



Uniform estimates for the finite-time ruin probability in the dependent renewal risk model [☆]

Yang Yang ^{a,b,c}, Remigijus Leipus ^{c,d}, Jonas Šiaulys ^{c,*}, Yuquan Cang ^a

^a School of Mathematics and Statistics, Nanjing Audit University, Nanjing 210029, China

^b Department of Mathematics, Southeast University, Nanjing 210096, China

^c Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania

^d Institute of Mathematics and Informatics, Vilnius University, Akademijos 4, Vilnius LT-08663, Lithuania

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ABSTRACT

This paper investigates the finite-time ruin probability in the dependent renewal risk model, where the claim sizes are independent and identically distributed random variables with strongly subexponential tails, and the interarrival times are negatively dependent. We establish an asymptotic estimate, which holds uniformly for the time horizon varying in the positive half line.

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1. Introduction

In the classical Sparre Andersen risk model, the *claim sizes* Z_1, Z_2, \dots form a sequence of independent and identically distributed (i.i.d.) nonnegative random variables (r.v.s) with common distribution $B(u) = P(Z_1 \leq u)$ and finite mean EZ_1 ; the *interarrival times* $\theta_1, \theta_2, \dots$ are i.i.d. nonnegative r.v.s with common finite positive mean $E\theta_1 = 1/\lambda$. In addition, it is assumed that $\{\theta_n, n \geq 1\}$ are independent of $\{Z_n, n \geq 1\}$.

In such a model, the times of successive claims $\{\tau_n \equiv \sum_{k=1}^n \theta_k, n \geq 1\}$ constitute a renewal counting process

$$\Theta(t) = \sup\{n \geq 0: \tau_n \leq t\}, \quad t \geq 0 \quad (1.1)$$

with a mean function $\lambda(t) = E\Theta(t)$, for which $\lambda(t) \sim \lambda t$ as $t \rightarrow \infty$. The surplus process of the insurance company is then expressed as

$$R(t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i, \quad t \geq 0, \quad (1.2)$$

where $x \geq 0$ is the initial risk reserve and $c > 0$ represents the constant premium rate. Denote the ruin probability within finite time t by

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* Corresponding author.

E-mail address: jonas.siaulys@mif.vu.lt (J. Šiaulys).

$$\begin{aligned}\Psi(x, t) &= P\left(\inf_{0 \leq s \leq t} R(s) < 0 \mid R(0) = x\right) \\ &= P\left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k (Z_i - c\theta_i) > x\right).\end{aligned}\quad (1.3)$$

Throughout the paper we assume the following safety loading condition:

$$\mu = cE\theta_1 - EZ_1 > 0.$$

The ruin probability has been one of the central research topics in insurance mathematics and applied probability. Most of early works assumed the light-tailed case, where the claims and interarrival times satisfy the Cramér–Lundberg condition. As for the asymptotic behavior of the *ultimate ruin probability* $\Psi(x, \infty)$ in the case of heavy-tails (i.e., when the Cramér–Lundberg condition is not satisfied) we can refer to [26,10] among others. Note that the asymptotic behavior of the ruin probability in this case is totally different from those when the Cramér–Lundberg condition holds.

In this paper, we are interested in the *finite-time ruin probability* $\Psi(x, t)$, which is more practical, although much harder to investigate than the ultimate ruin probability $\Psi(x, \infty)$. The study of the finite-time ruin probability in the renewal risk model has a long history and many methods have been developed. Some early contributions to the area, mostly in the case where the claim sizes are light tailed, are reviewed in [27,2,4,20] among others.

Throughout the paper, without special statement, all limit relationships hold for x tending to ∞ . For two positive functions $a(x)$ and $b(x)$, we write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$; $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$; $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$. Furthermore, for two positive bivariate functions $a(x, t)$ and $b(x, t)$, we write $a(x, t) \lesssim b(x, t)$ uniformly for $t \in \mathcal{T}$ if $\limsup_{t \in \mathcal{T}} a(x, t)/b(x, t) \leq 1$; write $a(x, t) \sim b(x, t)$ uniformly for $t \in \mathcal{T}$ if $a(x, t) \lesssim b(x, t)$ and $b(x, t) \lesssim a(x, t)$ uniformly for $t \in \mathcal{T}$. The indicator function of an event A is denoted by $\mathbf{1}_A$.

In this paper, we shall restrict the claim size distribution B to some heavy-tailed classes of distributions supported on $[0, \infty)$. A natural class of heavy-tailed distributions is the subexponential class, denoted by \mathcal{S} . A distribution V belongs to the class \mathcal{S} , if $\bar{V}^{*n}(x) \sim n\bar{V}(x)$ for any $n \geq 2$, where $\bar{V}(x) = 1 - V(x) > 0$ for all $x \geq 0$ and V^{*n} denotes the n -fold convolution of V . Closely related is a wider class \mathcal{L} of long-tailed distributions. A distribution V belongs to the class \mathcal{L} , if $\bar{V}(x+y) \sim \bar{V}(x)$ for any $y > 0$. Another commonly used class is the class \mathcal{C} of consistently-varying-tailed distributions. A distribution V belongs to the class \mathcal{C} , if $\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \bar{V}(xy)/\bar{V}(x) = 1$. Korshunov [15] introduced an important class of strongly subexponential distributions, denoted by \mathcal{S}_* . A distribution V , with finite $m_V := \int_0^\infty \bar{V}(x) dx$, belongs to the class \mathcal{S}_* , if $\bar{V}_u^{*2}(x) \sim 2\bar{V}_u(x)$ uniformly for $u \in [1, \infty)$, where $\bar{V}_u(x) = \min(1, \int_x^{x+u} \bar{V}(y) dy)$ for $x \geq 0$, and $\bar{V}_u(x) = 1$ for $x < 0$. In the case when the support of d.f. V is larger than $[0, \infty)$, we say that this d.f. belongs to the corresponding class if $V_+(x) = V(x)\mathbf{1}_{\{x \geq 0\}}$ is from that class. In case $m_V < \infty$, the following inclusion relationship holds

$$\mathcal{C} \subset \mathcal{S}_* \subset \mathcal{S} \subset \mathcal{L},$$

see, e.g., [9,14].

