. [29] proved the existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. Recently, L^1 stability for systems of hyperbolic conservation laws was investigated by Bressan et al. [30] (cf. [10,31–33]). Liu and Xin [34] proved the nonlinear stability of discrete shocks for systems of conservation laws. Dafermos [35] studied the entropy and the stability of classical solutions of hyperbolic systems of conservation laws. For a relaxation system in several space dimensions, Luo and Xin [36] proved the nonlinear stability of shock fronts. Liu and Xin [37] investigated the nonlinear stability of rarefaction waves for compressible Navier–Stokes equations. Hsiao and Pan [38] investigated the nonlinear stability of rarefaction waves for a rate-type viscoelastic system. Moreover, the nonlinear stability of an undercompressive shock for complex Burgers equation was studied by Liu and Zumbrun [39]. For the viscous conservation laws, the theory of nonlinear stability of shock waves was established (see [40,41] and the references therein).

This paper is organized as follows. For the sake of completeness, in Section 2, we briefly recall John's formula on the decomposition of waves with some supplements and give a generalized Hörmander Lemma. In Section 3, we first review the definition of contact discontinuity, and then analyze some properties of waves on the contact discontinuity curves, which will play an important role in our proof. The main result, Theorem 1.3 is proved in Section 4. Finally, some applications to quasilinear hyperbolic systems of conservation laws arising in the string theory and high energy physics, particularly to the system describing the motion of relativistic strings in Minkowski space R^{1+n}, are presented in Section 5.

2. John's formula, generalized Hörmander Lemma

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements, which will play an important role in our proof.

Let

$$w_i = l_i(u)u_x \quad (i = 1, \dots, n), \tag{2.1}$$

where $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ denotes the i-th left eigenvector.

By (1.5), it follows from (1.13) and (2.1) that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.2}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.3}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.4}$$

be the directional derivative along the i-th characteristic. We have [42,6,15,54]

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \dots, n), \tag{2.5}$$

where

$$\beta_{ijk}(u) = \big(\lambda_k(u) - \lambda_i(u)\big) l_i(u) \nabla r_j(u) r_k(u). \tag{2.6}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall i, j. \tag{2.7}$$

On the other hand, we have (cf. [42,6,15,54])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{2.8}$$

where

$$\gamma_{ijk}(u)=\frac{1}{2}\big\{\big(\lambda_j(u)-\lambda_k(u)\big)l_i(u)\nabla r_k(u)r_j(u)-\nabla\lambda_i(u)r_j(u)\delta_{ik}+(j|k)\big\}, \tag{2.9}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. Hence,

$$\gamma_{ijj}(u)\equiv 0,\quad \forall j\neq i\ (i,j=1,\dots,n) \tag{2.10}$$

and

$$\gamma_{iii}(u)\equiv -\nabla\lambda_i(u)r_i(u)\quad (i=1,\dots,n). \tag{2.11}$$

When the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of Lax, we have

$$\gamma_{iii}(u)\equiv 0. \tag{2.12}$$

Noting (2.3), by (2.8) we have (cf. [16,54])

$$\frac{\partial w_i}{\partial t}+\frac{\partial(\lambda_i(u)w_i)}{\partial x}=\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k \stackrel{\text{def}}{=} G_i(t,x), \tag{2.13}$$

equivalently,

$$d\big[w_i\big(dx-\lambda_i(u)\,dt\big)\big]=\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k\,dt\wedge dx=G_i(t,x)\,dt\wedge dx, \tag{2.14}$$

where

$$\Gamma_{ijk}(u)=\frac{1}{2}\big(\lambda_j(u)-\lambda_k(u)\big)l_i(u)\big[\nabla r_k(u)r_j(u)-\nabla r_j(u)r_k(u)\big]. \tag{2.15}$$

Hence, we have

$$\Gamma_{ijj}(u)\equiv 0,\quad \forall i,j. \tag{2.16}$$

Lemma 2.1 *(Generalized Hörmander Lemma). Suppose that $u=u(t,x)$ is a piecewise C^1 solution to system (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and $\mathcal{D}$ is the domain bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+. Suppose furthermore that the domain $\mathcal{D}$ contains m C^1 curves of discontinuity of u, denoted by $\widehat{C}_j$: $x=x_j(t)$ $(j=1,\dots,m)$, which are never tangent to the i-th characteristic direction. Then we have*

$$\int_{\tau_1}\big|w_i\big(dx-\lambda_i(u)\,dt\big)\big|\leqslant\int_{\tau_2}\big|w_i\big(dx-\lambda_i(u)\,dt\big)\big|+\sum_{j=1}^{m}\int_{\widehat{C}_j}\big|[w_i]\,dx-\big[w_i\lambda_i(u)\big]\,dt\big| + \iint_{\mathcal{D}}\Bigg|\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k\Bigg|\,dt\,dx, \tag{2.17}$$

where $\Gamma_{ijk}(u)$ is given by (2.15) and $[w_i]=w_i^+-w_i^-$ denotes the jump of w_i over the curve of discontinuity $\widehat{C}_j$ $(j=1,\dots,m)$, etc.

The proof can be found in Li and Kong [15].

Corollary 2.1. *If $\mathcal{D}$ is the domain bounded by two C^1 arcs τ_1 and τ_2 which are never tangent to the i-th characteristic direction, and one i-th characteristic curve L_i, then the conclusion of Lemma 2.1 still holds.*

Corollary 2.2. *If $\mathcal{D}$ is the domain bounded by three C^1 arcs τ_1, τ_2 and τ_3 which are never tangent to the i-th characteristic direction, and one i-th characteristic curve L_i, then we have*

$$\int_{\tau_1}\big|w_i\big(dx-\lambda_i(u)\,dt\big)\big|\leqslant\int_{\tau_2}\big|w_i\big(dx-\lambda_i(u)\,dt\big)\big|+\int_{\tau_3}\big|w_i\big(dx-\lambda_i(u)\,dt\big)\big|+\sum_{j=1}^{m}\int_{\widehat{C}_j}\big|[w_i]\,dx-\big[w_i\lambda_i(u)\big]\,dt\big| + \iint_{\mathcal{D}}\Bigg|\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k\Bigg|\,dt\,dx, \tag{2.18}$$

where $\Gamma_{ijk}(u)$ is given by (2.15) and $[w_i]=w_i^+-w_i^-$ denotes the jump of w_i over the curve of discontinuity $\widehat{C}_j$ $(j=1,\dots,m)$, etc.

3. Contact discontinuity

In this section, we first review the definition of contact discontinuity, and then analyze some properties of waves on the contact discontinuity curves, which will play an important role in our proof.

Definition 3.1. A piecewise C^1 vector function $u = u(t,x)$ defined on $\mathbf{R}^+ \times \mathbf{R}^+$ is called a piecewise C^1 solution containing a k-th contact discontinuity $x = x_k(t)$ $(x_k(0) = 0)$ for system (1.1), if $u = u(t,x)$ satisfies system (1.1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the Rankine–Hugoniot condition:

$$f\big(u^+\big) - f\big(u^-\big) = s\big(u^+ - u^-\big), \tag{3.1}$$

and

$$s = \lambda_k\big(u^+\big) = \lambda_k\big(u^-\big), \tag{3.2}$$

where $u^\pm = u^\pm(t, x_k(t)) \triangleq u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$.

Definition 3.1 can be found in [1] or [15].

The following lemma gives some properties of waves on the contact discontinuity curves.

Lemma 3.1. *Suppose that $|u^\pm|$ $(u^\pm = u(t, x_k(t) \pm 0))$ are suitably small. Then, on the k-th contact discontinuity $x = x_k(t)$ we have*

$$v_i^+ = v_i^- + O\big(\big|v^\pm\big|^2\big) \quad (i = 1, \ldots, k-1, k+1, \ldots, n) \tag{3.3}$$

and

$$w_i^+ = w_i^- + O\bigg(\big|u^+ - u^-\big| \cdot \sum_{j \neq k} \big|w_j^\pm\big|\bigg) \quad (i = 1, \ldots, k-1, k+1, \ldots, n), \tag{3.4}$$

where $v = (v_1, \ldots, v_n)^T$ is defined by (1.13) and $v^\pm \triangleq v(t, x_k(t) \pm 0)$, etc.

The proof can be found in Li and Kong [15].

Corollary 3.1. *On the k-th contact discontinuity $x = x_k(t)$, it holds that*

$$\big(w_i\lambda_i(u)\big)^+ = \big(w_i\lambda_i(u)\big)^- + O\bigg(\big|u^+ - u^-\big| \cdot \sum_{j \neq k} \big|w_j^\pm\big|\bigg) \quad (i = 1, \ldots, k-1, k+1, \ldots, n), \tag{3.5}$$

provided that $|u^\pm|$ is small.

Proof. Noting

$$\big(w_i\lambda_i(u)\big)^+ - \big(w_i\lambda_i(u)\big)^- = \big[w_i^+ - w_i^-\big]\big(\lambda_i(u)\big)^+ + w_i^-\big[\big(\lambda_i(u)\big)^+ - \big(\lambda_i(u)\big)^-\big], \tag{3.6}$$

from (3.4), we immediately get (3.5). □

4. Proof of Theorem 1.3

By the existence and uniqueness of local classical discontinuous solutions of quasilinear hyperbolic systems of conservation laws (see [4]), when $\eta > 0$ is suitably small, the generalized Riemann problem (1.1), (1.11) and (1.12) admits a unique piecewise C^1 solution $u = u(t,x)$ only containing $n - m$ contact discontinuities $x = x_i(t)$ $(x_i(0) = 0)$ $(i = m+1, \ldots, n)$ on the domain $[0, h] \times \mathbf{R}^+$, where $h > 0$ is a small number; moreover, this solution has a local structure similar to the one of the self-similar solution to the corresponding Riemann problem. In order to prove Theorem 1.3, it suffices to establish a uniform a priori estimate for the piecewise C^0 norm of u and u_x on any given domain of existence of the piecewise C^1 solution $u = u(t,x)$.

Noting (1.2) and (1.9), we have

$$\lambda_1(0) < \cdots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \cdots < \lambda_n(0). \tag{4.1}$$

Thus, there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geqslant \delta_0, \quad \forall |u|, |v| \leqslant \delta \ (i = 1, \ldots, n-1) \tag{4.2}$$

and

$$\left|\lambda_i(0)\right| \geqslant \delta_0 \quad (i=1,\dots,n). \tag{4.3}$$

For the time being it is supposed that on the domain of existence of the piecewise C^1 solution $u=u(t,x)$ to the generalized Riemann problem (1.1), (1.11) and (1.12), we have

$$\left|u(t,x)\right| \leqslant \delta. \tag{4.4}$$

At the end of the proof of Lemma 4.5, we will explain that this hypothesis is reasonable.

