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Existence of positive solutions for a class of semilinear and quasilinear elliptic equations with supercritical case ☆

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ABSTRACT

In this paper, we consider a class of semilinear elliptic Dirichlet problems in a bounded regular domain with cylindrical symmetry involving concave–convex nonlinearities with supercritical growth. Using a new Sobolev embedding theorem and variational method, we show the existence of two positive solutions of the problem. Additionally, we study the quasilinear elliptic equation and obtain a similar result.

1. Introduction

Let Ω be an open subset in $\mathbb{R}^N$ and consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = f_\lambda(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Caratheodory mapping with λ a real parameter. The existence, multiplicity, regularity of solutions of (P) have been extensively investigated, see [1,39] and the references therein.

When f_λ is sublinear, for example, $f_\lambda(x,u) = \lambda u^q$, $0 < q < 1$, sub-super solutions can easily provide the existence of (P) for all $\lambda > 0$.

When f_λ is the sum of a linear term and a superlinear term, such as, $f_\lambda(x,u) = \lambda u + u^p$, $1 < p < 2^* - 1$, $2^* = \frac{2N}{N-2}$. [3] showed that (P) has at least one positive solution if $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ under Dirichlet boundary condition. When $p = 2^* - 1$, say $f_\lambda(x,u) = \lambda u + u^{\frac{N+2}{N-2}}$, the problem becomes delicate because the lack of compactness. However, Brezis and Nirenburg [21] restored the compactness in bounded domain. They proved that the nontrivial bounded positive solution only exists for $0 < \lambda < \lambda_1$, $N > 3$, and $0 < \lambda^* < \lambda < \lambda_1$, $N = 3$, still by variational arguments. On the other hand, the well-known Pohozăev's identity showed the nonexistence for (P) with $\lambda \geqslant 0$ and $p \geqslant 2^* - 1$, if Ω is strictly star-shape. While Kazdan and Warner [24] showed the existence for all $p > 1$ if Ω is an annulus. Furthermore, Coron [25] used a variational approach to prove the solvability of (P), if Ω exhibits a small hole; while Rey [37] solved it for Ω exhibiting several small holes. Later, in [4], Bahri and Coron established that "nontrivial topology" (i.e., certain

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homology groups of Ω are nontrivial) guarantees the existence of a solution. Moreover, this nontriviality condition is only sufficient but not necessary, as some examples in [9,13,52].

When f_λ is the sum of a sublinear term and a superlinear term, for example, $f_\lambda(x,u) = \lambda u^q + u^p$, $0 < q < 1 < p$, the so-called concave–convex nonlinearity. Ambrosetti, Brezis and Cerami [2] showed that, if $p > 1$, there exists a constant $\Lambda > 0$, such that the problem has a minimal solution if $\lambda \in (0, \Lambda)$, has no solution if $\lambda > \Lambda$, and has a second solution if $p \in (1, 2^* - 1]$, $\lambda \in (0, \Lambda)$. For other results about this problem referring to [15,50,53].

When f_λ has supercritical growth, the general variational arguments can't be used directly because the corresponding functional is not well defined on the Sobolev space $H_0^1(\Omega)$. So we need some techniques. Merle and Peletier [16] studied the problem $f_\lambda(x,u) = u^p - \lambda u^q$, $q > p \geqslant 2^* - 1$, $\lambda > 0$, by defining a functional K on the set $H = \{v\colon \nabla v \in L^2(R^N),\ v \in L^{q+1}(R^N)\}$ they proved that the infimum of K on $H \cap \partial B$ was achieved where $\partial B = \{v \in L^{p+1}(R^N)\colon \int v^{p+1} = 1\}$. This contributes a solution for λ small enough.

In fact, most researchers solved supercritical problem by other methods. One is to take advantage of ODE techniques in symmetric domains. In 1973, Joseph and Lundgree [7] showed the first result in this aspect. In 1987, Budd and Norburg [6] proved for $N = 3$ the existence of a singular solution and infinite number of positive solutions. Later, Merle and Peletier [17] extended the singularity results to $N > 3$ and proved its uniqueness. In [38], Peihao Zhao and Chengkui Zhong considered the problem (P) for $f_\lambda(x,u) = \lambda u^q + u^p$, $0 < q < 1$, $p > 2^* - 1$ in a ball. They proved that there exist respectively unique constants $\lambda_*, \lambda^* > 0$, such that (P) has only one positive solution if $\lambda \in (0, \lambda_*)$; a unique singular solution and infinitely many positive solutions if $\lambda = \lambda^*$; at least two positive solutions if $\lambda \in (\lambda_*, \Lambda)$ (where Λ see [2]). Moreover, other results about this aspect see [5,29,41] in a ball, [18,40,54] in bounded domains, and [34] in all spaces R^N. Also see [22,48,49] for singular solution.

The other method is to consider the effect of geometry and topology of the domain. According to [4], Rabinowitz raised that whether there are suitable conditions on the topology of Ω. Some answers are given in [8,10–12].

In fact, nonexistence results of (P) hold for some $p > 2^* - 1$ in some nontrivial domains (see [10,11]); while an arbitrarily large number of solutions can be obtained in some contractible domains for all $p > 2^* - 1$ (see [8]). In [12], Passaseo considered the problem (P) for $f_\lambda(x,u) = \lambda |u|^{p-1} u$ with $p > 2^* - 1$ and $\lambda = 1$ in bounded domain Ω. Several perturbations have been used to construct a contractible domain for obtaining the number of positive solutions and nodal solutions. Later, the result in [42] showed that a unique perturbation can produce an arbitrarily large number of solutions. Also see [30,31,36] for other perturbed domains. Indeed, existence results have been obtained, even in some "nearly star-shaped" domains (see the definition introduced in [44]) for p sufficiently close to $\frac{N+2}{N-2}$ (see [45–47]) or for p large enough (see [43]). Moreover, a different definition of "nearly star-shaped" domain is used in [14] to extend Pohožaev's nonexistence result to nonstarshaped domains when p is large enough.