In case of heavy-tailed claim sizes, Tang [23] proved that if $B \in \mathcal{C}$ and some other mild conditions hold, then the finite-time ruin probability satisfies

$$\Psi(x, t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du \quad (1.4)$$

uniformly for $t \in \Lambda$, where $\Lambda = \{t > 0: \lambda(t) > 0\}$. He also pointed out that by the elementary renewal theorem

$$\Psi(x, t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda t} \bar{B}(u) du \quad (1.5)$$

uniformly for $t \in [f(x), \infty)$, where $f(x)$ is an arbitrary infinitely increasing function. Later, Leipus and Šiaulyš [18] investigated a more general case. Let $Q(u) = -\log \bar{B}(u)$, $u > 0$, be the hazard function of distribution B and assume that there exists a nonnegative function $q: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $Q(u) = \int_0^u q(v) dv$ and $r = \limsup_{u \rightarrow \infty} uq(u)/Q(u)$ is finite. Then function $q(u)$ and constant r are called the hazard rate and hazard ratio index of B , respectively (for more details, see [3]). Leipus and Šiaulyš [18] obtained that if $r < 1/2$, then (1.5) holds uniformly for $t \in [f(x), \gamma x]$ for an arbitrary infinitely increasing function $f(x)$ and an arbitrary positive constant γ . Recently, Leipus and Šiaulyš [19] substantially extended these results and established an asymptotic relation (1.5) under the assumption $B \in \mathcal{S}_*$, but the uniformity is for $t \in [f(x), \infty)$, where $f(x)$ is an arbitrary infinitely increasing function. Note also the recent papers of Jiang [11,12], where asymptotic relation (1.5) was established in the case of the compound Poisson model.

Motivated by the latter results, in this paper (see Theorem 2.1) we derive an asymptotic relation for $\Psi(x, t)$ equipped with uniformity similar as in [23], i.e. for $t \in [T, \infty)$, but for a significantly larger class for the claim size distribution.

Additionally, the interarrival times are not necessarily independent. We can replace the independence of $\{\theta_n, n \geq 1\}$ by a flavor of negative dependence, which was introduced by Ebrahimi and Ghosh [8] and Block et al. [5], showing that the asymptotics of $\Psi(x, t)$ are not affected by this dependence.

Recall some definitions. We say that r.v.s $\{\xi_k, k \geq 1\}$ are Upper Negatively Dependent (UND), if for each $n \geq 1$ and y_1, \dots, y_n

$$P\left(\bigcap_{k=1}^n \{\xi_k > y_k\}\right) \leq \prod_{k=1}^n P(\xi_k > y_k). \quad (1.6)$$

We say that r.v.s $\{\xi_k, k \geq 1\}$ are Lower Negatively Dependent (LND), if for each $n \geq 1$ and y_1, \dots, y_n

$$P\left(\bigcap_{k=1}^n \{\xi_k \leq y_k\}\right) \leq \prod_{k=1}^n P(\xi_k \leq y_k). \quad (1.7)$$

We say that r.v.s $\{\xi_k, k \geq 1\}$ are Negatively Dependent (ND), if both (1.6) and (1.7) hold for each $n \geq 1$ and y_1, \dots, y_n . Note that the ND structure is weaker the well-known negative association (see, e.g., [1,13]). For $n = 2$, the UND, LND, and ND structures are equivalent, in this case r.v.s ξ_1 and ξ_2 are called Negative Quadrant Dependent (NQD), according to Lehmann [17]. We say that r.v.s $\{\xi_k, k \geq 1\}$ are pairwise NQD, if for all positive integers $i \neq j$, ξ_i and ξ_j are NQD. Clearly, if r.v.s $\{\xi_k, k \geq 1\}$ are either UND or LND, they are also pairwise NQD.

The outline of the paper is as follows. Section 2 provides the assumptions and the main result of the paper. Section 3 proves some useful results related to ND r.v.s. Section 4 gives some auxiliary lemmas.

2. Main result

In this section, we formulate and provide a proof of our main result, which is based on the lemmas in Sections 3 and 4. We assume that claim sizes and interarrival times satisfy the following assumptions.

Assumption H₁. The claim sizes Z_1, Z_2, \dots form a sequence of i.i.d. nonnegative r.v.s with common distribution function $B(u) = P(Z_1 \leq u)$ and finite mean $\beta = EZ_1$.

Assumption H₂. The interarrival times $\theta_1, \theta_2, \dots$ are nonnegative, LND and identically distributed r.v.s with finite positive mean $E\theta_1 = 1/\lambda$.