For any fixed $T>0$, let

$$U_\infty(T)=\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}^+}\left|u(t,x)\right|, \tag{4.5}$$

$$V_\infty(T)=\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}^+}\left|v(t,x)\right|, \tag{4.6}$$

$$W_\infty(T)=\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}^+}\left|w(t,x)\right|, \tag{4.7}$$

$$\widetilde{W}_1(T)=\max_{i=1,\dots,n}\max_{j\neq i}\sup_{C_j}\int_{C_j}|w_i|\,dt, \tag{4.8}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $v=(v_1,\dots,v_n)^T$ and $w=(w_1,\dots,w_n)^T$ in which v_i and w_i are defined by (1.13) and (2.1) respectively, while C_j stands for any given j-th characteristic on the domain $[0,T]\times\mathbf{R}^+$. In (4.4)–(4.7), on any contact discontinuity curve $x=x_k(t)$ the values of $u(t,x)$, $v(t,x)$ and $w(t,x)$ are taken to be $u^\pm(t,x)=u(t,x_k(t)\pm 0)$, $v^\pm(t,x)=v(t,x_k(t)\pm 0)$ and $w^\pm(t,x)=w(t,x_k(t)\pm 0)$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

First we recall some basic L^1 estimates. They are essentially due to Schatzman [43,44] and Zhou [17].

Lemma 4.1. *Let $\phi=\phi(t,x)\in C^1$ satisfy*

$$\phi_t+\big(\lambda(t,x)\phi\big)_x=F(t,x),\quad 0\leqslant t\leqslant T,\ x\in R,\qquad \phi(0,x)=g(x),$$

where $\lambda\in C^1$. Then

$$\int_{-\infty}^{+\infty}\left|\phi(t,x)\right|dx\leqslant\int_{-\infty}^{+\infty}\left|g(x)\right|dx+\int_0^T\int_{-\infty}^{+\infty}\left|F(t,x)\right|dx\,dt,\quad \forall t\leqslant T, \tag{4.9}$$

provided that the right-hand side of the inequality is bounded.

Lemma 4.2. *Let $\phi=\phi(t,x)$ and $\psi=\psi(t,x)$ be C^1 functions satisfying*

$$\phi_t+\big(\lambda(t,x)\phi\big)_x=F_1(t,x),\quad 0\leqslant t\leqslant T,\ x\in R,\qquad \phi(0,x)=g_1(x),$$

and

$$\psi_t+\big(\mu(t,x)\psi\big)_x=F_2(t,x),\quad 0\leqslant t\leqslant T,\ x\in R,\qquad \psi(0,x)=g_2(x),$$

respectively, where $\lambda,\mu\in C^1$ such that there exists a positive constant δ_0 independent of T verifying

$$\mu(t,x)-\lambda(t,x)\geqslant\delta_0,\quad 0\leqslant t\leqslant T,\ x\in R.$$

Then

$$\begin{aligned}\int_0^T\int_{-\infty}^{+\infty}\left|\phi(t,x)\right|\left|\psi(t,x)\right|dx\,dt\leqslant{}&C\left(\int_{-\infty}^{+\infty}\left|g_1(x)\right|dx+\int_0^T\int_{-\infty}^{+\infty}\left|F_1(t,x)\right|dx\,dt\right)\\&\times\left(\int_{-\infty}^{+\infty}\left|g_2(x)\right|dx+\int_0^T\int_{-\infty}^{+\infty}\left|F_2(t,x)\right|dx\,dt\right),\end{aligned} \tag{4.10}$$

provided that the two factors on the right-hand side of the inequality is bounded.

In the present situation, similar to the above basic L^1 estimates (4.9)–(4.10), we have

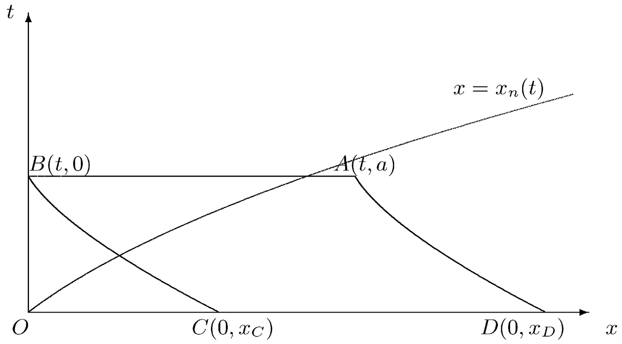

Fig. 1. The domain $ABCD$ in (t,x)-plane.

Lemma 4.3. *Under the assumptions of Theorem* 1.3*, on any given domain of existence* $[0,T]\times\mathbf{R}^+$ *of the piecewise* C^1 *solution* $u=u(t,x)$ *to the generalized Riemann problem* (1.1)*,* (1.11) *and* (1.12)*, there exists a positive constant* k_1 *independent of* ε*,* T *and* M *such that*

$$\int_0^{+\infty}\left|w_i(t,x)\right|dx\leqslant k_1\left\{\int_0^{+\infty}\left|\varphi'(x)\right|dx+\int_0^{+\infty}\left(\left|\alpha'(t)\right|+\left|h'(t)\right|\right)dt\right.$$
$$\left.+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\left|G(t,x)\right|dx\,dt\right\}\quad(i=1,\dots,n)\ \forall t\leqslant T,\tag{4.11}$$

provided that the right-hand side of the inequality is bounded and $G=(G_1,G_2,\dots,G_n)$*.*

Proof. To estimate $\int_0^{+\infty}|w_i(t,x)|\,dx$, we need only to estimate

$$\int_0^a\left|w_i(t,x)\right|dx\tag{4.12}$$

for any given $a>0$ and then let $a\to+\infty$.

(i) For $i=1,\dots,m$, for any given t with $0\leqslant t\leqslant T$, passing through point $A(t,a)$ $(a>x_n(t))$ (resp. $B(t,0)$), we draw the i-th backward characteristic which intersects the x-axis at a point $D(0,x_D)$ (resp. $C(0,x_C)$), see Fig. 1.

Then, applying (2.17) on the domain $ABCD$, we have

$$\int_{BA}\left|w_i(t,x)\right|dx\leqslant\int_{x_C}^{x_D}\left|w_i(0,x)\right|dx+\sum_{k=m+1}^n\int_{\widehat{C}_k}\left|[w_i]x_k'(t)-\left[w_i\lambda_i(u)\right]\right|dt+\iint_{ABCD}|G_i|\,dx\,dt,\tag{4.13}$$

where $\widehat{C}_k$: $x=x_k(t)$ stands for the k-th contact discontinuity passing through the origin, which is contained in the region $ABCD$. Therefore

$$\int_0^a\left|w_i(t,x)\right|dx\leqslant\int_0^{+\infty}\left|w_i(0,x)\right|dx+\sum_{k=m+1}^n\int_{\widehat{C}_k}\left|[w_i]x_k'(t)-\left[w_i\lambda_i(u)\right]\right|dt+\int_0^T\int_0^{+\infty}|G_i|\,dx\,dt.\tag{4.14}$$

Using (3.4)–(3.5) and (4.4), it is easy to see that

$$\int_0^a\left|w_i(t,x)\right|dx\leqslant c_1\left\{\int_0^{+\infty}\left|\varphi'(x)\right|dx+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}|G_i|\,dx\,dt\right\},\tag{4.15}$$

where here and henceforth, c_i $(i=1,2,\dots)$ will denote positive constants independent of ε, T and M.

Letting $a\to+\infty$, we immediately get the assertion in (4.11).

(ii) For $i=m+1,\dots,n$, for any given t with $0\leqslant t\leqslant T$, passing through point $A(t,a)$ $(a>x_n(t))$, we draw the i-th backward characteristic which intersects the x-axis at a point $C(0,x_C)$, see Fig. 2.

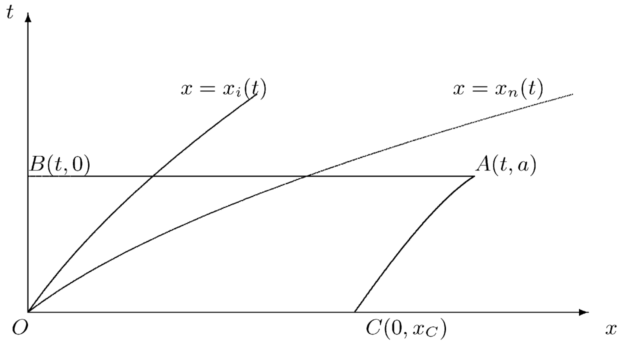

Fig. 2. The domain *ABOC* in (t,x)-plane.

Let B denote the point $(t,0)$. Then, applying (2.18) on the domain *ABOC*, we have

$$\int_{BA} \left|w_i(t,x)\right| dx \leqslant \int_0^{x_C} \left|w_i(0,x)\right| dx + \int_0^t \lambda_i\big(u(t,0)\big)\left|w_i(t,0)\right| dt$$
$$+ \sum_{k=m+1}^{n} \int_{\widehat{C}_k} \left|[w_i]x_k'(t) - \big[w_i\lambda_i(u)\big]\right| dt + \iint_{ABOC} |G_i|\, dx\, dt, \tag{4.16}$$

where $\widehat{C}_k$: $x = x_k(t)$ stands for the k-th contact discontinuity passing through the origin, which is contained in the region *ABOC*. Thus, noting (3.2), we get

$$\int_0^a \left|w_i(t,x)\right| dx \leqslant \int_0^{+\infty} \left|w_i(0,x)\right| dx + c_2 \int_0^T \left|w_i(t,0)\right| dt + \sum_{k=m+1,k\neq i}^{n} \int_{\widehat{C}_k} \left|[w_i]x_k'(t) - \big[w_i\lambda_i(u)\big]\right| dt$$
$$+ \int_0^T \int_0^{+\infty} |G_i|\, dx\, dt. \tag{4.17}$$

Using (3.4)–(3.5) and (4.4), it is easy to see that

$$\int_0^a \left|w_i(t,x)\right| dx \leqslant c_3 \left\{ \int_0^{+\infty} \left|\varphi'(x)\right| dx + \int_0^T \left|w_i(t,0)\right| dt + V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_0^{+\infty} |G_i|\, dx\, dt \right\}. \tag{4.18}$$

Similar to Lemma 3.2 in [3], by differentiating the nonlinear boundary condition (1.12) with respect to t, we get

$$x = 0: \quad \frac{\partial v_s}{\partial t} = \sum_{r=1}^{m} \frac{\partial G_s}{\partial v_r}\big(\alpha(t), v_1, \ldots, v_m\big)\frac{\partial v_r}{\partial t}$$
$$+ \sum_{i=1}^{k} \frac{\partial G_s}{\partial \alpha_i}\big(\alpha(t), v_1, \ldots, v_m\big)\alpha_i'(t) + h_s'(t) \quad (s = m+1, \ldots, n). \tag{4.19}$$

By (1.1), (1.3) and (2.3), it is easy to see that

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t}\big(l_i(u)u\big) = -\lambda_i(u)w_i + \sum_{k=1}^{n} a_{ik}(u)w_k \quad (i = 1, \ldots, n), \tag{4.20}$$

where

$$a_{ik}(u) = -\lambda_k(u)r_k^T(u)\nabla l_i(u)u. \tag{4.21}$$

Thus, noting (1.9) and (4.4), for sufficiently small $\delta > 0$, we easily see from (4.19)–(4.20) that

$$x = 0: \quad w_s = \sum_{j=1}^{m} f_{sj}(t,u)w_j + \sum_{i=1}^{k} \overline{f}_{si}(t,u)\alpha_i'(t) + \sum_{l=m+1}^{n} \widetilde{f}_{sl}(t,u)h_l'(t) \quad (s = m+1, \ldots, n), \tag{4.22}$$

where f_{sj}, $\overline{f}_{si}$ and $\widetilde{f}_{sl}$ are continuous functions of t and u.

