



On the norm convergence of a piecewise linear least squares method for Frobenius–Perron operators

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ABSTRACT

In this paper we prove both the L^1 -norm and the BV -norm convergence for a piecewise linear least squares approximation method that was recently proposed for computing stationary density functions of Frobenius–Perron operators associated with piecewise C^2 and stretching mappings of the unit interval. We also give a convergence rate analysis under the BV -norm.

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1. Introduction

In various applied fields of science and engineering, such as molecular dynamics and wireless communications, one has to explore the statistical property of chaotic transformations. This problem is often reduced to the existence and computation of a stationary density of the Frobenius–Perron operator associated with the transformation [6]. The first numerical method proposed for the computation of stationary densities of Frobenius–Perron operators is Ulam's piecewise constant approximation method [11]; see [1,8–10] and the references therein.

Because of slow convergence of Ulam's method, several higher order numerical methods have been proposed, such as piecewise linear Markov finite approximations (see [6]). Their convergence rate is at least linear in terms of the BV -norm which is stronger than the usual L^1 -norm, as compared to the $O(\ln n/n)$ rate for Ulam's method in L^1 -norm.

Recently the authors have proposed a piecewise linear least squares method that employs continuous piecewise linear functions for calculating stationary densities of Frobenius–Perron operators. Numerical experiments [5] of the new method show a faster convergence under the L^1 -norm than the previous ones, but only a weak convergence has been proved. The purpose of the current paper is to give the theoretical proof of the L^1 -norm convergence and the stronger BV -norm convergence along with establishing the convergence rate under the BV -norm for a class of one-dimensional mappings, based on arguments from linear algebra.

Let $S : [0, 1] \rightarrow [0, 1]$ be a nonsingular transformation in the sense that S is Lebesgue measurable and $m(S^{-1}(A)) = 0$ whenever $m(A) = 0$, where m is the Lebesgue measure. For $1 \leq p < \infty$ let $L^p(0, 1)$ be the Banach space of all functions f defined on $[0, 1]$ such that the function $|f|^p$ is Lebesgue integrable on $[0, 1]$ with the L^p -norm $\|f\|_p = (\int_0^1 |f|^p dm)^{1/p}$. In particular $L^2(0, 1)$ is a Hilbert space with the inner product $(f, g) = \int_0^1 f(x)g(x) dx$. We also write $(f, g) = \int_0^1 f(x)g(x) dx$ when $f \in L^1(0, 1)$ and $g \in L^\infty(0, 1)$.

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The Frobenius–Perron operator $P_S : L^1(0, 1) \rightarrow L^1(0, 1)$ associated with S is defined by

$$P_S f(x) = \frac{d}{dx} \int_{S^{-1}([0, x])} f(t) dt, \quad \forall x \in [0, 1],$$

for all $f \in L^1(0, 1)$. It is well known [6] that $\|P_S\|_1 = 1$ and $P_S f^* = f^*$ for a density function $f^* \in L^1(0, 1)$ if and only if the absolutely continuous probability measure μ , whose Radon–Nikodym derivative with respect to m is f^* , is S -invariant, that is, $\mu(S^{-1}(A)) = \mu(A)$ for all measurable $A \subset [0, 1]$. This f^* is called a *stationary density* of P_S .

We assume that $S : [0, 1] \rightarrow [0, 1]$ is piecewise C^2 and stretching. From the proof of the Lasota–Yorke theorem (see [6]), there are two constants $\alpha = 2/\inf|S'|$ and $\beta > 0$ such that for all functions f of bounded variation,

$$\bigvee_0^1 P_S f \leq \alpha \bigvee_0^1 f + \beta \|f\|_1. \quad (1)$$

Let

$$BV(0, 1) = \left\{ f \in L^1(0, 1) : \bigvee_0^1 f < \infty \right\}$$

with the BV -norm

$$\|f\|_{BV} = \|f\|_1 + \bigvee_0^1 f, \quad \forall f \in BV(0, 1).$$

Then $BV(0, 1)$ is a Banach space, and any closed bounded subset of $BV(0, 1)$ is strongly compact in $L^1(0, 1)$. Based on this compactness argument, it was proved that for the above class of mappings, Ulam's method and the higher order approximation methods are convergent [6].