Assumption H₃. The sequences $\{\theta_1, \theta_2, \dots\}$ and $\{Z_1, Z_2, \dots\}$ are mutually independent.

Theorem 2.1. Assume risk model (1.2), such that Assumptions H₁, H₂, H₃ are satisfied and $B \in \mathcal{S}_*$. Let the following conditions be satisfied

$$e^{-\delta\sqrt{x}} = o(\bar{B}(x)) \quad \forall \delta > 0, \quad \bar{B}(x - \sqrt{x}) \lesssim \bar{B}(x). \quad (2.1)$$

Then for any T , such that $\lambda(T) = E\Theta(T) > 0$, the asymptotic relation

$$\Psi(x, t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du$$

holds uniformly over $t \in [T, \infty)$.

Remark 2.1. As it follows from the proof of Theorem 2.1, the lower estimate

$$\Psi(x, t) \gtrsim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du \quad \text{uniformly for } t \in [T, \infty)$$

holds for wider class of d.f.s, i.e. for $B \in \mathcal{L} \supset \mathcal{S}_*$.

Remark 2.2. Since any d.f. from \mathcal{C} satisfies restrictions (2.1) (see Eq. (2.4) and Lemma 4.1 in [23]), Theorem 2.1 generalizes the corresponding result of Tang [23].

Proof of Theorem 2.1. Let $\epsilon \in (0, 1/2)$ and let $T_{1,\epsilon} > T$ be as in Lemma 4.1. Note that

$$\begin{aligned} & \sup_{t \in [T, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \\ & \leq \max \left\{ \sup_{t \in [T, T_{1,\epsilon}]} \frac{\Psi(x, t)}{\lambda(t) \bar{B}(x)}, \sup_{t \in [T, T_{1,\epsilon}]} \frac{\lambda(t) \bar{B}(x)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du}, \sup_{t \in [T_{1,\epsilon}, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \right\}. \end{aligned}$$

Lemmas 4.1 and 4.3 imply

$$\limsup_{x \rightarrow \infty} \sup_{t \in [T, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \leq \max \left\{ \limsup_{x \rightarrow \infty} \frac{\bar{B}(x)}{\bar{B}(x + \mu\lambda(T_{1,\epsilon}))}, 1 + \epsilon \right\} = 1 + \epsilon,$$

where the last equality holds by $B \in \mathcal{S}_* \subset \mathcal{L}$. As $\epsilon \in (0, 1/2)$ is arbitrarily chosen, the last inequality yields that

$$\Psi(x, t) \lesssim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du \quad (2.2)$$

uniformly for $t \in [T, \infty)$.

Similarly,

$$\begin{aligned} & \inf_{t \in [T, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \\ & \geq \min \left\{ \inf_{t \in [T, T_{2,\epsilon}]} \frac{\Psi(x, t)}{\lambda(t) \bar{B}(x)}, \inf_{t \in [T, T_{2,\epsilon}]} \frac{\lambda(t) \bar{B}(x)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du}, \inf_{t \in [T_{2,\epsilon}, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \right\} \\ & \geq \min \left\{ \inf_{t \in [T, T_{2,\epsilon}]} \frac{\Psi(x, t)}{\lambda(t) \bar{B}(x)}, \inf_{t \in [T_{2,\epsilon}, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \right\} \end{aligned}$$

with $T_{2,\epsilon} > T$ as in the statement of Lemma 4.2. Using Lemmas 4.2 and 4.4 we obtain

$$\liminf_{x \rightarrow \infty} \inf_{t \in [T, \infty)} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \geq 1 - \epsilon$$

for any $\epsilon \in (0, 1/2)$. Thus, uniformly for $t \in [T, \infty)$,

$$\Psi(x, t) \gtrsim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du \quad (2.3)$$

and the statement of the theorem follows from (2.2) and (2.3). \square

3. Some results related to negatively dependent r.v.s

In this section, we present two results related to negatively dependent r.v.s, which will be used in the following section.

The first result gives a property of the renewal counting process generated by LND r.v.s, which extends that of [16]. Recall that, as in (1.1), $\Theta(t)$ is a renewal counting process generated by interarrival times $\theta_1, \theta_2, \dots$.

Lemma 3.1. Let $\theta_1, \theta_2, \dots$ be (not necessarily identically distributed) LND r.v.s such that $P(\theta_n \geq \theta) = 1$ for all $n = 1, 2, \dots$ with some nonnegative r.v. θ having finite positive mean $E\theta = 1/\lambda$. Then for any $a > \lambda$, there exists $b > 1$ such that

$$\lim_{t \rightarrow \infty} \sum_{k > at} P(\Theta(t) \geq k) b^k = 0.$$

Proof. The proof only differs slightly from Theorem 1 of [16]. Indeed, since $\theta_1, \theta_2, \dots$ are nonnegative, for any $y > 0$ we have

$$\begin{aligned} P(\Theta(t) \geq k) &= P(\theta_1 + \dots + \theta_k \leq t) = P(e^{-y(\theta_1 + \dots + \theta_k)} \geq e^{-yt}) \\ &\leq e^{yt} E \left(\prod_{j=1}^k e^{-y\theta_j} \right). \end{aligned}$$