Noting (4.4), by (4.22) we have

$$\int_0^T |w_i(t,0)|\,dt = \sum_{r=1}^m \int_0^T \bigl|f_{ir}\bigl(t,u(t,0)\bigr)w_r(t,0)\bigr|\,dt + \sum_{j=1}^k \int_0^T \bigl|\overline{f}_{ij}\bigl(t,u(t,0)\bigr)\alpha'_j(t)\bigr|\,dt + \sum_{l=m+1}^n \int_0^T \bigl|\widetilde{f}_{il}\bigl(t,u(t,0)\bigr)h'_l(t)\bigr|\,dt$$
$$\leqslant c_4\left\{\sum_{r=1}^m \int_0^T |w_r(t,0)|\,dt + \int_0^{+\infty} \bigl(|\alpha'(t)| + |h'(t)|\bigr)\,dt\right\}. \tag{4.23}$$

Then, passing through the point $D(T,0)$, we draw the r-th characteristic C_r ($r \in \{1,\ldots,m\}$) which intersects the x-axis at point $E(0,x_E)$. Applying Corollary 2.1 on the domain DOE, we have

$$\int_0^T \bigl|w_r(t,0)\bigl(-\lambda_r(u)\bigr)\bigr|\,dt \leqslant \int_0^{x_E} |w_r(0,x)|\,dx + \sum_{k=m+1}^n \int_{\widehat{C}_k} \bigl|[w_r]x'_k(t) - \bigl[w_r\lambda_r(u)\bigr]\bigr|\,dt + \iint_{DOE} |G_r|\,dx\,dt$$
$$\leqslant c_5\left\{\int_0^{+\infty} |\varphi'(x)|\,dx + V_\infty(T)\widetilde{W}_1(T) + \int_0^T\int_0^{+\infty} |G_r|\,dx\,dt\right\}. \tag{4.24}$$

Noting (4.3) and (4.4), for sufficiently small $\delta > 0$, it is easy to see that

$$|\lambda_r(u)| \geqslant \frac{\delta_0}{2}. \tag{4.25}$$

Therefore, it follows from (4.24) that

$$\int_0^T |w_r(t,0)|\,dt \leqslant c_6\left\{\int_0^{+\infty} |\varphi'(x)|\,dx + V_\infty(T)\widetilde{W}_1(T) + \int_0^T\int_0^{+\infty} |G(t,x)|\,dx\,dt\right\}. \tag{4.26}$$

Combining (4.18) with (4.23) and (4.26), we obtain

$$\int_0^a |w_i(t,x)|\,dx \leqslant c_7\left\{\int_0^{+\infty} |\varphi'(x)|\,dx + \int_0^{+\infty} \bigl(|\alpha'(t)| + |h'(t)|\bigr)\,dt\right.$$
$$\left. + V_\infty(T)\widetilde{W}_1(T) + \int_0^T\int_0^{+\infty} |G(t,x)|\,dx\,dt\right\}. \tag{4.27}$$

Letting $a \to +\infty$, we immediately get the assertion in (4.11). The proof of Lemma 4.3 is finished. □

Lemma 4.4. *Under the assumptions of Theorem* 1.3, *on any given domain of existence* $[0,T] \times \mathbf{R}^+$ *of the piecewise* C^1 *solution* $u = u(t,x)$ *to the generalized Riemann problem* (1.1), (1.11) *and* (1.12), *there exists a positive constant* k_2 *independent of* ε, T *and* M *such that*

$$\int_0^T\int_0^{+\infty} |w_i(t,x)||w_j(t,x)|\,dx\,dt \leqslant k_2\left(\int_0^{+\infty} |\varphi'(x)|\,dx + \int_0^{+\infty} \bigl(|\alpha'(t)| + |h'(t)|\bigr)\,dt\right.$$
$$\left. + V_\infty(T)\widetilde{W}_1(T) + \int_0^T\int_{-\infty}^{+\infty} |G(t,x)|\,dx\,dt\right)^2, \quad \forall i \neq j\ (i,j = 1,\ldots,n), \tag{4.28}$$

provided that the right-hand side of the inequality is bounded and $G = (G_1, G_2, \ldots, G_n)$.

Proof. To estimate

$$\int_0^T\int_0^{+\infty} |w_i(t,x)||w_j(t,x)|\,dx\,dt, \tag{4.29}$$

it is enough to estimate

$$\int_0^T \int_0^L |w_i(t,x)||w_j(t,x)|\,dx\,dt \tag{4.30}$$

for any given $L > 0$ and then let $L \to +\infty$.

(i) For $i, j \in \{m+1, \dots, n\}$ and $i \neq j$, without loss of generality, we suppose that $i < j$. Let $x = x_i(t, L)$ $(0 \leqslant t \leqslant T)$ be the i-th forward characteristic passing through point $(0, L)$ $(L > x_n(T))$. Then, we draw the i-th backward characteristic $x = s_i(t)$ $(0 \leqslant t \leqslant T)$ passing through point (T, a) $(a > x_i(T, L))$.

We introduce the "continuous Glimm's functional" (cf. [12,45,17,18])

$$Q(t) = \iint_{0<x<y<s_i(t)} |w_j(t,x)||w_i(t,y)|\,dx\,dy. \tag{4.31}$$

Because of the piecewise C^1 solution $u = u(t,x)$ containing only $n-m$ contact discontinuities $x = x_k(t)$ $(x_k(0) = 0)$ $(k = m+1, \dots, n)$, we divide the bounded domain $\widetilde{\Omega} \triangleq \{(x,y) \mid 0 < x < y < s_i(t)\}$ by the straight lines $y = x_k(t)$ $(k = m+1, \dots, n)$ into some parts. Then, the straightforward calculations on all parts of the domain $\widetilde{\Omega}$ reveal that

$$\begin{aligned}
\frac{dQ(t)}{dt} &= s_i'(t)\big|w_i\big(t, s_i(t)\big)\big| \int_0^{s_i(t)} |w_j(t,x)|\,dx \\
&\quad + \sum_{k=m+1}^{n} x_k'(t)\big\{\big|w_i\big(t, x_k(t)-0\big)\big| - \big|w_i\big(t, x_k(t)+0\big)\big|\big\} \int_0^{x_k(t)} |w_j(t,x)|\,dx \\
&\quad + \iint_{0<x<y<s_i(t)} \frac{\partial}{\partial t}\big(|w_j(t,x)|\big)|w_i(t,y)|\,dx\,dy + \iint_{0<x<y<s_i(t)} |w_j(t,x)|\frac{\partial}{\partial t}\big(|w_i(t,y)|\big)\,dx\,dy \\
&= s_i'(t)\big|w_i\big(t, s_i(t)\big)\big| \int_0^{s_i(t)} |w_j(t,x)|\,dx \\
&\quad + \sum_{k=m+1}^{n} x_k'(t)\big\{\big|w_i\big(t, x_k(t)-0\big)\big| - \big|w_i\big(t, x_k(t)+0\big)\big|\big\} \int_0^{x_k(t)} |w_j(t,x)|\,dx \\
&\quad - \iint_{0<x<y<s_i(t)} \frac{\partial}{\partial x}\big(\lambda_j(u)|w_j(t,x)|\big)|w_i(t,y)|\,dx\,dy - \iint_{0<x<y<s_i(t)} |w_j(t,x)|\frac{\partial}{\partial y}\big(\lambda_i(u)|w_i(t,y)|\big)\,dx\,dy \\
&\quad + \iint_{0<x<y<s_i(t)} \operatorname{sgn}(w_j)G_j(t,x)|w_i(t,y)|\,dx\,dy + \iint_{0<x<y<s_i(t)} |w_j(t,x)|\operatorname{sgn}(w_i)G_i(t,y)\,dx\,dy \\
&= -\int_0^{s_i(t)} \big(\lambda_j\big(u(t,x)\big) - \lambda_i\big(u(t,x)\big)\big)|w_i(t,x)||w_j(t,x)|\,dx \\
&\quad + \big(s_i'(t) - \lambda_i\big(u\big(t, s_i(t)\big)\big)\big)\big|w_i\big(t, s_i(t)\big)\big| \int_0^{s_i(t)} |w_j(t,x)|\,dx + \lambda_j\big(u(t,0)\big)|w_j(t,0)| \int_0^{s_i(t)} |w_i(t,x)|\,dx \\
&\quad + \big(x_i'(t) - \lambda_i\big(u\big(t, x_i(t)-0\big)\big)\big)\big|w_i\big(t, x_i(t)-0\big)\big| \int_0^{x_i(t)} |w_j(t,x)|\,dx \\
&\quad + \big(\lambda_i\big(u\big(t, x_i(t)+0\big)\big) - x_i'(t)\big)\big|w_i\big(t, x_i(t)+0\big)\big| \int_0^{x_i(t)} |w_j(t,x)|\,dx \\
&\quad + \sum_{k=m+1, k\neq i}^{n} x_k'(t)\big\{\big|w_i\big(t, x_k(t)-0\big)\big| - \big|w_i\big(t, x_k(t)+0\big)\big|\big\} \int_0^{x_k(t)} |w_j(t,x)|\,dx
\end{aligned}$$

$$+\sum_{k=m+1,k\neq i}^{n}\bigl\{\lambda_i\bigl(u\bigl(t,x_k(t)+0\bigr)\bigr)\bigl|w_i\bigl(t,x_k(t)+0\bigr)\bigr|-\lambda_i\bigl(u\bigl(t,x_k(t)-0\bigr)\bigr)\bigl|w_i\bigl(t,x_k(t)-0\bigr)\bigr|\bigr\}\int_0^{x_k(t)}\bigl|w_j(t,x)\bigr|\,dx$$
$$+\iint_{0<x<y<s_i(t)}\operatorname{sgn}(w_j)G_j(t,x)\bigl|w_i(t,y)\bigr|\,dx\,dy+\iint_{0<x<y<s_i(t)}\bigl|w_j(t,x)\bigr|\operatorname{sgn}(w_i)G_i(t,y)\,dx\,dy. \tag{4.32}$$