Recently, Wenzhi Wang [51] establishes embedding results (see Lemma 2.1) in a cylindrically symmetric domain. He proves that functions having such symmetry and belonging to H_0^1 can be embedded compactly into some weighted L^p spaces, with p superior to the critical Sobolev exponent. Then, variational arguments is appropriate.

In this paper, we consider the semilinear elliptic equation with concave–convex nonlinearity, that is $f_\lambda(x,u) = \lambda u^q + h(x) u^p$. Our purpose is to use the embedding theorem in [51] and classical variational tools to solve the problem with supercritical growth.

This paper is organized as follows. Section 2 contains preliminaries and our main result. Section 3 gives the existence of two positive solutions for small λ. Section 4 provides the regular property for the two solutions. Section 5 proves the main result. Section 6 gives a similar existence result for the quasilinear elliptic equation.

2. Preliminary

Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$, with $\Omega_1 \subset \mathbb{R}^m$, $m \geqslant 1$ being a bounded regular domain, and Ω_2 being a $k \geqslant 2$ dimensional ball with radius R, centered at the origin.

Consider the problem

$$\begin{cases} -\Delta u = \lambda u^q + h(x) u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where $0 < q < 1 < p < 2^* - 1 + \tau$, and λ, τ are positive real parameters (τ is the constant obtained in [51]). Let $h(x)$ satisfy the following conditions:

(H1) $h(x)$ is a nonnegative Hölder continuous function in $\bar{\Omega}$, radially symmetric with respect to $x_2 \in \Omega_2$, satisfying $h(x_1, 0) = 0$.
(H2) $l_h > 0$, where $l_h = \sup\{\lambda > 0\colon |h(x)|/|x_2|^\lambda < \infty,\ x \in \Omega\}$.

Denote that

$H_s^1(\Omega) := \{u \in H_0^1(\Omega) \mid u(\cdot, x_2) = u(\cdot, |x_2|)\}$, with the norm $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$.

$L_h^p(\Omega) := \{u \in L^p(\Omega) \mid \int_\Omega h|u|^p\,dx < \infty\}$, with the norm $\|u\|_{h,p} = (\int_\Omega h|u|^p\,dx)^{1/p}$, $1 < p < \infty$.
$L^q(\Omega)$ is the Banach spaces for the form $\|u\|_q = (\int_\Omega |u|^q\,dx)^{1/q}$, $1 < q < \infty$.
$H_s^{-1}(\Omega)$ is the dual space of $H_s^1(\Omega)$, $\langle,\rangle$ denotes the pairing of $H_s^1(\Omega)$ and $H_s^{-1}(\Omega)$.
$c, C, C_1, C_2, \ldots$ are (possibly different) positive constants.
λ_1 is the first eigenvalue of the equation

$$\begin{cases} -\Delta\phi = \lambda,\phi & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases}$$

and ϕ_1 is the positive eigenfunction associated with λ_1.

Lemma 2.1. *(See [51].) Assume that $h(x)$ satisfies* (H1), (H2)*, then there exists a positive number $\tau = \tau(h, m, k)$ such that the embedding $H_s^1(\Omega) \hookrightarrow L_h^r(\Omega)$ is compact for all $r \in (1, 2^* + \tau)$.*

According to Lemma 2.1, the embedding mapping $i : H_s^1(\Omega) \hookrightarrow L_h^{p+1}(\Omega)$ is compact for $p + 1 < 2^* + \tau$. Hence, for $u \in H_s^1(\Omega)$, we have $u \in L_h^{p+1}(\Omega)$.

Let

$$f_\lambda(x, s) = \begin{cases} \lambda s^q + h(x)s^p, & s \geqslant 0, \\ 0, & s < 0, \end{cases}$$

and

$$F_\lambda(x, u) = \int_0^u f_\lambda(x, s)\,ds.$$

Let $u \in H_s^1(\Omega)$, $u^+ = \max_{x\in\Omega}\{u, 0\}$, define

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F_\lambda(x, u)\,dx = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int_\Omega \left|u^+\right|^{q+1} dx - \frac{1}{p+1}\int_\Omega h(x)\left|u^+\right|^{p+1} dx.$$

We know that the energy functional I_λ is well defined in $H_s^1(\Omega)$ and is of C^1.

Definition 2.2. We call $u \in H_s^1(\Omega)$ a weak solution of problem (P_λ), if u is a critical point of I_λ.

Our main result is the following:

Theorem 2.3. *Let $0 < q < 1 < p < 2^* - 1 + \tau$. If $h(x)$ satisfies* (H1), (H2)*, then there exists $\Lambda \in (0, \infty)$ such that*

(1) *for all $\lambda \in (0, \Lambda)$, problem (P_λ) has at least two classical solutions*;
(2) *for $\lambda = \Lambda$, problem (P_λ) has at least one weak solution $u_\Lambda \in H_s^1(\Omega) \cap L_h^{p+1}(\Omega)$*;
(3) *for all $\lambda > \Lambda$, problem (P_λ) has no solution.*

3. Existence of two solutions for λ small

In this section, we show existence of the first solution u_λ and the second solution v_λ of problem (P_λ) for $\lambda \in (0, \lambda_0)$. Moreover, we show that u_λ is a minimizer of $I_\lambda(u)$ in $H_s^1(\Omega)$ and v_λ is a mountain-pass type solution.

Lemma 3.1. *There exist $\lambda_0 > 0$ and $r_0, \rho > 0$, such that $I_\lambda(u) \geqslant \rho$ for all $u \in H_s^1(\Omega)$, $\|u\| = r_0$ and all $\lambda \in (0, \lambda_0)$.*

Proof. Indeed, for $u \in H_s^1(\Omega)$, we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int_\Omega \left|u^+\right|^{q+1} dx - \frac{1}{p+1}\int_\Omega h(x)\left|u^+\right|^{p+1} dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\left\|u^+\right\|_{q+1}^{q+1} - \frac{1}{p+1}\left\|u^+\right\|_{h,p+1}^{p+1}. \end{aligned} \tag{3.1}$$