The LND property of r.v.s $\theta_1, \theta_2, \dots$ implies that the r.v.s $e^{-y\theta_1}, e^{-y\theta_2}, \dots$ are UND. According to Lemma 2.2 in [25] we have

$$\mathbb{E} \left(\prod_{j=1}^k e^{-y\theta_j} \right) \leq \prod_{j=1}^k \mathbb{E} e^{-y\theta_j} \leq (\mathbb{E} e^{-y\theta})^k$$

for any $k = 1, 2, \dots$. The rest of the proof is identical to the proof of Theorem 1 of [16]. \square

Lemma 3.2. Let ξ_1, ξ_2, \dots be a sequence of UND (not necessarily identically distributed) r.v.s such that $P(\xi_n \leq \xi) = 1, n \geq 1$ with r.v. ξ having finite negative mean and let $\tau = \sup\{t \in \mathbb{R}: \mathbb{E} e^{t\xi} < \infty\}$ be positive. Then there exists positive $\tau^* \in (0, \tau)$ such that $\mathbb{E} e^{t\xi} < 1$ and

$$P \left(\sup_{n \geq 1} \sum_{i=1}^n \xi_i > x \right) \leq e^{-tx} \frac{\mathbb{E} e^{t\xi}}{1 - \mathbb{E} e^{t\xi}} \quad (3.1)$$

for all $x > 0$ and $t \in (0, \tau^*)$.

Proof. By Markov's inequality, for any positive t ,

$$\begin{aligned} P \left(\sup_{n \geq 1} \sum_{i=1}^n \xi_i > x \right) &= P \left(\bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^n \xi_i > x \right\} \right) \\ &\leq \sum_{n=1}^{\infty} P \left(\sum_{i=1}^n \xi_i > x \right) \\ &= \sum_{n=1}^{\infty} P(e^{t \sum_{i=1}^n \xi_i} > e^{tx}) \\ &\leq e^{-tx} \sum_{n=1}^{\infty} \mathbb{E} \left(\prod_{i=1}^n e^{t\xi_i} \right). \end{aligned}$$

Since the r.v.s ξ_1, ξ_2, \dots are UND and the function e^{tu} is strictly increasing in u , the r.v.s $e^{t\xi_1}, e^{t\xi_2}, \dots$ are UND too. Hence, according to [25, Lemma 2.2] and the conditions of the lemma we obtain

$$P \left(\sup_{n \geq 1} \sum_{i=1}^n \xi_i > x \right) \leq e^{-tx} \sum_{n=1}^{\infty} (\varphi(t))^n, \quad t \in (0, \tau), \quad (3.2)$$

where $\varphi(t) := \mathbb{E} e^{t\xi}$. For $t \in (0, \tau)$ we have $\varphi'(t) = \mathbb{E} \xi e^{t\xi} < \infty$, $\varphi'(0) = \mathbb{E} \xi < 0$ and $\varphi''(t) = \mathbb{E} \xi^2 e^{t\xi} \in (0, \infty)$. Therefore, $\varphi(t) < \varphi(0) = 1$ for $t \in (0, \tau^*)$ with some $\tau^* \in (0, \tau)$. Estimate (3.1) now follows from (3.2). \square

4. Auxiliary lemmas

Lemma 4.1. Assume risk model (1.2). If Assumptions H_1, H_2 and H_3 are satisfied with $B \in \mathcal{S}_*$, then for every $\epsilon > 0$ there exists $T_{1,\epsilon}$ such that

$$\Psi(x, t) \lesssim (1 + \epsilon) \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du$$

uniformly for $t \geq T_{1,\epsilon}$.

Proof. Suppose that $\epsilon \in (0, 1/2)$. For any $\Delta_1 > 0$ and $t > 0$, split

$$\Psi(x, t) = \Psi_1(x, t) + \Psi_2(x, t), \quad (4.1)$$

where

$$\begin{aligned} \Psi_1(x, t) &:= P \left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k (Z_i - c\theta_i) > x, \Theta(t) \leq \lambda(t)(1 + \Delta_1) \right), \\ \Psi_2(x, t) &:= P \left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k (Z_i - c\theta_i) > x, \Theta(t) > \lambda(t)(1 + \Delta_1) \right) \end{aligned}$$

with, as previously, $\lambda(t) = \mathbb{E} \Theta(t)$.

First, we estimate $\Psi_2(x, t)$. Observe that

$$\begin{aligned}\Psi_2(x, t) &\leq \sum_{n > \lambda(t)(1+\Delta_1)} \mathbb{P}\left(\max_{1 \leq k \leq n} \sum_{i=1}^k Z_i > x, \Theta(t) = n\right) \\ &= \sum_{n > \lambda(t)(1+\Delta_1)} \overline{B^{*n}}(x) \mathbb{P}(\Theta(t) = n).\end{aligned}\quad (4.2)$$

According to Lemma 1 in [14], we have $B \in \mathcal{S}_* \subset \mathcal{S}$. By Lemma 1.3.5(c) of [9], for any $\Delta_2 > 0$ there exists a constant $c_1(\Delta_2) > 0$ such that for all $x \geq 0$ and $n \geq 2$

$$\overline{B^{*n}}(x) \leq c_1(\Delta_2)(1 + \Delta_2)^n \overline{B}(x).$$

Therefore, we get from (4.2) that

$$\Psi_2(x, t) \leq c_1(\Delta_2) \overline{B}(x) \sum_{n > \lambda(t)(1+\Delta_1)} (1 + \Delta_2)^n \mathbb{P}(\Theta(t) = n)$$

for any $x > 0$, $t > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$.