Noting (3.2) and (4.1) and using (4.2), we get from (4.32) that

$$\begin{aligned}\frac{dQ(t)}{dt}\leqslant{}&-\delta_0\int_0^{s_i(t)}\bigl|w_i(t,x)\bigr|\bigl|w_j(t,x)\bigr|\,dx+\lambda_j\bigl(u(t,0)\bigr)\bigl|w_j(t,0)\bigr|\int_0^{s_i(t)}\bigl|w_i(t,x)\bigr|\,dx\\
&+\sum_{k=m+1,k\neq i}^{n}x_k'(t)\bigl\{\bigl|w_i\bigl(t,x_k(t)-0\bigr)-w_i\bigl(t,x_k(t)+0\bigr)\bigr|\bigr\}\int_0^{x_k(t)}\bigl|w_j(t,x)\bigr|\,dx\\
&+\sum_{k=m+1,k\neq i}^{n}\bigl\{\bigl|\lambda_i\bigl(u\bigl(t,x_k(t)+0\bigr)\bigr)w_i\bigl(t,x_k(t)+0\bigr)-\lambda_i\bigl(u\bigl(t,x_k(t)-0\bigr)\bigr)w_i\bigl(t,x_k(t)-0\bigr)\bigr|\bigr\}\int_0^{x_k(t)}\bigl|w_j(t,x)\bigr|\,dx\\
&+\int_0^{s_i(t)}\bigl|G_j(t,x)\bigr|\,dx\int_0^{s_i(t)}\bigl|w_i(t,x)\bigr|\,dx+\int_0^{s_i(t)}\bigl|G_i(t,x)\bigr|\,dx\int_0^{s_i(t)}\bigl|w_j(t,x)\bigr|\,dx\\
\leqslant{}&-\delta_0\int_0^{s_i(t)}\bigl|w_i(t,x)\bigr|\bigl|w_j(t,x)\bigr|\,dx+\lambda_j\bigl(u(t,0)\bigr)\bigl|w_j(t,0)\bigr|\int_0^{+\infty}\bigl|w_i(t,x)\bigr|\,dx\\
&+\sum_{k=m+1,k\neq i}^{n}x_k'(t)\bigl\{\bigl|w_i\bigl(t,x_k(t)-0\bigr)-w_i\bigl(t,x_k(t)+0\bigr)\bigr|\bigr\}\int_0^{+\infty}\bigl|w_j(t,x)\bigr|\,dx\\
&+\sum_{k=m+1,k\neq i}^{n}\bigl\{\bigl|\lambda_i\bigl(u\bigl(t,x_k(t)+0\bigr)\bigr)w_i\bigl(t,x_k(t)+0\bigr)-\lambda_i\bigl(u\bigl(t,x_k(t)-0\bigr)\bigr)w_i\bigl(t,x_k(t)-0\bigr)\bigr|\bigr\}\int_0^{+\infty}\bigl|w_j(t,x)\bigr|\,dx\\
&+\int_0^{+\infty}\bigl|G_j(t,x)\bigr|\,dx\int_0^{+\infty}\bigl|w_i(t,x)\bigr|\,dx+\int_0^{+\infty}\bigl|G_i(t,x)\bigr|\,dx\int_0^{+\infty}\bigl|w_j(t,x)\bigr|\,dx.\end{aligned} \tag{4.33}$$

It then follows from Lemma 4.3 that

$$\begin{aligned}&\frac{dQ(t)}{dt}+\delta_0\int_0^{s_i(t)}\bigl|w_i(t,x)\bigr|\bigl|w_j(t,x)\bigr|\,dx\\
&\leqslant c_8\Biggl(\lambda_j\bigl(u(t,0)\bigr)\bigl|w_j(t,0)\bigr|+\sum_{k=m+1,k\neq i}^{n}x_k'(t)\bigl\{\bigl|w_i\bigl(t,x_k(t)-0\bigr)-w_i\bigl(t,x_k(t)+0\bigr)\bigr|\bigr\}\\
&\quad+\sum_{k=m+1,k\neq i}^{n}\bigl\{\bigl|\lambda_i\bigl(u\bigl(t,x_k(t)+0\bigr)\bigr)w_i\bigl(t,x_k(t)+0\bigr)-\lambda_i\bigl(u\bigl(t,x_k(t)-0\bigr)\bigr)w_i\bigl(t,x_k(t)-0\bigr)\bigr|\bigr\}+\int_0^{+\infty}\bigl|G(t,x)\bigr|\,dx\Biggr)\\
&\quad\times\Biggl(\int_0^{+\infty}\bigl|\varphi'(x)\bigr|\,dx+\int_0^{+\infty}\bigl(\bigl|\alpha'(t)\bigr|+\bigl|h'(t)\bigr|\bigr)\,dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\bigl|G(t,x)\bigr|\,dx\,dt\Biggr).\end{aligned} \tag{4.34}$$

Therefore

$$\delta_0\int_0^T\int_0^{s_i(t)}\bigl|w_i(t,x)\bigr|\bigl|w_j(t,x)\bigr|\,dx\,dt\leqslant Q(0)+c_9\Biggl(\int_0^T\lambda_j\bigl(u(t,0)\bigr)\bigl|w_j(t,0)\bigr|\,dt+\sum_{k=m+1,k\neq i}^{n}\int_{\widehat{C}_k}\bigl|[w_i]\bigr|\lambda_k\bigl(u^{\pm}\bigr)\,dt$$

$$+\sum_{k=m+1,k\neq i}^{n}\int_{\widehat{C}_k}\big|[w_i\lambda_i(u)]\big|\,dt+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg)$$

$$\times\Bigg(\int_0^{+\infty}\big|\varphi'(x)\big|\,dx+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt$$

$$+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg). \tag{4.35}$$

Using (3.4)–(3.5) and noting (4.4), we obtain

$$\delta_0\int_0^T\int_0^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt\leqslant Q(0)+c_{10}\Bigg(\int_0^T\big|w_j(t,0)\big|\,dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg)$$

$$\times\Bigg(\int_0^{+\infty}\big|\varphi'(x)\big|\,dx+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt$$

$$+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg). \tag{4.36}$$

By exploiting the same arguments as in Lemma 4.3, we can deduce that

$$\int_0^T\big|w_j(t,0)\big|\,dt\leqslant c_{11}\Bigg\{\sum_{r=1}^{m}\int_0^T\big|w_r(t,0)\big|dt+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt\Bigg\}$$

$$\leqslant c_{12}\Bigg(\int_0^{+\infty}\big|\varphi'(x)\big|\,dx+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg). \tag{4.37}$$

Then, noting

$$Q(0)\leqslant\int_0^{+\infty}\big|w_i(0,x)\big|\,dx\int_0^{+\infty}\big|w_j(0,x)\big|\,dx, \tag{4.38}$$

it follows from (4.36)–(4.37) that

$$\delta_0\int_0^T\int_0^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt$$

$$\leqslant c_{13}\Bigg(\int_0^{+\infty}\big|\varphi'(x)\big|\,dx+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg)^2. \tag{4.39}$$

It then follows

$$\int_0^T\int_0^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt$$

$$\leqslant\frac{c_{13}}{\delta_0}\Bigg(\int_0^{+\infty}\big|\varphi'(x)\big|\,dx+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\Bigg)^2. \tag{4.40}$$

Therefore

$$\int_0^T\int_0^L |w_i(t,x)||w_j(t,x)|\,dx\,dt$$
$$\leqslant \frac{c_{13}}{\delta_0}\left(\int_0^{+\infty}|\varphi'(x)|\,dx+\int_0^{+\infty}\left(|\alpha'(t)|+|h'(t)|\right)dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}|G(t,x)|\,dx\,dt\right)^2 \tag{4.41}$$

and the desired conclusion follows by taking $L\to+\infty$.

(ii) For $i,j\in\{1,\dots,m\}$ and $i\neq j$, without loss of generality, we suppose that $i<j$, and passing through point (T,L) $(L>x_n(T))$, we draw the i-th backward characteristic $x=s_i(t)$ $(0\leqslant t\leqslant T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional" (cf. [12,45,17,18])

$$Q(t)=\iint\limits_{0<x<y<s_i(t)}|w_j(t,x)||w_i(t,y)|\,dx\,dy.$$

The argument in step (ii) is similar to the one in step (i), so instead of giving all the details one can just refer to (i) and briefly describe the changes one needs to introduce. Instead of formula (4.36) we have (cf. [56])

$$\delta_0\int_0^T\int_0^{s_i(t)}|w_i(t,x)||w_j(t,x)|\,dx\,dt$$
$$\leqslant Q(0)+c_{14}\left(V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}|G(t,x)|\,dx\,dt\right)$$
$$\times\left(\int_0^{+\infty}|\varphi'(x)|\,dx+\int_0^{+\infty}\left(|\alpha'(t)|+|h'(t)|\right)dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}|G(t,x)|\,dx\,dt\right). \tag{4.42}$$

Noting

$$Q(0)\leqslant\int_0^{+\infty}|w_i(0,x)|\,dx\int_0^{+\infty}|w_j(0,x)|\,dx, \tag{4.43}$$

we get

$$\delta_0\int_0^T\int_0^{s_i(t)}|w_i(t,x)||w_j(t,x)|\,dx\,dt$$
$$\leqslant c_{15}\left(\int_0^{+\infty}|\varphi'(x)|\,dx+\int_0^{+\infty}\left(|\alpha'(t)|+|h'(t)|\right)dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}|G(t,x)|\,dx\,dt\right)^2. \tag{4.44}$$

It is easy to deduce from (4.44) formula (4.28).