Our results presented in the paper can be summarized as follows. First we show that the sequence of the approximating operators from the least squares method is uniformly bounded in the L^1 -norm and we also establish some "local error" estimates for different classes of functions. Then we prove the uniform boundedness of the variation sequence of the operators. Based on such results we prove the convergence of our method under both the L^1 -norm and the BV -norm, and finally we give BV -norm error estimates.

In the next section we briefly introduce the least square method. Sections 3 and 4 will be devoted to the consistency and stability studies, respectively. The proof of the L^1 -norm and BV -norm convergence will be given in Section 5. A convergence rate analysis under the BV -norm will be presented in Section 6. Section 7 contains numerical results, and we conclude in Section 8.

2. Piecewise linear least squares method for Frobenius–Perron operators

We first introduce the piecewise linear least squares method. Divide the interval $[0, 1]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$ with the length $h = 1/n$. Denote by Δ_n the corresponding vector space of continuous piecewise linear functions. The dimension of Δ_n is clearly $n + 1$. A canonical basis for Δ_n consists of the hat-shaped functions

$$\phi_i(x) = \phi\left(\frac{x - x_i}{h}\right), \quad i = 0, 1, \dots, n,$$

where

$$\phi(x) = \begin{cases} 1 + x, & x \in [-1, 0], \\ 1 - x, & x \in [0, 1], \\ 0, & x \notin [-1, 1] \end{cases}$$

is the standard hat function. The support of ϕ_i , which is the set of all x values such that $\phi_i(x) > 0$, is denoted by $\text{supp } \phi_i$.

Now we define the corresponding piecewise linear projection operator $Q_n : L^1(0, 1) \rightarrow L^1(0, 1)$ by letting $Q_n f \in \Delta_n$ and

$$(Q_n f, \phi_i) = (f, \phi_i), \quad i = 0, 1, \dots, n. \quad (2)$$

If we write

$$Q_n f = \sum_{j=0}^n c_j \phi_j, \quad (3)$$

then from the above equations, the coefficients c_0, c_1, \dots, c_n can be determined uniquely by solving the linear equations

$$\sum_{j=0}^n c_j (\phi_j, \phi_i) = (f, \phi_i), \quad i = 0, 1, \dots, n. \quad (4)$$

It is well known that for any $f \in L^2(0, 1)$, the unique solution $Q_n f$ of the above system is the least squares solution as follows:

$$\begin{aligned} \|f - Q_n f\|_2 &= \min\{\|f - g\|_2 : g \in \Delta_n\} \\ &= \min\left\{\left\|f - \sum_{i=0}^n c_i \phi_i\right\|_2 : (c_0, c_1, \dots, c_n)^T \in R^{n+1}\right\}. \end{aligned} \quad (5)$$

Thus, the projection method is called the *least squares approximation*.

Our piecewise linear least squares method for numerically computing Frobenius–Perron operator's fixed point equation

$$P_S f^* = f^*$$

is to solve the following *discretized* operator equation

$$P_n f_n = f_n, \quad f_n \in \Delta_n, \quad (6)$$

where $P_n \equiv Q_n P_S : L^1(0, 1) \rightarrow \Delta_n$.

The finite-dimensional operator equation (6) is equivalent to the following $n + 1$ scalar equations

$$(Q_n P_S f_n, \phi_i) = (f_n, \phi_i), \quad i = 0, 1, \dots, n$$

for $f_n \in \Delta_n$. Since $(Q_n P_S f_n, \phi_i) = (P_S f_n, \phi_i)$, the above equations become

$$(P_S f_n, \phi_i) = (f_n, \phi_i), \quad i = 0, 1, \dots, n. \quad (7)$$

If we let

$$f_n(x) = \sum_{j=0}^n v_j \phi_j(x),$$

then (7) can be written as

$$\sum_{j=0}^n v_j (P_S \phi_j, \phi_i) = \sum_{j=0}^n v_j (\phi_j, \phi_i), \quad i = 0, 1, \dots, n. \quad (8)$$