According to the proof of Proposition 5.1 in [7] (see also Theorem 4.2 in [6]), for nonnegative, identically distributed LND r.v.s $\theta_1, \theta_2, \dots$ with finite, positive mean $1/\lambda$ it holds that

$$\Theta(t)/(\lambda t) \rightarrow 1 \quad \text{a.s. and} \quad \mathbb{E}\Theta(t) \sim \lambda t \quad (4.3)$$

as $t \rightarrow \infty$. Thus, for sufficiently large t (say, $t \geq T_3$) we have $\lambda(t) \geq \lambda t(1 - \Delta_1/(2(1 + \Delta_1))) \geq \lambda t/2$. Hence, for $x > 0$, $t \geq T_3$, $\Delta_1 > 0$ and $\Delta_2 > 0$,

$$\Psi_2(x, t) \leq c_1(\Delta_2) \overline{B}(x) \sum_{n > \lambda t(1+\Delta_1/2)} (1 + \Delta_2)^n \mathbb{P}(\Theta(t) \geq n). \quad (4.4)$$

Using Lemma 3.1 we have that for any $\Delta_1 > 0$ there exists $\Delta_2 > 0$ such that

$$\mathbb{E}(1 + \Delta_2)^{\Theta(t)} \mathbf{1}_{\{\Theta(t) \geq \lambda t(1+\Delta_1/2)\}} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Since $\lambda(T_3)$ is finite, we have that for any $\Delta_1 > 0$ and special $\Delta_2 > 0$

$$\mathbb{E}(1 + \Delta_2)^{\Theta(t)} \mathbf{1}_{\{\Theta(t) \geq \lambda t(1+\Delta_1/2)\}} \leq \frac{\epsilon \lambda(T_3)}{2c_1(\Delta_2)}$$

if $t \geq T_4 = T_4(\epsilon) \geq T_3$. This and (4.4) imply that

$$\Psi_2(x, t) \leq \epsilon \frac{\lambda(T_3)}{2} \overline{B}(x). \quad (4.5)$$

Clearly, if $t \geq T_4$ then

$$\frac{\overline{B}(x)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \overline{B}(u) du} \leq \frac{\overline{B}(x)}{\lambda(T_3) \overline{B}(x + \mu\lambda(T_4))}. \quad (4.6)$$

Combining (4.5) and (4.6), together with $B \in \mathcal{S}_* \subset \mathcal{L}$, we obtain

$$\Psi_2(x, t) \lesssim \frac{\epsilon}{2} \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \overline{B}(u) du \quad (4.7)$$

uniformly for $t \in [T_4, \infty)$.

Now we deal with $\Psi_1(x, t)$. For all $\Delta_1 > 0$, $\Delta_3 \in (0, 1)$, $x > 0$ and sufficiently large t (e.g., $t \geq T_3$) we have

$$\begin{aligned}\Psi_1(x, t) &\leq \mathbb{P}\left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k \left(Z_i - \frac{c}{\lambda}(1 - \Delta_3)\right) + c \sup_{k \geq 1} \sum_{i=1}^k \left(\frac{1 - \Delta_3}{\lambda} - \theta_i\right) > x, \Theta(t) \leq \lambda(t)(1 + \Delta_1)\right) \\ &\leq \mathbb{P}(\zeta_t + \eta > x) \leq \mathbb{P}(\zeta_t + \eta^+ > x),\end{aligned}$$

where

$$\begin{aligned}\zeta_t &:= \max_{1 \leq k \leq \lambda(t)(1+\Delta_1)} \sum_{i=1}^k \left(Z_i - \frac{c}{\lambda}(1 - \Delta_3)\right), \\ \eta &:= c \sup_{k \geq 1} \sum_{i=1}^k \left(\frac{1 - \Delta_3}{\lambda} - \theta_i\right), \quad \eta^+ := \max\{\eta, 0\}.\end{aligned}$$

By the independence of r.v.s ζ_t and η^+ , for arbitrary $x > 0$ we have

$$\begin{aligned}\Psi_1(x, t) &\leq \int_0^{x/2} P(\zeta_t > x - u) dP(\eta^+ \leq u) + P(\eta^+ > x/2) \\ &=: \Psi_{11}(x, t) + \Psi_{12}(x).\end{aligned}\quad (4.8)$$

As $\tilde{\beta} = E(Z_1 - \frac{c}{\lambda}(1 - \Delta_3)) = -\mu + (\mu + \beta)\Delta_3 < 0$ with $\Delta_3 \in (0, \mu/(\mu + \beta))$, from Lemma 9 of [15] we obtain that for any $\epsilon_1 \in (0, 1/2)$

$$\begin{aligned}P(\zeta_t > x - u) &\leq \frac{1 + \epsilon_1}{|\tilde{\beta}|} \int_{x-u}^{x-u+\mu\lambda(t)(1+\Delta_1)} \bar{B}\left(v + \frac{c}{\lambda}(1 - \Delta_3)\right) dv \\ &\leq \frac{1 + \epsilon_1}{|\tilde{\beta}|} \int_x^{x+\mu\lambda(t)(1+\Delta_1)} \bar{B}(w - u) dw,\end{aligned}$$

where $x - u \geq x/2$, x is sufficiently large and $t \geq T_3$. By Fubini's theorem,

$$\begin{aligned}\Psi_{11}(x, t) &\leq \frac{1 + \epsilon_1}{|\tilde{\beta}|} \int_0^{x/2} \left(\int_x^{x+\mu\lambda(t)(1+\Delta_1)} \bar{B}(w - u) dw \right) dP(\eta^+ \leq u) \\ &\leq \frac{1 + \epsilon_1}{|\tilde{\beta}|} \int_x^{x+\mu\lambda(t)(1+\Delta_1)} \bar{B} * F_{\eta^+}(w) dw\end{aligned}\quad (4.9)$$

for sufficiently large x and $t \geq T_3$, where $F_{\eta^+}(w)$ is a d.f. of r.v. η^+ .