(iii) For $i\in\{m+1,\dots,n\}$ and $j\in\{1,\dots,m\}$, passing through point (T,L) $(L>x_n(T))$, we draw the j-th backward characteristic $x=s_j(t)$ $(0\leqslant t\leqslant T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional" (cf. [12,45,17,18])

$$Q(t)=\iint\limits_{0<y<x<s_j(t)}|w_i(t,x)||w_j(t,y)|\,dx\,dy. \tag{4.45}$$

The argument in step (iii) is similar to the one in step (i), so instead of giving all the details one can just refer to (i) and briefly describe the changes one needs to introduce. Instead of formula (4.32) we have

$$\frac{dQ(t)}{dt}=-\int_0^{s_j(t)}\left(\lambda_i(u(t,x))-\lambda_j(u(t,x))\right)|w_i(t,x)||w_j(t,x)|\,dx$$

$$+\big(s_j'(t)-\lambda_j\big(u\big(t,s_j(t)\big)\big)\big)\big|w_j\big(t,s_j(t)\big)\big|\int_0^{s_j(t)}\big|w_i(t,x)\big|\,dx+\lambda_i\big(u(t,0)\big)\big|w_i(t,0)\big|\int_0^{s_j(t)}\big|w_j(t,x)\big|\,dx$$

$$+\sum_{k=m+1}^{n}x_k'(t)\big\{\big|w_j\big(t,x_k(t)-0\big)\big|-\big|w_j\big(t,x_k(t)+0\big)\big|\big\}\int_0^{x_k(t)}\big|w_i(t,x)\big|\,dx$$

$$+\sum_{k=m+1}^{n}\big\{\lambda_j\big(u\big(t,x_k(t)+0\big)\big)\big|w_j\big(t,x_k(t)+0\big)\big|-\lambda_j\big(u\big(t,x_k(t)-0\big)\big)\big|w_j\big(t,x_k(t)-0\big)\big|\big\}\int_0^{x_k(t)}\big|w_i(t,x)\big|\,dx$$

$$+\iint_{0<x<y<s_j(t)}\operatorname{sgn}(w_i)G_i(t,x)\big|w_j(t,y)\big|\,dx\,dy+\iint_{0<x<y<s_j(t)}\big|w_i(t,x)\big|\operatorname{sgn}(w_j)G_j(t,y)\,dx\,dy. \tag{4.46}$$

By exploiting the same arguments as in (i), we can deduce that

$$\int_0^T\int_0^{s_j(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt$$
$$\leqslant\frac{c_4}{\delta_0}\left(\int_0^{+\infty}\big|\varphi'(x)\big|\,dx+\int_0^{+\infty}\big(\big|\alpha'(t)\big|+\big|h'(t)\big|\big)\,dt+V_\infty(T)\widetilde{W}_1(T)+\int_0^T\int_0^{+\infty}\big|G(t,x)\big|\,dx\,dt\right)^2. \tag{4.47}$$

It is easy to deduce from (4.47) formula (4.28).

(iv) For $i\in\{1,\dots,m\}$ and $j\in\{m+1,\dots,n\}$, passing through the point (T,L) $(L>x_n(T))$, we draw the i-th backward characteristic $x=s_i(t)$ $(0\leqslant t\leqslant T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional" (cf. [12,45,17,18])

$$Q(t)=\iint_{0<x<y<s_i(t)}\big|w_j(t,x)\big|\big|w_i(t,y)\big|\,dx\,dy.$$

The argument in step (iv) is similar to the one in step (i), so instead of giving all the details one can just refer to (i) and briefly describe the changes one needs to introduce. Instead of formula (4.32) we have

$$\frac{dQ(t)}{dt}=-\int_0^{s_i(t)}\big(\lambda_j\big(u(t,x)\big)-\lambda_i\big(u(t,x)\big)\big)\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx$$

$$+\big(s_i'(t)-\lambda_i\big(u\big(t,s_i(t)\big)\big)\big)\big|w_i\big(t,s_i(t)\big)\big|\int_0^{s_i(t)}\big|w_j(t,x)\big|\,dx+\lambda_j\big(u(t,0)\big)\big|w_j(t,0)\big|\int_0^{s_i(t)}\big|w_i(t,x)\big|\,dx$$

$$+\sum_{k=m+1}^{n}x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)\big|-\big|w_i\big(t,x_k(t)+0\big)\big|\big\}\int_0^{x_k(t)}\big|w_j(t,x)\big|\,dx$$

$$+\sum_{k=m+1}^{n}\big\{\lambda_i\big(u\big(t,x_k(t)+0\big)\big)\big|w_i\big(t,x_k(t)+0\big)\big|-\lambda_i\big(u\big(t,x_k(t)-0\big)\big)\big|w_i\big(t,x_k(t)-0\big)\big|\big\}\int_0^{x_k(t)}\big|w_j(t,x)\big|\,dx$$

$$+\iint_{0<x<y<s_i(t)}\operatorname{sgn}(w_j)G_j(t,x)\big|w_i(t,y)\big|\,dx\,dy+\iint_{0<x<y<s_i(t)}\big|w_j(t,x)\big|\operatorname{sgn}(w_i)G_i(t,y)\,dx\,dy.$$

By exploiting the same arguments as in (i), we can deduce that

$$\int_0^T \int_0^{s_i(t)} |w_i(t,x)||w_j(t,x)|\,dx\,dt$$
$$\leqslant c_4\left(\int_0^{+\infty} |\varphi'(x)|\,dx + \int_0^{+\infty} \left(|\alpha'(t)| + |h'(t)|\right) dt + V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_0^{+\infty} |G(t,x)|\,dx\,dt\right)^2. \tag{4.48}$$

It is easy to deduce from (4.48) formula (4.28). The proof of Lemma 4.4 is finished. □

Lemma 4.5. *Under the assumptions of Theorem* 1.3, *for small* $\eta > 0$ *there exists a constant* $\varepsilon > 0$ *so small that on any given domain of existence* $[0,T] \times \mathbf{R}^+$ *of the piecewise* C^1 *solution* $u = u(t,x)$ *to the generalized Riemann problem* (1.1), (1.11) *and* (1.12), *there exist positive constants* k_3, k_4 *and* k_5 *independent of* η, ε, T *and* M, *such that the following uniform a priori estimates hold*:

$$\widetilde{W}_1(T) \leqslant k_3\varepsilon, \tag{4.49}$$
$$U_\infty(T), V_\infty(T) \leqslant k_4\eta \tag{4.50}$$

and

$$W_\infty(T) \leqslant k_5 M. \tag{4.51}$$

Proof. We introduce

$$Q_W(T) = \sum_{j=1}^n \sum_{i\neq j} \int_0^T \int_0^{+\infty} |w_i(t,x)||w_j(t,x)|\,dx\,dt. \tag{4.52}$$

By (2.13), it follows from Lemma 4.4 that

$$Q_W(T) \leqslant c_1\left(\int_0^{+\infty} |\varphi'(x)|\,dx + \int_0^{+\infty} \left(|\alpha'(t)| + |h'(t)|\right) dt + V_\infty(T)\widetilde{W}_1(T) + \int_0^T \int_0^{+\infty} |G(t,x)|\,dx\,dt\right)^2, \tag{4.53}$$

where here and henceforth c_i $(i = 1,2,\ldots)$ will denote positive constants independent of η, ε, T and M.

Noting (2.16), we have

$$\int_0^T \int_0^{+\infty} |G(t,x)|\,dx\,dt \leqslant c_2 Q_W(T). \tag{4.54}$$

Substituting (4.54) into (4.53) and noting (1.21), we obtain

$$Q_W(T) \leqslant c_3\left(\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\right)^2. \tag{4.55}$$

We next estimate $\widetilde{W}_1(T)$.

To estimate $\widetilde{W}_1(T)$, we need to estimate

$$\int_{C_j} |w_i(t,x)|\,dt,$$

where C_j stands for any given j-th characteristic on the domain $[0,T] \times \mathbf{R}^+$. Without loss of generality, we assume that C_j intersects the x-axis with point $A(0,\alpha)$, and intersects the line $t = T$ with point B.

(i) For $i = 1,\ldots,m$, passing through point B, we draw the i-th backward characteristic C_i which intersects the x-axis at a point $C(0,\beta)$. For fixing the idea, we may suppose that $\alpha < \beta$. Then, applying Corollary 2.1 on the domain ABC, we have

$$\int_{C_j} |w_i(t,x)||\lambda_j(u) - \lambda_i(u)|\,dt \leqslant \int_\alpha^\beta |w_i(0,x)|\,dx + \sum_{k\in S_1} \int_{\widehat{C}_k} \left|\left([w_i]x_k'(t) - [w_i\lambda_i(u)]\right) dt\right|$$
$$+ \iint_{ABC} \sum_{j\neq k} |\Gamma_{ijk}(u) w_j w_k|\,dx\,dt, \tag{4.56}$$

where S_1 stands for the set of all indices k such that the k-th contact discontinuity $\widehat{C}_k$: $x=x_k(t)$ is partly contained in the domain ABC. Using (1.21), (3.4), (3.5) and (4.4), and noting the fact that $i \notin S_1$, we obtain

$$\int_{C_j} \left|w_i(t,x)\right|\left|\lambda_j(u)-\lambda_i(u)\right| dt \leqslant c_4\left\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\right\}. \tag{4.57}$$

In the definition of $\widetilde{W}_1(T)$, $j \neq i$, thus we have from (4.2) that

$$\left|\lambda_j(u)-\lambda_i(u)\right| \geqslant \delta_0. \tag{4.58}$$

Therefore, it follows that

$$\int_{C_j} \left|w_i(t,x)\right| dt \leqslant c_5\left\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\right\}. \tag{4.59}$$

(ii) For $i=m+1,\dots,n$, we draw the i-th backward characteristic C_i passing through point B. Here, there are only two possibilities:

(a) The i-th backward characteristic C_i intersects the t-axis at a point $C(\beta,0)$. Applying (2.18) on the domain $OABC$ and noting (3.2), we have

$$\begin{aligned}
&\int_{C_j} \left|w_i(t,x)\right|\left|\lambda_j(u)-\lambda_i(u)\right| dt \\
&\quad\leqslant \int_0^\alpha \left|w_i(0,x)\right| dx + \int_0^\beta \left|\lambda_i\big(u(t,0)\big)\right|\left|w_i(t,0)\right| dt \\
&\qquad + \sum_{k=m+1,k\neq i}^{n} \int_{\widehat{C}_k} \left|\big([w_i]x_k'(t) - \left[w_i\lambda_i(u)\right]\big)\,dt\right| + \iint_{OABC} \sum_{j\neq k} \left|\Gamma_{ijk}(u)w_jw_k\right| dt\,dx,
\end{aligned} \tag{4.60}$$

where $\widehat{C}_k$: $x=x_k(t)$ stands for the k-th contact discontinuity passing through the origin, which is contained in the region $OABC$. Using (1.21), (3.4), (3.5) and (4.4), we obtain

$$\int_{C_j} \left|w_i(t,x)\right|\left|\lambda_j(u)-\lambda_i(u)\right| dt \leqslant c_6\left\{\varepsilon + \int_0^\beta \left|w_i(t,0)\right| dt + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\right\}. \tag{4.61}$$

Therefore, it follows from (4.58) that

$$\int_{C_j} \left|w_i(t,x)\right| dt \leqslant c_7\left\{\varepsilon + \int_0^\beta \left|w_i(t,0)\right| dt + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\right\}. \tag{4.62}$$

Noting (4.4), by (4.22) we have

$$\begin{aligned}
\int_0^\beta \left|w_i(t,0)\right| dt &= \sum_{r=1}^{m} \int_0^\beta \left|f_{ir}(t,u)w_r(t,0)\right| dt + \sum_{j=1}^{k} \int_0^\beta \left|\bar{f}_{ij}(t,u)\alpha_j'(t)\right| dt + \sum_{l=m+1}^{n} \int_0^\beta \left|\widetilde{f}_{il}(t,u)h_l'(t)\right| dt \\
&\leqslant c_8\left\{\sum_{r=1}^{m} \int_0^\beta \left|w_r(t,0)\right| dt + \int_0^{+\infty} \big(\left|\alpha'(t)\right| + \left|h'(t)\right|\big)\,dt\right\}.
\end{aligned} \tag{4.63}$$