Since $1 < q + 1 < 2 < p + 1 < 2^* + \tau$, it follows from Lemma 2.1 that the embedding $H_s^1(\Omega) \hookrightarrow L_h^{p+1}(\Omega)$ is compact, and also $H_s^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$. Thus, there exists a constant C such that

$$\|u\|_{h,p+1} \leqslant C\|u\|, \tag{3.2}$$

$$\|u\|_{q+1} \leqslant C\|u\|. \tag{3.3}$$

Using (3.1), (3.2) and (3.3), we infer that

$$\begin{aligned}
I_\lambda(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{C\lambda}{q+1}\|u^+\|^{q+1} - \frac{C}{p+1}\|u^+\|^{p+1} \\
&= \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\|u^+\|^2 - \lambda\big(C\|u^+\|\big)^{q+1} - \big(C\|u^+\|\big)^{p+1} \\
&= \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\|u^+\|\big(\|u^+\| - \lambda\big(C\|u^+\|\big)^q - \big(C\|u^+\|\big)^p\big).
\end{aligned}$$

Since $0 < q < 1 < p$, we can find λ_0 such that for all $0 < \lambda \leqslant \lambda_0$ there exists $M = M(\lambda) > 0$ satisfying

$$M > \lambda(CM)^q + (CM)^p.$$

As a consequence, there exist $r_0 > 0$ and a small enough constant $\rho > 0$ such that

$$I_\lambda(u) \geqslant \rho > 0 \quad \text{for every } \|u\| = r_0. \qquad \square$$

Lemma 3.2. *For all $\lambda \in (0, \lambda_0)$, I_λ possesses a local minimum close to the origin.*

Proof. Let $\lambda \in (0, \lambda_0)$, we note that

$$\begin{aligned}
I_\lambda(tu) &= \frac{1}{2}\|tu\|^2 - \frac{\lambda}{q+1}\int_\Omega |tu^+|^{q+1}\,dx - \frac{1}{p+1}\int_\Omega h(x)|tu^+|^{p+1}\,dx \\
&= \frac{1}{2}t^2\|u\|^2 - \frac{\lambda}{q+1}t^{q+1}\|u^+\|_{q+1}^{q+1} - \frac{1}{p+1}t^{p+1}\|u^+\|_{h,p+1}^{p+1}.
\end{aligned}$$

Clearly, $I_\lambda(tu) < 0$ for $t > 0$ small enough and any $u \in H_s^1(\Omega)$ with $\|u^+\| \neq 0$. Set

$$A = \{u \in H_s^1(\Omega) \mid \|u\| \leqslant r_0\},$$

then we have

$$m := \min_{u\in A} I_\lambda(u) < 0.$$

We claim that this minimum can be achieved at some u_λ. To see this, select a minimizing sequence $\{u_n\}_{n=1}^\infty$, then

$$I_\lambda(u_n) \to m.$$

And let $u_n \rightharpoonup u_\lambda$ in $H_s^1(\Omega)$, we have

$$\int_\Omega |\nabla u_\lambda|^2\,dx \leqslant \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx.$$

By compact embedding theorem (Lemma 2.1), we obtain that

$$\int_\Omega F_\lambda(x, u_n)\,dx \to \int_\Omega F_\lambda(x, u_\lambda)\,dx.$$

Thus,

$$\begin{aligned}
I_\lambda(u_\lambda) &= \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2\,dx - \int_\Omega F_\lambda(x, u_\lambda)\,dx \\
&\leqslant \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2\,dx + \lim_{n\to\infty}\int_\Omega F_\lambda(x, u_n)\,dx \\
&= \liminf_{n\to\infty} I_\lambda(u_n) = m.
\end{aligned}$$

Since $u_\lambda \in A$, it follows that

$$I_\lambda(u_\lambda) = m = \min_{u\in A} I(u).$$

That is, u_λ is a minimizer for I_λ in A, and hence $I_\lambda(u_\lambda)$ is a local minimum. $\square$

Remark 3.3. Here we get the first solution u_λ for $0<q<1<p<2^*-1+\tau$ and $\lambda\in(0,\lambda_0)$. In fact, applying sub-super solutions in [2], we can obtain u_λ for all $0<q<1<p$ and $\lambda\in(0,\lambda_0')$, where λ_0' is a small positive constant.

In the sequel, we will show the existence of the second solution v_λ by the Mountain Pass Theorem.

Definition 3.4. (See [32,33].) Let $c\in\mathbb{R}$. A C^1 functional $\Phi: X\to\mathbb{R}$ satisfies the $(PS)_c$ condition if every sequence $\{v_k\}_{k=1}^\infty$ in X such that $\Phi(v_k)\to c$ and $\Phi'(v_k)\to 0$ has a convergence subsequence.

Lemma 3.5. *I_λ satisfies the $(PS)_c$ condition.*

Proof. We must show that for all sequences $\{v_k\}_{k=1}^\infty\subset H_s^1(\Omega)$, which satisfy

$$I_\lambda(v_k)\to c,\qquad I_\lambda'(v_k)\to 0, \tag{3.4}$$

there exists a convergence subsequence $\{v_{k_j}\}_{k_j=1}^\infty$ such that $v_{k_j}\to v$ in $H_s^1(\Omega)$.

From (3.4), we can obtain that $\{v_k\}_{k=1}^\infty$ satisfy

$$\frac{1}{2}\int_\Omega|\nabla v_k|^2\,dx-\frac{\lambda}{q+1}\int_\Omega\left|v_k^+\right|^{q+1}dx-\frac{1}{p+1}\int_\Omega h(x)\left|v_k^+\right|^{p+1}dx=c+o(1), \tag{3.5}$$

and

$$-\Delta v_k-\lambda\left(v_k^+\right)^q-h(x)\left(v_k^+\right)^p=\xi_k,\qquad \xi_k\to 0\quad\text{in } H_s^{-1}(\Omega). \tag{3.6}$$

Multiplying (3.6) by v_k and integrating in Ω, we have

$$\int_\Omega|\nabla v_k|^2\,dx-\lambda\int_\Omega\left|v_k^+\right|^{q+1}dx-\int_\Omega h(x)\left|v_k^+\right|^{p+1}dx=\langle\xi_k,v_k\rangle. \tag{3.7}$$