Define two $(n + 1) \times (n + 1)$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ by

$$a_{ij} = (P_S \phi_j, \phi_i), \quad b_{ij} = (\phi_j, \phi_i), \quad i, j = 0, 1, \dots, n.$$

Then the system (8) of linear algebraic equations has the matrix form

$$(A - B)v = 0, \quad v = (v_0, v_1, \dots, v_n)^T \in R^{n+1}. \quad (9)$$

The (i, j) -entry a_{ij} of the matrix A can be calculated by the formula [5]

$$a_{ij} = \int_{\text{supp } \phi_j} \phi_i(S(x)) \phi_j(x) dx, \quad \forall i, j = 1, \dots, n.$$

The $(n + 1) \times (n + 1)$ matrix $B = (b_{ij})$ is a symmetric tridiagonal one. A direct computation gives that

$$b_{00} = b_{nn} = \frac{h}{3}, \quad b_{i,i+1} = b_{i+1,i} = \frac{h}{6}, \quad b_{ii} = \frac{2h}{3}, \quad i = 1, 2, \dots, n - 1.$$

It was shown in [5] that Eq. (9) has a nontrivial solution for any positive integer n . Hence, if we normalize a nonzero solution $(v_0^*, v_1^*, \dots, v_n^*)^T$ of (9) so that the resulting function

$$f_n(x) = \sum_{j=0}^n v_j^* \phi_j(x)$$

satisfies $\|f_n\|_1 = 1$, then $f_n \in \Delta_n$ satisfies Eq. (6). Since we want f_n to be a piecewise linear least squares approximation to a stationary density f^* of the Frobenius–Perron operator P_S , we also require f_n to satisfy

$$\int_0^1 f_n^+(x) dx \geq \int_0^1 f_n^-(x) dx. \quad (10)$$

Note that it is always possible to fulfill (10) because if f_n does not fulfill (10) we simply replace f_n by $-f_n$.

To prove the norm convergence of our method, we need to establish two facts about the sequence $\{Q_n\}$ of finite-dimensional operators. The first is the *consistency*, which means that $\lim_{n \rightarrow \infty} \|Q_n f - f\|_1 = 0$ for all $f \in L^1(0, 1)$. The second fact about $\{Q_n\}$ is the *stability*, which claims that there is a constant γ such that the variation of $Q_n f$ is uniformly bounded by γ times the variation of f for all $f \in BV(0, 1)$.

3. The consistency of the method

In this section we investigate the consistency problem of $\{Q_n\}$. The system (4) of linear equations can be written as

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = 6 \begin{bmatrix} \int_0^1 (1-x)f(hx) dx \\ \int_0^1 (1-x)[f(x_1-hx) + f(x_1+hx)] dx \\ \vdots \\ \int_0^1 (1-x)[f(x_{n-1}-hx) + f(x_{n-1}+hx)] dx \\ \int_0^1 (1-x)f(1-hx) dx \end{bmatrix}, \quad (11)$$

so its solution is

$$c = (c_0, c_1, \dots, c_n)^T = C^{-1}b,$$

where C and $b = (b_0, b_1, \dots, b_n)^T$ are the $(n+1) \times (n+1)$ coefficient matrix and the right hand side of (11), respectively.

Using the diagonal dominance of the matrix C , we can estimate the L^1 -norm of its inverse C^{-1} quite easily.

Lemma 3.1. *The matrix C is nonsingular and $\|C^{-1}\|_1 \leq 1$ for all n .*

Proof. Let $y = Cx$. Then

$$\begin{aligned} \|y\|_1 &= \sum_{i=0}^n |y_i| = |2x_0 + x_1| + \sum_{i=1}^{n-1} |x_{i-1} + 4x_i + x_{i+1}| + |x_{n-1} + 2x_n| \\ &\geq 2|x_0| - |x_1| + \sum_{i=1}^{n-1} (4|x_i| - |x_{i-1}| - |x_{i+1}|) + 2|x_n| - |x_{n-1}| \\ &= 2 \sum_{i=1}^{n-1} |x_i| + |x_0| + |x_n| \geq \sum_{i=0}^n |x_i| = \|x\|_1. \end{aligned}$$

Thus, C^{-1} exists and $\|C^{-1}y\|_1 = \|x\|_1 \leq \|y\|_1$, so $\|C^{-1}\|_1 \leq 1$. \square

The following lemma gives the consistency of $\{Q_n\}$.