Then, passing through $C(\beta,0)$, we draw the r-th characteristic C_r $(r \in \{1,\dots,m\})$ which intersects the x-axis at point $D(0,x_D)$. Applying Corollary 2.1 on the domain COD, we get

$$\begin{aligned}
\int_0^\beta \left|w_r(t,0)\big(-\lambda_r(u)\,dt\big)\right| &\leqslant \int_0^{x_D} \left|w_r(0,x)\right| dx + \sum_{k=m+1}^{n} \int_{\widehat{C}_k} \left|\big([w_r]x_k'(t) - \left[w_r\lambda_r(u)\right]\big)\,dt\right| + \iint_{COD} \sum_{j\neq k} \left|\Gamma_{rjk}(u)w_jw_k\right| dx\,dt \\
&\leqslant c_9\left\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\right\}.
\end{aligned} \tag{4.64}$$

Therefore, noting (4.25), it follows that

$$\int_0^\beta \big|w_r(t,0)\big|\,dt \leqslant c_{10}\big\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\big\}. \tag{4.65}$$

Substituting (4.65) into (4.63) and noting (1.21) and (4.62), we have

$$\int_{C_j} \big|w_i(t,x)\big|\,dt \leqslant c_{11}\big\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\big\}. \tag{4.66}$$

(b) The i-th backward characteristic C_i intersects the x-axis at a point $(0,\beta)$. By exploiting the same arguments as in (i), we can deduce that

$$\int_{C_j} \big|w_i(t,x)\big|\,dt \leqslant c_{12}\big\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\big\}. \tag{4.67}$$

Combining (4.59) with (4.66) and (4.67), we have

$$\widetilde{W}_1(T) \leqslant c_{13}\big\{\varepsilon + V_\infty(T)\widetilde{W}_1(T) + Q_W(T)\big\}. \tag{4.68}$$

We next estimate $U_\infty(T)$ and $V_\infty(T)$.

(i) For $i = 1,\dots,m$, passing through any fixed point $(t,x) \in [0,T] \times \mathbf{R}^+$, we draw the i-th backward characteristic C_i which intersects the x-axis at a point $(0,\alpha)$. Integrating (2.5) along this characteristic C_i and noting (2.7) yields

$$v_i(t,x) = v_i(0,\alpha) + \sum_{k\in S_2}[v_i]_k + \int_{C_i} \sum_{j,k=1,k\neq i}^{n} \beta_{ijk}(u)v_j w_k\,dt, \tag{4.69}$$

where S_2 denotes the set of all indices k such that this characteristic C_i intersects the k-th contact discontinuity $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[v_i]_k = v_i(t_k, x_k(t_k)+0) - v_i(t_k, x_k(t_k)-0)$. Noting Remark 1.1 and using (1.21), we have

$$\big|\varphi(x)\big| \leqslant \big|\varphi(0)\big| + \int_0^{+\infty} \big|\varphi'(x)\big|\,dx \leqslant \eta + \varepsilon, \quad \forall x \in \mathbf{R}^+. \tag{4.70}$$

Therefore, noting the fact that $i \notin S_2$, and using (1.13), (3.3) and (4.4), we get from (4.69)–(4.70) that

$$\big|v_i(t,x)\big| \leqslant c_{14}\big\{\eta + \varepsilon + V_\infty(T)\big[V_\infty(T) + \widetilde{W}_1(T)\big]\big\}. \tag{4.71}$$

(ii) For $i = m+1,\dots,n$, passing through any fixed point $(t,x) \in [0,T] \times \mathbf{R}^+$, we draw the i-th backward characteristic C_i: $x = x_i(s;t,x)$. Here, there are only two possibilities:

(a) The i-th backward characteristic C_i intersects the t-axis at a point $(t_0,0)$. Integrating (2.5) along this characteristic C_i and noting (2.7) yields

$$v_i(t,x) = v_i(t_0,0) + \sum_{k\in S_3}[v_i]_k + \int_{t_0}^{t} \sum_{j,k=1,k\neq i}^{n} \beta_{ijk}(u)v_j w_k\big(s, x_i(s;t,x)\big)\,ds, \tag{4.72}$$

where S_3 denotes the set of all indices k such that this characteristic C_i intersects the k-th contact discontinuity $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[v_i]_k = v_i(t_k, x_k(t_k)+0) - v_i(t_k, x_k(t_k)-0)$. Noting (1.15), by (1.12), it is easy to see that

$$v_i(t_0,0) = \sum_{r=1}^{m} g_{ir}(t_0)v_r(t_0,0) + h_i(t_0), \tag{4.73}$$

where

$$g_{ir}(t_0) = \int_0^1 \frac{\partial G_i}{\partial v_r}\big(\alpha(t_0), \tau v_1(t_0,0), \dots, \tau v_m(t_0,0)\big)\,d\tau. \tag{4.74}$$

Noting Remark 1.1 and using (1.21), we have

$$\big|h_s(t)\big| \leqslant \big|h_s(0)\big| + \int_0^{+\infty} \big|h_s'(t)\big|\,dt \leqslant \eta + \varepsilon \quad (s = m+1,\dots,n). \tag{4.75}$$

Therefore, using (3.3), (4.4) and noting the fact that $i \notin S_3$, we obtain from (4.71)–(4.72) that

$$\begin{aligned}\big|v_i(t,x)\big| &\leqslant c_{15}\left\{\eta+\varepsilon+\sum_{r=1}^{m}\big|v_r(t_0,0)\big|+V_\infty(T)\big[V_\infty(T)+\widetilde{W}_1(T)\big]\right\}\\ &\leqslant c_{16}\big\{\eta+\varepsilon+V_\infty(T)\big[V_\infty(T)+\widetilde{W}_1(T)\big]\big\}.\end{aligned}\tag{4.76}$$

(b) The i-th backward characteristic C_i intersects the x-axis at a point $(0,\alpha)$. By exploiting the same arguments as in (i) and noting the fact that $i \notin S_2$, we can deduce that

$$\big|v_i(t,x)\big| \leqslant c_{17}\big\{\eta+\varepsilon+V_\infty(T)\big[V_\infty(T)+\widetilde{W}_1(T)\big]\big\}.\tag{4.77}$$

Combining (4.71) and (4.76), (4.77), we have

$$V_\infty(T) \leqslant c_{18}\big\{\eta+\varepsilon+V_\infty(T)\big[V_\infty(T)+\widetilde{W}_1(T)\big]\big\}.\tag{4.78}$$

We now prove (4.49)–(4.50) and

$$Q_W(T) \leqslant k_6\varepsilon^2,\tag{4.79}$$

where k_6 is a positive constant independent of η, ε and T.

Recalling (4.70), evidently we have

$$U_\infty(0), V_\infty(0) \leqslant c_{19}\eta\tag{4.80}$$

and

$$Q_W(0) = \widetilde{W}_1(0) = 0,\tag{4.81}$$

provided that $\varepsilon \ll \eta$. Thus, by continuity there exist positive constants k_3, k_4 and k_6 independent of η, ε and T such that (4.49)–(4.50) and (4.79) hold at least for $0 \leqslant T \leqslant \tau_0$, where τ_0 is a small positive number. Hence, in order to prove (4.49)–(4.50) and (4.79) it suffices to show that we can choose k_3, k_4 and k_6 in such a way that for any fixed T_0 $(0 < T_0 \leqslant T)$ such that

$$\widetilde{W}_1(T_0) \leqslant 2k_3\varepsilon,\tag{4.82}$$

$$V_\infty(T_0) \leqslant 2k_4\eta,\tag{4.83}$$

$$Q_W(T_0) \leqslant 2k_6\varepsilon^2,\tag{4.84}$$

we have

$$\widetilde{W}_1(T_0) \leqslant k_3\varepsilon,\tag{4.85}$$

$$V_\infty(T_0) \leqslant k_4\eta,\tag{4.86}$$

$$Q_W(T_0) \leqslant k_6\varepsilon^2.\tag{4.87}$$

To this end, substituting (4.82)–(4.84) into the right-hand side of (4.55), (4.68) and (4.78) (in which we take $T = T_0$), it is easy to see that, when $\eta > 0$ is suitably small, we have

$$Q_W(T_0) \leqslant 4c_3\varepsilon^2,\tag{4.88}$$

$$\widetilde{W}_1(T_0) \leqslant 2c_{13}\varepsilon,\tag{4.89}$$

$$V_\infty(T_0) \leqslant 3c_{18}\eta,\tag{4.90}$$

provided that $\varepsilon \ll \eta$.

Hence, if $k_3 \geqslant 2c_{13}$, $k_4 \geqslant 3c_{18}$ and $k_6 \geqslant 4c_3$, then we get (4.85)–(4.87), provided that η is suitably small. This proves (4.49)–(4.50) and (4.79).

We finally estimate $W_\infty(T)$.

(i) For $i = 1,\dots,m$, passing through any fixed point $(t,x) \in [0,T] \times \mathbf{R}^+$, we draw the i-th backward characteristic C_i which intersects the x-axis at a point $(0,y)$. Integrating (2.8) along this characteristic C_i and noting (2.10) and (2.12) yields

$$w_i(t,x) = w_i(0,y) + \sum_{k\in S_4}[w_i]_k + \int_{C_i}\sum_{j,l=1,\,j\neq l}^{n}\gamma_{ijl}(u)w_jw_l\,dt,\tag{4.91}$$

where S_4 denotes the set of all indices k such that this characteristic C_i intersects the k-th contact discontinuity $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[w_i]_k = w_i(t_k, x_k(t_k)+0) - w_i(t_k, x_k(t_k)-0)$. Using (3.4) and (4.4) and noting the fact that $i \notin S_4$, we have

$$\big|w_i(t,x)\big| \leqslant W_\infty(0) + c_{20}\big\{V_\infty(T)W_\infty(T) + W_\infty(T)\widetilde{W}_1(T)\big\}. \tag{4.92}$$

Hence, noting (1.14), (4.49) and (4.50), it is easy to see that

$$\big|w_i(t,x)\big| \leqslant c_{21}\big\{M + \eta W_\infty(T) + \varepsilon W_\infty(T)\big\}. \tag{4.93}$$

(ii) For $i = m+1,\dots,n$, for any fixed point $(t,x) \in [0,T] \times \mathbf{R}^+$, we draw the i-th backward characteristic C_i passing through this point. Here, there are only two possibilities:

(a) The i-th backward characteristic C_i intersects the x-axis at a point $(0,\alpha)$. By exploiting the same arguments as in (i) and noting the fact that $i \notin S_4$, we can deduce that

$$\big|w_i(t,x)\big| \leqslant c_{22}\big\{M + \eta W_\infty(T) + \varepsilon W_\infty(T)\big\}. \tag{4.94}$$