Taking a computation with (3.5) and (3.7), we get

$$\frac{\lambda(q-1)}{2(q+1)}\int_\Omega\left|v_k^+\right|^{q+1}dx+\frac{p-1}{2(p+1)}\int_\Omega h(x)\left|v_k^+\right|^{p+1}dx=c-\frac{1}{2}\langle\xi_k,v_k\rangle.$$

Let $C_1=\frac{p-1}{2(p+1)}$, $C_2=\frac{\lambda(1-q)}{2(q+1)}$. Clearly $C_1,C_2>0$ for $0<q<1<p$. Thus,

$$C_1\int_\Omega h(x)\left|v_k^+\right|^{p+1}dx\leqslant C_2\int_\Omega\left|v_k^+\right|^{q+1}dx+c+\|\xi_k\|_{H_s^{-1}}\|v_k\|,$$

that is,

$$\int_\Omega h(x)\left|v_k^+\right|^{p+1}dx\leqslant C\left\|v_k^+\right\|_{q+1}^{q+1}+c+C\|\xi_k\|_{H_s^{-1}}\|v_k\|. \tag{3.8}$$

Combining (3.5) and (3.8), we have

$$\begin{aligned}\frac{1}{2}\|v_k\|^2&=\frac{\lambda}{q+1}\left\|v_k^+\right\|_{q+1}^{q+1}+\frac{1}{p+1}\left\|v_k^+\right\|_{h,p+1}^{p+1}+c+o(1)\\&\leqslant\frac{\lambda}{q+1}\left\|v_k^+\right\|_{q+1}^{q+1}+C\left\|v_k^+\right\|_{q+1}^{q+1}+c+C\|\xi_k\|_{H_s^{-1}}\|v_k\|\\&=C\left\|v_k^+\right\|_{q+1}^{q+1}+c+C\|\xi_k\|_{H_s^{-1}}\|v_k\|.\end{aligned}$$

According to (3.3), it follows that

$$\|v_k\|^2\leqslant C\left\|v_k^+\right\|^{q+1}+C+C\|\xi_k\|_{H_s^{-1}}\|v_k\|\leqslant C\left\|v_k^+\right\|^{q+1}+C+C\|v_k\|.$$

We deduce that there exists a constant C, such that $\|v_k\|<C$. Consequently, $\{v_k\}_{k=1}^\infty$ is bounded in $H_s^1(\Omega)$, and then, there exists a weakly convergence subsequence $\{v_{k_j}\}_{k_j=1}^\infty\subset\{v_k\}_{k=1}^\infty$. Moreover, we have another inequality

$$\left\|v_k^+\right\|_{h,p+1}^{p+1}\leqslant C.$$

From above, we can deduce the following:

$$\begin{aligned}
&v_{k_j} \rightharpoonup v \quad \text{weakly in } H_s^1(\Omega),\\
&v_{k_j} \to v \quad \text{strongly in } L_h^{p+1}(\Omega) \text{ for } p+1 < 2^* + \tau,\\
&v_{k_j} \to v \quad \text{strongly in } L^q(\Omega) \text{ for } q < 2^*,\\
&v_{k_j} \to v \quad \text{a.e in } \Omega.
\end{aligned}$$

Obviously we have

$$\lambda v_{k_j}^q + h(x) v_{k_j}^p \to \lambda v^q + v^p \quad \text{in } H_s^{-1}(\Omega).$$

From the Lax–Milgram Theorem, for each $f_\lambda(v) \in H_s^{-1}(\Omega)$, the problem

$$\begin{cases} -\Delta u = f_\lambda(v) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H_s^1(\Omega)$. Writing $u = K(f_\lambda(v))$, so that

$$K : H_s^{-1}(\Omega) \to H_s^1(\Omega) \quad \text{is an isometry.}$$

Therefore, we have

$$K\big[f_\lambda(v_{k_j})\big] \to K\big[f_\lambda(v)\big] \quad \text{in } H_s^1(\Omega).$$

As

$$I'_\lambda(v_k) = v_k - K\big(f_\lambda(v_k)\big) \to 0 \quad \text{in } H_s^1(\Omega),$$

consequently,

$$v_{k_j} \to v \quad \text{in } H_s^1(\Omega).$$

This completes the proof. $\square$

Lemma 3.6. *For all $\lambda \in (0, \lambda_0)$, I_λ has the second solution v_λ of mountain-pass type.*

Proof. Let $v \in H_s^1(\Omega)$. From Lemma 3.1, we can obtain that for all $\lambda \in (0, \lambda_0)$, there exist constants $r_0, \rho > 0$ such that $I_\lambda(v) \geqslant \rho$, for all $\|v\| = r_0$.

Next, we verify that $I_\lambda(0) < \rho$ and there exists $v \notin A$ such that $I_\lambda(v) < \rho$. Clearly, $I_\lambda(0) = 0 < \rho$. Now, fix some element $v \in H_s^1(\Omega)$, $v \not\equiv 0$. Write $\omega := tv$ for $t > 0$ to be selected. From (3.1) we have

$$I_\lambda(\omega) = \frac{1}{2} t^2 \|v\|^2 - \frac{\lambda}{q+1} t^{q+1} \|v^+\|_{q+1}^{q+1} - \frac{1}{p+1} t^{p+1} \|v^+\|_{h,p+1}^{p+1}.$$

Since $q+1 < 2 < p+1$, it follows that

$$I_\lambda(\omega) = I_\lambda(tv) \to -\infty, \quad \text{as } t \to +\infty.$$

Therefore, there exists $T > 0$ such that

$$\begin{aligned}
&\|\omega\| = \|Tv\| > r_0 \quad (\text{that is } v \notin \partial A),\\
&I_\lambda(\omega) = I_\lambda(tv) < \rho.
\end{aligned}$$

Consequently, there is a function $v_\lambda \in H_s^1(\Omega)$, $v_\lambda \not\equiv 0$, such that $I_\lambda(v_\lambda) = c \geqslant \rho > 0$, $I'_\lambda(v_\lambda) = 0$. That is, v_λ is a nontrivial critical point of (P_λ). For $I_\lambda(u_\lambda) < 0$, $I_\lambda(v_\lambda) > 0$, v_λ is the second solution of the problem (P_λ). $\square$

4. Regularity

The solutions u_λ, v_λ we have found in $H_s^1(\Omega)$ are weak solutions. In the following, we will apply bootstrap iteration [35] to improve their regularity.