Lemma 3.2. *The sequence $\{Q_n\}$ satisfies the following properties:*

- (i) $\|Q_n\|_1 \leq 6$ uniformly.
- (ii) If $f \in BV(0, 1)$, then

$$\|Q_n f - f\|_1 \leq \frac{1}{\sqrt{n}} \bigvee_0^1 f. \quad (12)$$

- (iii) If $f \in C^2[0, 1]$, then

$$\|Q_n f - f\|_1 \leq \frac{1}{8n^2} \|f''\|_\infty. \quad (13)$$

- (iv) For any $f \in L^1(0, 1)$, under the L^1 -norm,

$$\lim_{n \rightarrow \infty} Q_n f = f. \quad (14)$$

Proof. (i) Let $\hat{b} = ((f, \phi_0), (f, \phi_1), \dots, (f, \phi_n))^T$ for any $f \in L^1(0, 1)$. Then $b = 6n\hat{b}$. If $f \geq 0$, then $\hat{b} \geq 0$ and

$$\|\hat{b}\|_1 = \sum_{i=0}^n (f, \phi_i) = \left(f, \sum_{i=0}^n \phi_i \right) = (f, 1) = \|f\|_1.$$

It follows from (3) and Lemma 3.1 that

$$\begin{aligned} \|Q_n f\|_1 &= \left\| \sum_{i=0}^n c_i \phi_i \right\|_1 \leq \sum_{i=0}^n |c_i| \|\phi_i\|_1 \leq \frac{1}{n} \sum_{i=0}^n |c_i| \\ &= \frac{1}{n} \|c\|_1 \leq \frac{1}{n} \|C^{-1}\|_1 \|b\|_1 = 6 \|\hat{b}\|_1 \leq 6 \|f\|_1. \end{aligned}$$

Writing $f = f^+ - f^-$ for arbitrary $f \in L^1(0, 1)$, we obtain

$$\begin{aligned}\|Q_n f\|_1 &= \|Q_n f^+ - Q_n f^-\|_1 \leq \|Q_n f^+\|_1 + \|Q_n f^-\|_1 \\ &\leq 6\|f^+\|_1 + 6\|f^-\|_1 = 6\|f\|_1.\end{aligned}$$

Therefore $\|Q_n\|_1 \leq 6$ uniformly.

(ii) Now suppose $f \in BV(0, 1)$. The Cauchy-Schwarz inequality implies $\|Q_n f - f\|_1 \leq \|Q_n f - f\|_2$. So, since $\phi_{i-1}(x) + \phi_i(x) \equiv 1$ on $[x_{i-1}, x_i]$, from (5),

$$\begin{aligned}\|Q_n f - f\|_1^2 &\leq \|Q_n f - f\|_2^2 \leq \|L_n f - f\|_2^2 = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (L_n f(x) - f(x))^2 dx \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} [(f(x_{i-1}) - f(x))\phi_{i-1}(x) + (f(x_i) - f(x))\phi_i(x)]^2 dx \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \left(\bigvee_{x_{i-1}}^{x_i} f\right)^2 dx = \frac{1}{n} \sum_{i=1}^n \left(\bigvee_{x_{i-1}}^{x_i} f\right)^2 = \frac{1}{n} \left(\bigvee_0^1 f\right)^2,\end{aligned}$$

where L_n is the Lagrange interpolation operator defined by

$$L_n f(x) = \sum_{i=0}^n f(x_i) \phi_i(x).$$

(iii) Let $f \in C^2[0, 1]$. Then

$$\begin{aligned}\|Q_n f - f\|_1 &\leq \|Q_n f - f\|_2 \leq \|L_n f - f\|_2 \\ &\leq \|L_n f - f\|_\infty \leq \frac{1}{8n^2} \|f''\|_\infty,\end{aligned}$$

where the last inequality is from the standard result in interpolation theory. This completes the proof.