(b) The i-th backward characteristic C_i intersects the t-axis at a point $(t_0,0)$. Integrating (2.8) along this characteristic C_i from t_0 to t and noting (2.10) and (2.12) yields

$$w_i(t,x) = w_i(t_0,0) + \sum_{k\in S_5}[w_i]_k + \int_{C_i}\sum_{j,l=1,\, j\neq l}^{n}\gamma_{ijl}(u)w_jw_l\,dt, \tag{4.95}$$

where S_5 denotes the set of all indices k such that this characteristic C_i intersects the k-th contact discontinuity $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[w_i]_k = w_i(t_k, x_k(t_k)+0) - w_i(t_k, x_k(t_k)-0)$. It follows from (4.22) that

$$w_i(t_0,0) = \sum_{r=1}^{m} f_{ir}\big(t_0, u(t_0,0)\big)w_r(t_0,0) + \sum_{j=1}^{k}\overline{f}_{ij}\big(t_0,u(t_0,0)\big)\alpha_j'(t_0) + \sum_{l=m+1}^{n}\widetilde{f}_{il}\big(t_0,u(t_0,0)\big)h_l'(t_0). \tag{4.96}$$

Then, noting (1.14), (4.4) and (4.93), we have

$$\big|w_i(t_0,0)\big| \leqslant c_{23}\big\{M + \eta W_\infty(T) + \varepsilon W_\infty(T)\big\}. \tag{4.97}$$

Substituting (4.97) into (4.95) and noting (3.4), (4.4), (4.49), (4.50) and the fact that $i \notin S_5$, we obtain

$$\big|w_i(t,x)\big| \leqslant c_{24}\big\{M + \eta W_\infty(T) + \varepsilon W_\infty(T)\big\}. \tag{4.98}$$

Combining (4.93) with (4.94) and (4.98) gives

$$W_\infty(T) \leqslant c_{25}\big\{M + \eta W_\infty(T) + \varepsilon W_\infty(T)\big\}, \tag{4.99}$$

which implies (4.51).

Finally, we observe that when $\eta > 0$ is suitably small, by (4.50) we have

$$U_\infty(T) \leqslant k_4\eta \leqslant \frac{1}{2}\delta. \tag{4.100}$$

This implies the validity of hypothesis (4.4). The proof of Lemma 4.5 is finished. □

Proof of Theorem 1.3. Under the assumptions of Theorem 1.3, from (4.50) and (4.51), we know that for small $\eta > 0$ there exists $\varepsilon > 0$ suitably small such that on any given domain of existence $[0,T] \times \mathbf{R}^+$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (1.1), (1.11) and (1.12), the piecewise C^1 norm of the solution possesses a uniform a priori estimate independent of T. This leads to the conclusion of Theorem 1.3 immediately. The proof of Theorem 1.3 is finished. □

5. Applications

5.1. System of the planar motion of an elastic string

Consider the following generalized initial–boundary Riemann problem for the system of the planar motion of an elastic string (cf. [3,15,46])

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\dfrac{r-1}{r}u\right)_x = 0 \end{cases} \tag{5.1}$$

with the initial condition

$$t=0:\quad u=\widetilde{u}_0+u_0(x),\qquad v=v_0(x)\quad (x\geqslant 0) \tag{5.2}$$

and the boundary condition on the fixed end

$$x=0:\quad v=0. \tag{5.3}$$

Here, $u=(u_1,u_2)^T$, $v=(v_1,v_2)^T$, $r=|u|=\sqrt{u_1^2+u_2^2}$, $\widetilde{u}_0=(\widetilde{u}_1^0,\widetilde{u}_2^0)^T$ is a constant vector with $\widetilde{r}_0=|\widetilde{u}_0|=\sqrt{(\widetilde{u}_1^0)^2+(\widetilde{u}_2^0)^2}>1$, $(u_0(x)^T,v_0(x)^T)\in C^1$. Suppose that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$.

Let

$$U=\begin{pmatrix}u\\ v\end{pmatrix}. \tag{5.4}$$

It is easy to see that in a neighborhood of $U_0=\left(\begin{smallmatrix}\widetilde{u}_0\\ 0\end{smallmatrix}\right)$, system (5.1) is strictly hyperbolic and has the following four distinct real eigenvalues:

$$\lambda_1(U)=-1<\lambda_2(U)=-\sqrt{\frac{r-1}{r}}<0<\lambda_3(U)=\sqrt{\frac{r-1}{r}}<\lambda_4(U)=1. \tag{5.5}$$

The corresponding left and right eigenvectors are

$$\begin{aligned}
&l_1(U)=\left(u^T,u^T\right),\qquad l_2(U)=\left(\sqrt{\frac{r-1}{r}}w^T,w^T\right),\\
&l_3(U)=\left(\sqrt{\frac{r-1}{r}}w^T,-w^T\right),\qquad l_4(U)=\left(u^T,-u^T\right)
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&r_1(U)=\begin{pmatrix}u\\ u\end{pmatrix},\qquad r_2(U)=\begin{pmatrix}w\\ \sqrt{\frac{r-1}{r}}w\end{pmatrix},\\
&r_3(U)=\begin{pmatrix}-w\\ \sqrt{\frac{r-1}{r}}w\end{pmatrix},\qquad r_4(U)=\begin{pmatrix}-u\\ u\end{pmatrix},
\end{aligned} \tag{5.7}$$

respectively, where

$$w=(-u_2,u_1)^T. \tag{5.8}$$

It is easy to see that all characteristic fields are linearly degenerate, i.e.,

$$\nabla\lambda_i(U)r_i(U)\equiv 0\quad (i=1,\dots,4). \tag{5.9}$$

Let

$$V_i=l_i(U)(U-U_0)\quad (i=1,\dots,4). \tag{5.10}$$

Then, the boundary condition (5.3) can be rewritten as

$$x=0:\quad V_3=V_2,\qquad V_4=V_1. \tag{5.11}$$

By Theorem 1.3 we get

Theorem 5.1. *Suppose that u_0, v_0 are C^1 functions with respect to their arguments, for which there is a constant $M>0$ such that*

$$\left\|u_0(x)\right\|_{C^1},\ \left\|v_0(x)\right\|_{C^1}\leqslant M. \tag{5.12}$$

Suppose furthermore that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$. Suppose finally that

$$\eta\triangleq\left|\left(u_0(0),v_0(0)\right)\right|>0\quad \text{is suitably small.} \tag{5.13}$$

Then for any constant $M>0$, there exists a positive constant ε so small that if

$$\int_0^{+\infty} |u_0'(x)|\,dx,\ \int_0^{+\infty} |v_0'(x)|\,dx \leqslant \varepsilon, \tag{5.14}$$

then the generalized Riemann problem (5.1)–(5.3) *admits a unique global piecewise* C^1 *solution* $U = U(t,x)$ *containing only two contact discontinuities* $x = x_i(t)$ $(x_i(0) = 0)$ $(i = 3,4)$ *on the domain* $\{(t,x) \mid t \geqslant 0,\ x \geqslant 0\}$. *This solution has a global structure similar to the one of the self-similar solution to the corresponding Riemann problem.*

Suppose, now, that the initial condition (5.2) is replaced by

$$t = 0: \quad u = \widetilde{u}_0 + u_0(x), \qquad v = \widetilde{v}_0 + v_0(x) \quad (x \geqslant 0) \tag{5.15}$$

and the boundary condition (5.3) is replaced by the following dissipative boundary condition

$$x = 0: \quad \frac{r-1}{r}u = \alpha v \quad (\alpha > 0 \text{ is a constant}), \tag{5.16}$$

where $\widetilde{u}_0 = (\widetilde{u}_1^0, \widetilde{u}_2^0)^T$ and $\widetilde{v}_0 = (\widetilde{v}_1^0, \widetilde{v}_2^0)^T$ are constant vectors such that $\widetilde{r}_0 = |\widetilde{u}_0| > 1$ and

$$\frac{\widetilde{r}_0 - 1}{\widetilde{r}_0}\widetilde{u}_0 = \alpha \widetilde{v}_0, \tag{5.17}$$

$(u_0(x)^T, v_0(x)^T) \in C^1$. Suppose that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$.

Let

$$V_i = l_i(U)(U - U_0) \quad (i = 1, \dots, 4), \tag{5.18}$$

in which

$$U_0 = \begin{pmatrix} \widetilde{u}_0 \\ \widetilde{v}_0 \end{pmatrix}. \tag{5.19}$$

Then, the boundary condition (5.16) can be rewritten as

$$x = 0: \quad V_3 = f_3(V_1, V_2), \qquad V_4 = f_4(V_1, V_2), \tag{5.20}$$

where f_3 and f_4 are C^1 functions with respect to their arguments, which satisfy

$$f_3(0,0) = f_4(0,0) = 0, \tag{5.21}$$

see [3].

By Theorem 1.3 we get

Theorem 5.2. *Suppose that* u_0, v_0 *are* C^1 *functions with respect to their arguments, for which there is a constant* $M > 0$ *such that*

$$\|u_0(x)\|_{C^1},\ \|v_0(x)\|_{C^1} \leqslant M.$$

Suppose furthermore that the conditions of C^0 *compatibility are not satisfied at the point* $(0,0)$. *Suppose finally that*

$$\eta \triangleq \big|\big(u_0(0), v_0(0)\big)\big| > 0 \quad \textit{is suitably small.}$$

Then for any constant $M > 0$, *there exists a positive constant* ε *so small that if*

$$\int_0^{+\infty} |u_0'(x)|\,dx,\ \int_0^{+\infty} |v_0'(x)|\,dx \leqslant \varepsilon,$$

then the generalized Riemann problem (5.1), (5.15) *and* (5.16) *admits a unique global piecewise* C^1 *solution* $U = U(t,x)$ *containing only two contact discontinuities* $x = x_i(t)$ $(x_i(0) = 0)$ $(i = 3,4)$ *on the domain* $\{(t,x) \mid t \geqslant 0,\ x \geqslant 0\}$. *This solution has a global structure similar to the one of the self-similar solution to the corresponding Riemann problem.*

5.2. System of the motion of relativistic strings in the Minkowski space R^{1+n}

Consider the following generalized initial–boundary Riemann problem for the system of the motion of relativistic strings in the Minkowski space R^{1+n} (cf. [47,55]):

$$\begin{cases} \left(\dfrac{|v|^2u-\langle u,v\rangle v}{\sqrt{\langle u,v\rangle^2-(|u|^2-1)|v|^2}}\right)_t-\left(\dfrac{\langle u,v\rangle u-(|u|^2-1)v}{\sqrt{\langle u,v\rangle^2-(|u|^2-1)|v|^2}}\right)_\theta=0, \\ v_t-u_\theta=0 \end{cases} \tag{5.22}$$

with the initial condition

$$t=0:\quad u=u_0(\theta),\qquad v=\widetilde{v}_0+v_0(\theta)\quad (\theta\geqslant 0) \tag{5.23}$$

and the boundary condition

$$\theta=0:\quad u=0\quad (t\geqslant 0), \tag{5.24}$$

where $u=(u_1,\dots,u_n)^T$, $v=(v_1,\dots,v_n)^T$, $\widetilde{v}_0=(\widetilde{v}_1^0,\dots,\widetilde{v}_n^0)^T$ is a constant vector with $|\widetilde{v}_0|=\sqrt{(\widetilde{v}_1^0)^2+\cdots+(\widetilde{v}_n^0)^2}>0$, $(u_0(\theta)^T,v_0(\theta)^T)\in C^1$. Suppose that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$.