Note that $E(\frac{1}{\lambda}(1 - \Delta_3) - \theta_1) = -\frac{\Delta_3}{\lambda} < 0$, $\frac{1}{\lambda}(1 - \Delta_3) - \theta_1 \leq \frac{1}{\lambda}(1 - \Delta_3)$ and the r.v.s $\frac{1}{\lambda}(1 - \theta_i)$ are UND. Hence, by Lemma 3.2,

$$P(\eta^+ > x) \leq c_2 e^{-c_3 x} \quad (4.10)$$

for some positive constants c_2 and c_3 (possibly depending on Δ_3). Since $B \in \mathcal{S}_* \subset \mathcal{S}$, Corollary 2 of [22] implies that $\bar{B} * F_{\eta^+}(w) \sim \bar{B}(w)$ as $w \rightarrow \infty$. Thus, by (4.9),

$$\begin{aligned}\Psi_{11}(x, t) &\leq \frac{(1 + \epsilon_1)^2}{|\tilde{\beta}|} \int_x^{x+\mu\lambda(t)(1+\Delta_1)} \bar{B}(w) dw \\ &= \frac{(1 + \epsilon_1)^2}{|\tilde{\beta}|} \int_x^{x+\mu\lambda(t)} \bar{B}(w) dw \left(1 + \frac{\int_{x+\mu\lambda(t)}^{x+\mu\lambda(t)(1+\Delta_1)} \bar{B}(v) dv}{\int_x^{x+\mu\lambda(t)} \bar{B}(v) dv} \right) \\ &\leq \frac{(1 + \epsilon_1)^2(1 + \Delta_1)}{|\tilde{\beta}|} \int_x^{x+\mu\lambda(t)} \bar{B}(w) dw\end{aligned}\quad (4.11)$$

for $\Delta_1 > 0$, $\Delta_3 \in (0, \mu/(\mu + \beta))$, sufficiently large x and $t \geq T_3$.

It remains to estimate $\Psi_{12}(x)$. If $t \geq T_3$, then, by (4.10),

$$\frac{\Psi_{12}(x)}{\int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \leq \frac{P(\eta^+ > x/2)}{\int_x^{x+\mu\lambda(T_3)} \bar{B}(u) du} \leq \frac{c_2 e^{-c_3 x/2}}{\mu\lambda(T_3) \bar{B}(x + \mu\lambda(T_3))}.$$

By $B \in \mathcal{S}_* \subset \mathcal{L}$ and Lemma 1.3.5(b) of [9] we have that for sufficiently large x

$$\frac{\Psi_{12}(x)}{\int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \leq \frac{\epsilon_1}{|\tilde{\beta}|}. \quad (4.12)$$

Now, from estimates (4.8), (4.11) and (4.12) we have that

$$\Psi_1(x, t) \lesssim \left(\frac{(1 + \epsilon_1)^2(1 + \Delta_1) + \epsilon_1}{1 - \Delta_3(\mu + \beta)/\mu} \right) \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du$$

for $t \geq T_3$. Taking $\epsilon_1 = \epsilon/7$ we have $(1 + \epsilon_1)^2 + \epsilon_1 \leq 1 + \frac{\epsilon}{2}$. So that, the arbitrariness of $\Delta_1 > 0$ and $\Delta_3 \in (0, \mu/(\mu + \beta))$ gives that

$$\Psi_1(x, t) \lesssim \left(1 + \frac{\epsilon}{2}\right) \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du \quad (4.13)$$

uniformly for $t \in [T_3, \infty)$ and the statement of the lemma follows from (4.1), (4.7) and (4.13). \square

Lemma 4.2. Let Assumptions H_1, H_2 and H_3 be satisfied with $B \in \mathcal{L}$. Then for every $\epsilon > 0$ there exists $T_{2,\epsilon}$ such that

$$\Psi(x, t) \gtrsim (1 - \epsilon) \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du$$

uniformly for $t \geq T_{2,\epsilon}$.

Proof. Suppose again that $\epsilon \in (0, 1/2)$. For any $x > 0, t > 0$ and any positive constants Δ_4, Δ_5 we have

$$\begin{aligned} \Psi(x, t) &= P\left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k (Z_i - c\theta_i) > x\right) \\ &\geq P\left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k \left(Z_i - \frac{c}{\lambda}(1 + \Delta_4)\right) > x + \frac{c}{\Delta_5}, \min_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k \left(\frac{1}{\lambda}(1 + \Delta_4) - \theta_i\right) > -\frac{1}{\Delta_5}\right) \\ &= \sum_{n=1}^{\infty} P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k \left(Z_i - \frac{c}{\lambda}(1 + \Delta_4)\right) > x + \frac{c}{\Delta_5}\right) P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k \left(\theta_i - \frac{1}{\lambda}(1 + \Delta_4)\right) < \frac{1}{\Delta_5}, \Theta(t) = n\right), \end{aligned}$$

where the last equality holds by the independence of $\{Z_1, Z_2, \dots\}$ and $\{\theta_1, \theta_2, \dots\}$.