Lemma 4.1. *Let $v \in H_s^1(\Omega)$ be a weak solution of problem (P_λ), then $v \in C^{2,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.*

Proof. For a solution $v \in H_s^1(\Omega)$ of (P_λ), we denote

$$f_\lambda(x) = f_\lambda\big(x, v(x)\big) = \lambda v(x)^q + h(x)v(x)^p.$$

Because of Lemma 2.1,

$$H_s^1(\Omega) \hookrightarrow L_h^r(\Omega) \subset L^r(\Omega), \quad r = 2^* + \tau,$$

we have,

$$f_\lambda(x) \in L^\sigma(\Omega), \quad \text{with } \sigma = \frac{r}{p}.$$

Since $1 < p < 2^* - 1 + \tau$, we get

$$\sigma > \frac{2^* + \tau}{2^* - 1 + \tau}.$$

Thus,

$$\sigma = \frac{2^* + \tau}{2^* - 1 + \tau}(1 + \varepsilon), \quad \text{for some } \varepsilon > 0.$$

We can write the origin equation as a linear nonhomogeneous elliptic equation

$$\begin{cases} -\Delta v = f_\lambda(x), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

According to the boundary regularity theorem of linear nonhomogeneous elliptic equation (see [27]), and $f_\lambda(x) \in L^\sigma(\Omega)$, we can obtain that $v \in W_0^{2,\sigma}(\Omega)$. If $2\sigma > n$, we are done. Otherwise, from Sobolev embedding theorem, we have

$$W_0^{2,\sigma}(\Omega) \hookrightarrow L^{r_1}(\Omega), \quad r_1 = \frac{N\sigma}{N - 2\sigma}.$$

It easily follows that

$$f_\lambda(x) \in L^{\sigma_1}(\Omega), \quad \sigma_1 = \frac{r_1}{p}.$$

Then, $v \in W_0^{2,\sigma_1}(\Omega)$.

Next, we need to show that the regularity of v has been improved, that is to show that

$$\frac{\sigma_1}{\sigma} = \frac{r_1}{r} > 1.$$

By computation, we obtain

$$\frac{r_1}{r} = \frac{N(1+\varepsilon)}{N(2^* + \tau - 1) - 2(1+\varepsilon)(2^* + \tau)}.$$

Thus, we only need to check

$$\begin{cases} N\big(2^* + \tau - 1\big) - 2(1+\varepsilon)\big(2^* + \tau\big) > 0 \quad \text{(a)}, \\ N(1+\varepsilon) > N\big(2^* + \tau - 1\big) - 2(1+\varepsilon)\big(2^* + \tau\big) \quad \text{(b)}. \end{cases}$$

We can easily get from (a), (b) that

$$\frac{N(2^* + \tau - 1)}{N + 2(2^* + \tau)} - 1 < \varepsilon < \frac{N(2^* + \tau - 1)}{2(2^* + \tau)} - 1.$$

Clearly, we can find $\varepsilon > 0$ satisfying (4.1). Consequently, we indeed have

$$\frac{\sigma_1}{\sigma} > 1.$$

From boundary regularity theorem, not only for $\sigma_1 > \sigma$, $v \in W_0^{2,\sigma_1}(\Omega)$, but also for any σ_k large enough, $v \in W_0^{2,\sigma_k}(\Omega)$.

When $2\sigma_k > n$, from Sobolev embedding theorem, we can obtain that $v \in C^{0,\theta}(\Omega)$, $f_\lambda(x) \in C^{0,\theta}(\Omega)$, then $v \in C^{2,\theta}(\Omega)$ by Schauder regularity theorem. $\square$

5. Existence of solutions for $\lambda \in (0, \Lambda)$

Lemma 5.1. *Let $\Lambda = \sup\{\lambda > 0\colon (P_\lambda) \text{ has a solution}\}$, then $\Lambda \in (0, \infty)$.*

Proof. Let $a > 0$ and choose $\Omega_1 \subset \Omega$ such that $h(x) \geqslant a$ in Ω_1. Define

$$\eta(x) = \begin{cases} \phi_1(x), & x \in \Omega_1, \\ 0, & x \in \Omega \setminus \Omega_1. \end{cases}$$

Multiplying (P_λ) by $\eta(x)$ and using integrations by parts, we can get

$$\int_{\Omega_1} \lambda_1 u\phi_1\, dx = \int_{\Omega_1} \bigl(\lambda u^q + h(x)u^p\bigr)\phi_1\, dx.$$

Let $\bar\lambda$ satisfy

$$\lambda_1 t < \bar\lambda t^q + at^p \quad \text{for any } t > 0.$$

We can obtain that

$$\int_{\Omega} \bigl(\lambda u^q + au^p\bigr)\eta(x)\, dx < \int_{\Omega_1} \bigl(\lambda u^q + h(x)u^p\bigr)\phi_1\, dx = \int_{\Omega_1} \lambda_1 u\phi_1\, dx < \int_{\Omega_1} \bigl(\bar\lambda u^q + au^p\bigr)\phi_1\, dx$$
$$< \int_{\Omega} \bigl(\bar\lambda u^q + h(x)u^p\bigr)\eta(x)\, dx.$$

Clearly, $\lambda < \bar\lambda$.