Let

$$U=\begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.25}$$

We can rewrite system (5.22) as

$$U_t+A(U)U_\theta=0, \tag{5.26}$$

where

$$A(U)=\begin{bmatrix} -\frac{2\langle u,v\rangle}{|v|^2}I_{n\times n} & \frac{|u|^2-1}{|v|^2}I_{n\times n} \\ -I_{n\times n} & 0 \end{bmatrix}. \tag{5.27}$$

It is easy to see that in a neighborhood of $U_0=\binom{0}{\widetilde{v}_0}$, (5.22) is a hyperbolic system with the following real eigenvalues:

$$\lambda_1(U)\equiv\cdots\equiv\lambda_n(U)=\lambda_-<0<\lambda_{n+1}(U)\equiv\cdots\equiv\lambda_{2n}(U)=\lambda_+, \tag{5.28}$$

where

$$\lambda_\pm=\frac{-\langle u,v\rangle\pm\sqrt{\langle u,v\rangle^2-(|u|^2-1)|v|^2}}{|v|^2}. \tag{5.29}$$

The corresponding left and right eigenvectors are

$$l_i(U)=(e_i,\lambda_+e_i)\quad (i=1,\dots,n),\qquad l_i(U)=(e_{i-n},\lambda_-e_{i-n})\quad (i=n+1,\dots,2n) \tag{5.30}$$

and

$$r_i(U)=(-\lambda_-e_i,e_i)^T\quad (i=1,\dots,n),\qquad r_i(U)=(-\lambda_+e_{i-n},e_{i-n})^T\quad (i=n+1,\dots,2n) \tag{5.31}$$

respectively, where

$$e_i=\big(0,\dots,0,\overset{(i)}{1},0,\dots,0\big)\quad (i=1,\dots,n). \tag{5.32}$$

When $n=1$, (5.22) is a strictly hyperbolic system; while, when $n\geqslant 2$, (5.22) is a non-strictly hyperbolic system with characteristics with constant multiplicity. It is easy to see that all characteristic fields are linearly degenerate in the sense of Lax, i.e.,

$$\nabla\lambda_i(U)r_i(U)\equiv 0\quad (i=1,\dots,2n), \tag{5.33}$$

see [47].

Let

$$V_i=l_i(U)(U-U_0)\quad (i=1,\dots,2n). \tag{5.34}$$

Then, the boundary condition (5.24) can be rewritten as

$$\theta=0:\quad V_{n+i}=-V_i\quad (i=1,\dots,n). \tag{5.35}$$

By Theorem 1.3 we get

Theorem 5.3. *Suppose that u_0, v_0 are all C^1 functions with respect to their arguments, for which there is a constant $M > 0$ such that*

$$\|u_0(\theta)\|_{C^0}, \|v_0(\theta)\|_{C^0}, \|u_0'(\theta)\|_{C^0}, \|v_0'(\theta)\|_{C^0} \leqslant M. \tag{5.36}$$

Suppose furthermore that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$. Suppose finally that

$$\eta \triangleq \left|\left(u_0(0), v_0(0)\right)\right| > 0 \quad \text{is suitably small.} \tag{5.37}$$

Then for any constant $M > 0$, there exists a positive constant ε so small that if

$$\int_0^{+\infty} \left|u_0'(\theta)\right| d\theta, \int_0^{+\infty} \left|v_0'(\theta)\right| d\theta \leqslant \varepsilon, \tag{5.38}$$

then the generalized Riemann problem (5.22)–(5.24) admits a unique global piecewise C^1 solution $U = U(t,\theta)$ containing only one contact discontinuity $\theta = \theta(t)$ ($\theta(0) = 0$) with constant multiplicity n on the domain $\{(t,\theta) \mid t \geqslant 0,\ \theta \geqslant 0\}$. This solution has a global structure similar to the one of the self-similar solution to the corresponding Riemann problem.

5.3. The Born–Infeld system

The Born–Infeld model is a nonlinear version of Maxwell's theory, it was introduced by Born and Infeld [48] in the 1930s to cutoff (in a nonlinear fashion) the singularities created by point particles in classical Electrodynamics. Recently, the Born–Infeld system has attracted considerable attention because of its new applications in the string theory and high energy physics. We refer the reader to Boillat et al. [49], Brenier [50] and Serre [51] for mathematical analysis of the BI system and to Gibbons [52] for its impact in modern high energy physics and string theory. The one-dimensional Born–Infeld system reads (cf. [50]):

$$\begin{cases} \partial_t D_2 + \partial_x\left(\dfrac{B_3 + D_2 P_1 - D_1 P_2}{h}\right) = 0, \\ \partial_t D_3 + \partial_x\left(\dfrac{-B_2 + D_3 P_1 - D_1 P_3}{h}\right) = 0, \\ \partial_t B_2 + \partial_x\left(\dfrac{-D_3 + B_2 P_1 - B_1 P_2}{h}\right) = 0, \\ \partial_t B_3 + \partial_x\left(\dfrac{D_2 + B_3 P_1 - B_1 P_3}{h}\right) = 0, \\ P(u) = D \times B, \quad h(u) = \sqrt{1 + |B|^2 + |D|^2 + |D \times B|^2}, \end{cases} \tag{5.39}$$

where $u = (D_2, D_3, B_2, B_3)^T$ are the unknown variables, B_1, D_1 are real constants and

$$B = (B_1, B_2, B_3)^T, \qquad D = (D_1, D_2, D_3)^T, \qquad P = (P_1, P_2, P_3)^T.$$

We are interested in the generalized initial–boundary Riemann problem for system (5.39) with the initial condition

$$t = 0: \quad u = u^0(x) = \left(\widetilde{D_2^0} + D_2^0(x), \widetilde{D_3^0} + D_3^0(x), \widetilde{B_2^0} + B_2^0(x), \widetilde{B_3^0} + B_3^0(x)\right)^T \quad (x \geqslant 0) \tag{5.40}$$

and the boundary condition

$$x = 0: \quad D_2 = D_3 = 0 \quad (t \geqslant 0), \tag{5.41}$$

where $\widetilde{D_2^0}$, $\widetilde{D_3^0}$, $\widetilde{B_2^0}$, and $\widetilde{B_3^0}$ are real constants, $(D_2^0(x), D_3^0(x), B_2^0(x), B_3^0(x))^T \in C^1$, such that

$$\|D_2^0(x)\|_{C^1}, \|D_3^0(x)\|_{C^1}, \|B_2^0(x)\|_{C^1}, \|B_3^0(x)\|_{C^1} \leqslant M, \tag{5.42}$$

for some positive constant M (bounded but possibly large). Also, we assume that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$ and

$$\widetilde{D_2^0}\widetilde{B_3^0} - \widetilde{D_3^0}\widetilde{B_2^0} < \sqrt{1 + B_1^2 + D_1^2} \tag{5.43}$$

holds.

From Li et al. [53], we know that in a neighborhood of $u^0 = (\widetilde{D_2^0}, \widetilde{D_3^0}, \widetilde{B_2^0}, \widetilde{B_3^0})^T$, (5.39) is a linearly degenerate hyperbolic system with the following real eigenvalues:

$$\lambda_1(u) = \lambda_2(u) = \frac{P_1 - a}{h} < 0 < \lambda_3(u) = \lambda_4(u) = \frac{P_1 + a}{h}. \tag{5.44}$$

The corresponding left eigenvectors are

$$\begin{aligned} &l_1(u) = (a, \beta_1, 0, -\beta_3), \qquad l_2(u) = (-\beta_1, a, \beta_3, 0), \\ &l_3(u) = (0, -\beta_2, a, \beta_1), \qquad l_4(u) = (\beta_2, 0, -\beta_1, a), \end{aligned} \tag{5.45}$$

in which

$$\beta_1 = B_1 D_1, \qquad \beta_2 = 1 + B_1^2, \qquad \beta_3 = 1 + D_1^2 \quad \text{and} \quad a = \sqrt{1 + B_1^2 + D_1^2} > 0. \tag{5.46}$$

Hence, system (5.39) is non-strictly hyperbolic but with characteristics with constant multiplicity. They found that it enjoys many interesting properties like non-strictly hyperbolicity, constant multiplicity of eigenvalues, linear degeneracy of all characteristic fields, richness, existence of entropy–entropy flux pairs, etc.

Let

$$V_i = l_i(u)\big(u - u^0\big) \quad (1 \leqslant i \leqslant 4). \tag{5.47}$$

Then, the boundary condition (5.41) can be rewritten as

$$x = 0: \quad V_3 = -\frac{\beta_1}{\beta_3} V_1 + \frac{a}{\beta_3} V_2, \qquad V_4 = -\frac{a}{\beta_3} V_1 - \frac{\beta_1}{\beta_3} V_2, \quad t \geqslant 0. \tag{5.48}$$

By Theorem 1.3 we get

Theorem 5.4. *Suppose that $D_2^0(x)$, $D_3^0(x)$, $B_2^0(x)$ and $B_3^0(x)$ are C^1 functions with respect to their arguments, for which there is a constant $M > 0$ such that*

$$\big\|D_2^0(x)\big\|_{C^1}, \big\|D_3^0(x)\big\|_{C^1}, \big\|B_2^0(x)\big\|_{C^1}, \big\|B_3^0(x)\big\|_{C^1} \leqslant M.$$

Suppose furthermore that the conditions of C^0 compatibility are not satisfied at the point $(0,0)$ and (5.43) holds. Suppose finally that

$$\eta \triangleq \big|\big(D_2^0(0), D_3^0(0), B_2^0(0), B_3^0(0)\big)\big| > 0 \quad \textit{is suitably small.} \tag{5.49}$$

Then for any constant $M > 0$, there exists a positive constant ε so small that if

$$\int_0^{+\infty} \big|D_2^{0'}(x)\big|\,dx, \int_0^{+\infty} \big|D_3^{0'}(x)\big|\,dx, \int_0^{+\infty} \big|B_2^{0'}(x)\big|\,dx, \int_0^{+\infty} \big|B_3^{0'}(x)\big|\,dx \leqslant \varepsilon, \tag{5.50}$$

then the generalized Riemann problem (5.39)–(5.41) admits a unique global piecewise C^1 solution $u = u(t,x)$ containing only one contact discontinuity $x = x(t)$ $(x(0) = 0)$ with constant multiplicity 2 on the domain $\{(t,x) \mid t \geqslant 0,\ x \geqslant 0\}$. This solution has a global structure similar to the one of the self-similar solution to the corresponding Riemann problem.

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