Since $B \in \mathcal{L}$ and $\hat{\beta} := E(Z_1 - \frac{c}{\lambda}(1 + \Delta_4)) = -\mu - (\mu + \beta)\Delta_4 < 0$, we obtain from [15, Lemma 1] that

$$\Psi(x, t) \geq \frac{1 - \epsilon_2}{|\hat{\beta}|} \sum_{n=1}^{\infty} \int_{x+c/\Delta_5}^{x+c/\Delta_5+n|\hat{\beta}|} \bar{B}\left(u + \frac{c}{\lambda}(1 + \Delta_4)\right) du P\left(\chi < \frac{1}{\Delta_5}, \Theta(t) = n\right) \quad (4.14)$$

for any $\epsilon_2 \in (0, 1/2)$, positive Δ_4, Δ_5, t and sufficiently large x , where

$$\chi := \sup_{k \geq 1} \sum_{i=1}^k \left(\theta_i - \frac{1 + \Delta_4}{\lambda}\right).$$

Clearly, the condition $B \in \mathcal{L}$ implies that

$$\inf_{u \geq x} \frac{\bar{B}(u + \frac{c}{\Delta_5} + \frac{c}{\lambda}(1 + \Delta_4))}{\bar{B}(u + \frac{c}{\Delta_5})} \geq 1 - \epsilon_2$$

for sufficiently large x . Hence, (4.14) implies

$$\begin{aligned} \Psi(x, t) &\geq \frac{(1 - \epsilon_2)^2}{|\hat{\beta}|} \sum_{n=1}^{\infty} \int_x^{x+n\mu} \bar{B}\left(u + \frac{c}{\Delta_5}\right) du P\left(\chi < \frac{1}{\Delta_5}, \Theta(t) = n\right) \\ &\geq \frac{(1 - \epsilon_2)^2}{|\hat{\beta}|} \int_x^{x+\mu\lambda(t)(1-\epsilon_2)} \bar{B}\left(u + \frac{c}{\Delta_5}\right) du P\left(\Theta(t) \geq \lambda(t)(1 - \epsilon_2), \chi < \frac{1}{\Delta_5}\right) \end{aligned} \quad (4.15)$$

for positive $\Delta_4, \Delta_5, t \geq T_3$ and sufficiently large x .

According to Theorem 1 of [21],

$$\frac{\sum_{i=1}^k (\theta_i - \frac{1+\Delta_4}{\lambda})}{k} \rightarrow -\frac{\Delta_4}{\lambda} < 0 \quad \text{a.s.}$$

as $k \rightarrow \infty$, implying that

$$\lim_{\Delta_5 \downarrow 0} \mathbb{P}\left(\chi < \frac{1}{\Delta_5}\right) = 1. \quad (4.16)$$

On the other hand, by (4.3), almost surely $\Theta(t)/\lambda(t) \rightarrow 1$ as $t \rightarrow \infty$, so that

$$\lim_{t \rightarrow \infty} \mathbb{P}(\Theta(t) \geq \lambda(t)(1 - \epsilon_2)) = 1. \quad (4.17)$$

Relations (4.16) and (4.17) imply that

$$\lim_{\Delta_5 \downarrow 0} \lim_{t \rightarrow \infty} \mathbb{P}\left(\chi < \frac{1}{\Delta_5}, \Theta(t) \geq \lambda(t)(1 - \epsilon_2)\right) = 1.$$

Hence, there exists $T_5 = T_5(\epsilon) \geq T_3$ such that for $t \geq T_5$ and sufficiently small Δ_5

$$\mathbb{P}\left(\chi < \frac{1}{\Delta_5}, \Theta(t) \geq \lambda(t)(1 - \epsilon_2)\right) \geq 1 - \epsilon_2. \quad (4.18)$$

From lower estimates (4.15) and (4.18) we have that

$$\begin{aligned} \frac{\Psi(x, t)}{\mu^{-1} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du} &\geq \frac{(1 - \epsilon_2)^3}{\mu^{-1} |\hat{\beta}|} \frac{\int_x^{x+\mu\lambda(t)(1-\epsilon_2)} \bar{B}(u + \frac{c}{\Delta_5}) du}{\int_x^{x+\mu\lambda(t)} \bar{B}(u) du} \\ &\geq \frac{(1 - \epsilon_2)^3}{1 + \frac{\mu+\beta}{\mu} \Delta_4} \inf_{u \geq x} \frac{\bar{B}(u + \frac{c}{\Delta_5})}{\bar{B}(u)} \left(1 - \frac{\int_x^{x+\mu\lambda(t)(1-\epsilon_2)} \bar{B}(u) du}{\int_x^{x+\mu\lambda(t)} \bar{B}(u) du}\right) \end{aligned} \quad (4.19)$$

for sufficiently large x , arbitrary Δ_4 , sufficiently small $\Delta_5 > 0$ and all $t \geq T_5$.