Moreover, we have obtained a solution u_λ of (P_λ) for $\lambda \in (0, \lambda_0)$. Hence, $0 < \lambda_0 \leqslant \Lambda \leqslant \bar\lambda < \infty$. □

Lemma 5.2. *(P_λ) has a solution for all $\lambda \in (0, \Lambda)$.*

Proof. Given $0 < \lambda < \mu < \Lambda$. Let u_μ be a solution of (P_μ), then

$$-\Delta u_\mu = \mu u_\mu^q + h(x)u_\mu^p > \lambda u_\mu^q + h(x)u_\mu^p,$$

that is, u_μ is a supersolution of (P_λ). Furthermore, $\varepsilon\phi_1$ is a subsolution of (P_λ), and $\varepsilon\phi_1 < u_\mu$ for ε small enough. Therefore, there exists a solution u_λ of (P_λ) satisfying $\varepsilon\phi_1 \leqslant u_\lambda \leqslant u_\mu$. Consequently, for all $\lambda \in (0, \Lambda)$, (P_λ) has a solution. □

Lemma 5.3. *For all $\lambda \in (0, \Lambda)$, (P_λ) has a local minimum in the C^1 topology.*

Proof. Fix $\lambda \in (0, \Lambda)$. Choose $\lambda < \lambda_1 < \Lambda$ such that (P_λ) has a solution u_1. Let u_0 be the unique positive solution of

$$\begin{cases} -\Delta u = \lambda u^q, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Because

$$-\Delta u_1 = \lambda_1 u_1^q + h(x)u_1^p > \lambda u_1^q,$$

u_1 is a supersolution of (5.1). Moreover, $u_0 \neq u_1$, so $u_0 < u_1$ in Ω.

Set

$$\tilde f_\lambda(x, s) = \begin{cases} f_\lambda(u_0), & s \leqslant u_0, \\ f_\lambda(x, s), & u_0 < s < u_1, \\ f_\lambda(u_1), & s \geqslant u_1, \end{cases}$$

$$\tilde F_\lambda(x, u) = \int_0^u \tilde f_\lambda(x, s)\, ds,$$

and the functional $\tilde I_\lambda : H_s^1(\Omega) \to \mathbb{R}$ is given by

$$\tilde I_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} \tilde F_\lambda(x, u)\, dx.$$

Clearly, $\tilde{I_\lambda}$ is coercive and bounded below, then it achieve its global minimum at some $u_\lambda \in H_s^1(\Omega)$. Thus,

$$\begin{cases} -\Delta u_\lambda = \tilde{f}_\lambda(x, u_\lambda), & x \in \Omega, \\ u_\lambda > 0, & x \in \Omega, \\ u_\lambda = 0, & x \in \partial\Omega. \end{cases} \tag{5.2}$$

Since $\tilde{f}_\lambda(x, u_\lambda) \geqslant f_\lambda(u_0) \geqslant \lambda u_0^q$, we get $u_0 < u_\lambda$ in Ω.

Furthermore, we have

$$\begin{cases} -\Delta(u_\lambda - u_1) = \tilde{f}_\lambda(u_\lambda) - f_\lambda(u_1) \leqslant 0, & x \in \Omega, \\ u_\lambda - u_1 = 0, & x \in \partial\Omega, \end{cases} \tag{5.3}$$

the strong maximum principle yields $u_\lambda < u_1$ in Ω. Thus, $u_0 < u_\lambda < u_1$ in Ω.

From Lemma 4.1, $u_\lambda \in C^{2,\alpha}(\Omega)$, $\alpha \in (0,1)$. For $\|u - u_\lambda\|_{C^1} = \varepsilon$ with $\varepsilon > 0$ small enough, we have $u_0 < u < u_1$ in Ω. Hence, $\tilde{f}_\lambda(u) = f_\lambda(u)$, and also $\tilde{I_\lambda}(u) = I_\lambda(u)$ for $u_0 < u < u_1$. Therefore, u_λ is a local minimizer for I_λ in C^1 topology. □

Lemma 5.4. *Let $\lambda \in (0, \Lambda)$. u_λ is a local minimizer for I_λ in $H_s^1(\Omega)$.*

Proof. Follows from [20]. □

In the sequel we fix $\lambda \in (0, \Lambda)$, and hope to find a mountain-pass type solution of the form $v_\lambda = u_\lambda + v$, where u_λ is the local minimizer we have found in Lemma 5.4, and $v > 0$ in Ω.

Let v satisfy the equation

$$\begin{cases} -\Delta v = f_\lambda(u_\lambda + v) - f_\lambda(u_\lambda), & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{5.4}$$

Then, we define

$$g_\lambda(x, s) = \begin{cases} f_\lambda(u_\lambda + s) - f_\lambda(u_\lambda), & s \geqslant 0, \\ 0, & s < 0, \end{cases}$$

and

$$G_\lambda(v) = \int_0^v g_\lambda(x, s)\, ds,$$

$$J_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G_\lambda(v)\, dx.$$

Clearly, $J_\lambda : H_s^1(\Omega) \to \mathbb{R}^+$ is a C^1 functional. Moreover, if v is a nontrivial critical point of J_λ, then $v_\lambda = u_\lambda + v$ is a solution of (P_λ), and $v_\lambda \neq u_\lambda$.

Lemma 5.5. *Let $\lambda \in (0, \Lambda)$. J_λ has a nontrivial critical point.*

Proof. Clearly, $J_\lambda(0) = 0$. Moreover, it can be easily checked that $v = 0$ is a local minimizer of J_λ for all $\lambda \in (0, \Lambda)$. Recalling Section 3, we can similarly obtain that J_λ satisfies the $(PS)_c$ condition, and $J_\lambda(tv) \to \infty$ as $t \to \infty$. Hence, applying the Mountain Pass Theorem, we can obtain a critical point $v_0 \in H_s^1(\Omega)$ with $v_0 > 0$ in Ω. □

Proof of Theorem 2.3. (i) Lemma 5.4 and Lemma 5.5 prove point 1.

(ii) For all $\lambda \in (0, \Lambda)$, it is easy to verify that there exists a solution u_λ such that $I_\lambda(u_\lambda) < 0$. Choose a sequence $\{\lambda_n\}$ such that $\lambda_n \to \Lambda$ as $n \to \infty$. We denote the corresponding solutions to λ_n be u_{λ_n}. Then, they satisfy

$$I_{\lambda_n}(u_{\lambda_n}) < 0, \qquad I'_{\lambda_n}(u_\lambda) = 0.$$

That is

$$\frac{1}{2}\|u_{\lambda_n}\|^2 - \frac{\lambda_n}{q+1}\|u_{\lambda_n}^+\|_{q+1}^{q+1} - \frac{1}{p+1}\|u_{\lambda_n}^+\|_{h,p+1}^{p+1} < 0, \tag{5.5}$$

$$\|u_{\lambda_n}\|^2 - \lambda_n\|u_{\lambda_n}^+\|_{q+1}^{q+1} - \|u_{\lambda_n}^+\|_{h,p+1}^{p+1} = 0. \tag{5.6}$$

It follows from (5.5), (5.6) that, there exists $C > 0$ such that

$$\|u_{\lambda_n}\| < C.$$

Hence, there exists a convergence subsequence of $\{u_{\lambda_n}\}$, denoted by $\{u_{\lambda_m}\}$. Then,

$$u_{\lambda_m} \rightharpoonup u_\Lambda \quad \text{in } H_s^1(\Omega),$$
$$u_{\lambda_m} \to u_\Lambda \quad \text{in } L_h^{p+1}(\Omega).$$

Such a u_Λ is a weak solution of (P_λ) for $\lambda = \Lambda$.

(iii) Point 3 follows from the definition of Λ. $\square$

6. Existence results for the *p*-Laplace equation

Consider the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u \equiv -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f_\lambda(x,u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{$\bar{P}$}$$

where $1 < p < N$, $p^* = \frac{Np}{N-p}$; $\lambda > 0$.

When f_λ is sublinear, for example, $f_\lambda(x,u) = \lambda u^\alpha$, $0 < \alpha < p-1$, the sub-super solutions still can provide the existence of a unique solution of $(\bar{P})$ for all $\lambda > 0$, see [55].

When f_λ is the concave–convex nonlinearity, for instance, $f_\lambda(x,u) = \lambda u^\alpha + u^\beta$, $0 < \alpha < p-1 < \beta < p^*-1$, see [56, 23], they showed that the local minimizers of a class of functionals in the C^1 topology are still their local minimizers in $W_0^{1,p}(\Omega)$. Applying this fact, they obtained that $(\bar{P})$ has at least two solutions for all $\lambda \in (0,\Lambda)$, no solution for $\lambda > \Lambda$, at least one solution for $\lambda = \Lambda$. Such a kind of problems also has been studied in [59] by variational method and genus, in [28] by sub-super solutions.

When f_λ has supercritical growth, there are few papers about this aspect. The papers still take advantage of ODE techniques in balls. Zongming Guo [57,58] considered the problem $(\bar{P})$ for $f_\lambda(x,u) = u^\alpha - \lambda u^\beta$, $p^*-1 \leqslant \alpha < \beta$. He obtained that there are at least two positive radial solutions of $(\bar{P})$ for λ sufficiently small, and showed their asymptotic behavior as $\lambda \to 0$. In [19], the authors studied $(\bar{P})$ in a ball B_R for $f_\lambda(x,u) = u^\alpha + u^\beta$, where $p-1 < \alpha < p^*-1 < \beta$, $\lambda = 1$. They proved that there exists R_* such that $(\bar{P})$ has at least two distinct radial solutions provided $R > R_*$ and at least one radial solution provided $R = R_*$.

In this section, we give the existence result of positive solutions for quasilinear elliptic equation with supercritical growth. In fact, we can extend the results of semilinear elliptic problem naturally to quasilinear elliptic problem by similar methods.

In the following, we study the p-Laplacian equation

$$\begin{cases} -\Delta_p u = \lambda u^\alpha + h(x)u^\beta, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{$\bar{P}_\lambda$}$$

where $0 < \alpha < p-1 < \beta < p^*-1+\tau$, and λ, τ are positive parameters (τ is the constant obtained in [26]). $h(x)$ satisfies (H1), (H2).

We denote that

$$W_{0,s}^{1,p}(\Omega) = \big\{u \in W_0^{1,p}(\Omega) \bigm| u(\cdot,x_2) = u\big(\cdot,|x_2|\big),\ \forall x \in \Omega_2\big\},$$

with the norm $\|u\|_p = (\int_\Omega |\nabla u|^p\,dx)^{\frac{1}{p}}$.

$\bar{\lambda}_1$ is the first eigenvalue of $-\Delta_p$ in Ω with Dirichlet boundary condition, and $\bar{\phi}_1$ is the associated eigenfunction such that $\bar{\phi}_1 > 0$ in Ω.

Lemma 6.1. *(See [26].) Assume that $h(x)$ satisfies* (H1), (H2)*, then there exists a positive number $\tau = \tau(h,p,m,k)$ such that the embedding $W_{0,s}^{1,p}(\Omega) \hookrightarrow L_h^r(\Omega)$ is compact for all $r \in (1, p^*+\tau)$.*

According to Lemma 6.1, the embedding mapping $i : W_{0,s}^{1,p}(\Omega) \hookrightarrow L_h^{\beta+1}(\Omega)$ is compact for $\beta+1 < p^*+\tau$. Hence, for $u \in W_{0,s}^{1,p}(\Omega)$, we have $u \in L_h^{\beta+1}(\Omega)$. Then we can define the functional $\bar{I}_\lambda : W_{0,s}^{1,p}(\Omega) \to \mathbb{R}^+$ by

$$\bar{I}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \bar{F}_\lambda(x,u)\,dx,$$

where

$$\bar{F}_\lambda(x,u) = \int_0^u \bar{f}_\lambda(x,s)\,ds \quad \text{and} \quad \bar{f}_\lambda(x,s) = \begin{cases} \lambda s^\alpha + h(x)s^\beta, & s \geqslant 0, \\ 0, & s < 0. \end{cases}$$

We know that the energy functional $\bar{I}_\lambda$ is of class C^1.

Now, we give the existence result for $(\bar{P}_\lambda)$:

Theorem 6.2. *Let* $0 < \alpha < p-1 < \beta < p^* - 1 + \tau$. *If* $h(x)$ *satisfies* (H1), (H2), *then there exists* $\bar{\Lambda} \in (0, \infty)$, *such that*

(1) *for all* $\lambda \in (0, \bar{\Lambda})$, *problem* $(\bar{P}_\lambda)$ *has at least two weak solutions*;
(2) *for* $\lambda = \bar{\Lambda}$, *problem* $(\bar{P}_\lambda)$ *has at least one weak solution* $u_\Lambda \in W_{0,s}^{1,p}(\Omega) \cap L_h^{\beta+1}(\Omega)$;
(3) *for all* $\lambda > \bar{\Lambda}$, *problem* $(\bar{P}_\lambda)$ *has no solution.*

Lemma 6.3. *Let* v *be a weak solution of* $(\bar{P}_\lambda)$, *then* $v \in C^{1,\theta}(\Omega)$ *for some* $\theta \in (0,1)$.