To finish the proof, note that, since $B \in \mathcal{L}$, for sufficiently large x

$$\inf_{u \geq x} \frac{\bar{B}(u + \frac{c}{\Delta_5})}{\bar{B}(u)} \geq 1 - \epsilon_2. \quad (4.20)$$

In addition,

$$\begin{aligned} \int_{x+\mu\lambda(t)(1-\epsilon_2)}^{x+\mu\lambda(t)} \bar{B}(u) du &\leq \mu\lambda(t)\epsilon_2 \bar{B}(x + \mu\lambda(t)(1 - \epsilon_2)) \\ &\leq \frac{\epsilon_2}{1 - \epsilon_2} \int_x^{x+\mu\lambda(t)(1-\epsilon_2)} \bar{B}(u) du \\ &\leq \frac{\epsilon_2}{1 - \epsilon_2} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du. \end{aligned} \quad (4.21)$$

Combining now (4.20) and (4.21), we obtain from (4.19) that for arbitrary Δ_4

$$\Psi(x, t) \gtrsim \left(\frac{(1 - \epsilon_2)^3(1 - 2\epsilon_2)}{1 + \frac{\mu+\beta}{\mu} \Delta_4}\right) \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{B}(u) du$$

uniformly for $t \in [T_5, \infty)$.

The last estimate implies the statement of the lemma noticing that $(1 - \epsilon_2)^3(1 - 2\epsilon_2) \geq 1 - \epsilon$ if $\epsilon_2 = \epsilon^3$ and $\epsilon \in (0, 1/2)$. \square

Lemma 4.3. Let Assumptions H_1 , H_2 and H_3 be satisfied, $B \in \mathcal{S}_*$ and let conditions (2.1) hold. Then for any T , such that $\lambda(T) > 0$,

$$\Psi(x, t) \lesssim \lambda(t) \bar{B}(x)$$

uniformly for $t \geq T$.

Proof. Along the line of [18] (see their estimate (4.1)), for any $\Delta_6 > 0$

$$\begin{aligned}\Psi(x, t) &\leq \mathbb{P}\left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k \left(Z_i - \frac{c}{\lambda}(1 - \Delta_6)\right) > x - \sqrt{x}\right) + \mathbb{P}\left(\sup_{k \geq 1} \sum_{i=1}^k \left(\frac{1}{\lambda}(1 - \Delta_6) - \theta_i\right) > \frac{\sqrt{x}}{c}\right) \\ &=: \widehat{\Psi}_1(x, t) + \widehat{\Psi}_2(x).\end{aligned}\quad (4.22)$$

The r.v.s $\frac{1}{\lambda}(1 - \Delta_6) - \theta_i$ satisfy the conditions of Lemma 3.2, implying that for some positive constants c_4 and c_5 , possibly depending on Δ_6 , the following inequality holds:

$$\widehat{\Psi}_2(x) \leq c_4 e^{-c_5 \sqrt{x}}.$$

Hence, according to the conditions of the lemma,

$$\limsup_{x \rightarrow \infty} \sup_{t \geq T} \frac{\widehat{\Psi}_2(x)}{\lambda(t)\bar{B}(x)} = \frac{1}{\lambda(T)} \limsup_{x \rightarrow \infty} \frac{\widehat{\Psi}_2(x)}{\bar{B}(x)} = 0. \quad (4.23)$$

Since $\beta^* = E(Z_1 - \frac{c}{\lambda}(1 - \Delta_6)) = -\mu + (\mu + \beta)\Delta_6 < 0$ for $\Delta_6 \in (0, \mu/(\mu + \beta))$, Lemma 9 of [15] implies that

$$\begin{aligned}\widehat{\Psi}_1(x, t) &= \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq k \leq n} \sum_{i=1}^k \left(Z_i - \frac{c}{\lambda}(1 - \Delta_6)\right) > x - \sqrt{x}\right) \mathbb{P}(\Theta(t) = n) \\ &\leq \frac{1 + o(1)}{|\beta^*|} \sum_{n=1}^{\infty} \int_{x - \sqrt{x}}^{x - \sqrt{x} + n|\beta^*|} \bar{B}\left(u + \frac{c}{\lambda}(1 - \Delta_6)\right) du \mathbb{P}(\Theta(t) = n) \\ &\leq (1 + o(1)) \bar{B}\left(x - \sqrt{x} + \frac{c}{\lambda}(1 - \Delta_6)\right) \sum_{n=1}^{\infty} n \mathbb{P}(\Theta(t) = n) \\ &\leq (1 + o(1)) \bar{B}(x - \sqrt{x}) \lambda(t).\end{aligned}$$

Hence, uniformly for $t \geq T$,

$$\widehat{\Psi}_1(x, t) \lesssim \lambda(t) \bar{B}(x). \quad (4.24)$$

Relations (4.22), (4.23) and (4.24) imply the estimate of the lemma. \square

Lemma 4.4. Let Assumptions H_1 , H_2 and H_3 be satisfied and $B \in \mathcal{L}$. Then for any $T_6 < T_7 < \infty$, such that $\lambda(T_6) > 0$, it holds uniformly for $t \in [T_6, T_7]$ that

$$\Psi(x, t) \gtrsim \lambda(t) \bar{B}(x).$$

Proof. For any $t \in [T_6, T_7]$ and any $\Delta_7 > 0$ it holds

$$\begin{aligned}\Psi(x, t) &\geq \mathbb{P}\left(\sum_{i=1}^{\Theta(t)} Z_i > x + ct\right) \\ &\geq \mathbb{P}\left(\max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^k (Z_i - (1 + \Delta_7)EZ_i) > x + cT_7\right).\end{aligned}$$

Here $E(Z_i - (1 + \Delta_7)EZ_i) < 0$, so that the result of the lemma is an easy consequence of Lemma 3.2 in [24]. \square

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