Proof. $v \in W_{0,s}^{1,p}(\Omega)$ is a solution of $(\bar{P}_\lambda)$, applying Lemma 6.1, we get

$$v \in W_{0,s}^{1,p}(\Omega) \hookrightarrow L_h^r(\Omega) \subset L^r(\Omega), \quad r = p^* + \tau.$$

Then, we obtain

$$f_\lambda(x) = f_\lambda\big(x, v(x)\big) = \lambda v(x)^\alpha + h(x)v(x)^\beta \in L^\sigma(\Omega), \quad \sigma = \frac{r}{\beta}.$$

Since $1 < \beta < p^* - 1 + \tau$, we have

$$\sigma > \frac{p^* + \tau}{p^* + \tau - 1}.$$

We can denote

$$\sigma = \frac{p^* + \tau}{p^* + \tau - 1}(1 + \varepsilon), \quad \text{for some } \varepsilon > 0.$$

Clearly, we also have

$$-\mathrm{div}\big(|\nabla v|^{p-2}\nabla v\big) \in L^\sigma(\Omega),$$

and then

$$|\nabla v|^{p-1} \in W^{1,\sigma}(\Omega).$$

If $\sigma > N$, we are done. Otherwise, for $1 < \sigma < N$, we get from the Sobolev embedding theorem that

$$W^{1,\sigma}(\Omega) \hookrightarrow L^s(\Omega), \quad s = \frac{N\sigma}{N - \sigma}.$$

Thus,

$$|\nabla v|^{p-1} \in L^s(\Omega) \quad \text{and} \quad |\nabla v| \in L^{s(p-1)}(\Omega),$$

where $s(p-1) < N$ for $1 < p < N$ and $1 < \sigma < N$. Then, we get

$$v \in W^{1,s(p-1)}(\Omega) \hookrightarrow L^{r_1}(\Omega), \quad r_1 = \frac{Ns(p-1)}{N - s(p-1)}.$$

Clearly,

$$f_\lambda(x) \in L^{\sigma_1}(\Omega), \quad \sigma_1 = \frac{r_1}{\beta}.$$

We assert that

$$\frac{\sigma_1}{\sigma} > 1.$$

Indeed,

$$\frac{\sigma_1}{\sigma} = \frac{r_1}{r}, \quad \text{and} \quad r_1 > 0,\ r > 0.$$

Thus, we only need to check

$$r_1 > r,$$

that is,

$$\frac{Ns(p-1)}{N - s(p-1)} > p^* + \tau.$$

By computation, we get

$$\big[N^2(p-1) + Np + p(N-p)\tau\big]\varepsilon > (N-p)^2\tau.$$

Since $1 < p < N$, we have

$$\varepsilon > \frac{(N-p)^2\tau}{N^2(p-1) + Np + p(N-p)\tau} > 0.$$

Because of the arbitrary of ε, we conclude $r_1 > r$, and then $\frac{\sigma_1}{\sigma} > 1$.

Applying bootstrap argument, we can get $f_\lambda(x) \in L^{\sigma_k}(\Omega)$ for σ_k large enough. When $\sigma_k > n$, according to the Sobolev embedding theorem, we obtain

$$|\nabla v|^{p-1} \in W^{1,\sigma_k}(\Omega) \hookrightarrow C^{0,\theta}(\Omega), \quad \theta \in (0,1).$$

It follows that

$$|\nabla v| \in C^{0,\theta}(\Omega),$$

and then we conclude

$$v \in C^{1,\theta}(\Omega). \qquad \square$$

Remark 6.4. In fact, the proof of Theorem 6.2 is similar to the proof of Theorem 2.3. Only the following lemma we should give a different proof. Other proofs we skip them here.

Lemma 6.5. *Let $\bar{\Lambda} = \sup\{\lambda > 0\colon (\bar{P}_\lambda)$ has a solution$\}$, then $\bar{\Lambda} \in (0,\infty)$.*

Proof. We know that $\bar{\lambda}_1$ is isolated in bounded domain, that is, there exists $\delta > 0$ such that for every $\mu \in (\bar{\lambda}_1, \bar{\lambda}_1 + \delta)$, the problem

$$\begin{cases} -\Delta_p u = \mu |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

has no nontrivial solution.

Suppose that $v \in W^{1,p}_{0,s}(\Omega)$ is a solution of $(\bar{P}_\lambda)$, then it follows from Lemma 6.3 that $v \in C^{1,\theta}(\Omega)$. And there exists a small enough $\varepsilon > 0$ such that

$$0 < \varepsilon\bar{\phi}_1 \leqslant v \quad \text{in } \Omega. \tag{6.2}$$

Denote $\psi = \varepsilon\bar{\phi}_1$, we obtain

$$-\Delta_p \psi = \bar{\lambda}_1 \psi^{p-1} \leqslant \mu\psi^{p-1}. \tag{6.3}$$

However, let $\tilde{\lambda}$ be large enough such that for all $\lambda > \tilde{\lambda}$, we get

$$(\bar{\lambda}_1 + \delta)v^{p-1} \leqslant \lambda v^\alpha + h(x)v^\beta.$$

Thus, we have

$$-\Delta_p v \geqslant (\bar{\lambda}_1 + \delta)v^{p-1} \geqslant \mu v^{p-1}. \tag{6.4}$$

From (6.2), (6.3) and (6.4), we can construct a solution $\psi \leqslant u \leqslant v$ of the problem (6.1) by sub-super solutions. But this is a contradiction. Hence, we conclude that there exists $\tilde{\lambda}$ such that $\bar{\Lambda} \leqslant \tilde{\lambda} < \infty$.

Moreover, we can also obtain solutions of $(\bar{P}_\lambda)$ for small λ similar to Section 3, then $\bar{\Lambda} > 0$. So, $\bar{\Lambda} \in (0,\infty)$. $\square$

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