(iv) The strong convergence of $\{Q_n\}$ on $L^1(0, 1)$ follows from (iii) because of the uniform boundedness of $\{\|Q_n\|_1\}$ by (i). \square

4. The stability of the method

Now we investigate the stability problem of $\{Q_n\}$ for the least squares method. For this purpose, we need to estimate $\bigvee_0^1 Q_n f$ in terms of $\bigvee_0^1 f$ for any $f \in BV(0, 1)$. Since the piecewise linear function $Q_n f$ is determined by its values c_i at the node points x_i , $i = 0, 1, \dots, n$, from the definition of the variation, we see immediately that

$$\bigvee_0^1 Q_n f = \sum_{i=1}^n |c_i - c_{i-1}|.$$

Suitable row operations to the system (11) of equations give rise to the following system for $c_1 - c_0, c_2 - c_1, \dots, c_n - c_{n-1}$:

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 - c_0 \\ c_2 - c_1 \\ \vdots \\ c_{n-1} - c_{n-2} \\ c_n - c_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 - 2b_0 \\ b_2 - b_1 \\ \vdots \\ b_{n-1} - b_{n-2} \\ 2b_n - b_{n-1} \end{bmatrix}. \quad (15)$$

Let E be the $n \times n$ coefficient matrix of above system.

Lemma 4.1. E^{-1} exists and $\|E^{-1}\|_1 \leq 1/2$ for all n .

Proof. The proof is similar to that of Lemma 3.1. Let $y = Ex$. Then

$$\begin{aligned}\|y\|_1 &\geq 3|x_1| - |x_2| + \sum_{i=2}^{n-1} (4|x_i| - |x_{i-1}| - |x_{i+1}|) + 3|x_n| - |x_{n-1}| \\ &= 2 \sum_{i=2}^{n-1} |x_i| + 2|x_1| + 2|x_n| = 2 \sum_{i=1}^n |x_i| = 2\|x\|_1.\end{aligned}$$

So $\|E^{-1}y\|_1 = \|x\|_1 \leq \|y\|_1/2$, which implies $\|E^{-1}\|_1 \leq 1/2$. \square

Lemma 4.2. Let $f \in BV(0, 1)$. Then

$$\bigvee_0^1 Q_n f \leq 6 \bigvee_0^1 f, \quad \forall n.$$

Proof. Let $z, w \in R^n$ be defined by $z_i = c_i - c_{i-1}$ for $i = 1, 2, \dots, n$ and $w_i = b_i - b_{i-1}$ for $i = 2, 3, \dots, n-1$, and $w_1 = b_1 - 2b_0$ and $w_n = 2b_n - b_{n-1}$ in (15). Then, by Lemma 4.1,

$$\bigvee_0^1 Q_n f = \sum_{i=1}^n |c_i - c_{i-1}| = \|z\|_1 \leq \|E^{-1}\|_1 \|w\|_1 = \frac{1}{2} \|w\|_1.$$

Now, from the expressions of b_i , we have

$$\begin{aligned}\frac{1}{6} \|w\|_1 &= \frac{1}{6} \left(|b_1 - 2b_0| + \sum_{i=2}^{n-1} |b_i - b_{i-1}| + |2b_n - b_{n-1}| \right) \\ &= \left| \int_0^1 (1-x) [f(x_1 - hx) + f(x_1 + hx) - 2f(hx)] dx \right| \\ &\quad + \sum_{i=2}^{n-1} \left| \int_0^1 (1-x) [f(x_i - hx) - f(x_{i-1} - hx) + f(x_i + hx) - f(x_{i-1} + hx)] dx \right| \\ &\quad + \left| \int_0^1 (1-x) \{2f(1-hx) dx - [f(x_{n-1} - hx) + f(x_{n-1} + hx)]\} dx \right| \\ &\leq \int_0^1 (1-x) |f(x_1 - hx) + f(x_1 + hx) - 2f(hx)| dx \\ &\quad + \sum_{i=2}^{n-1} \int_0^1 (1-x) |f(x_i - hx) - f(x_{i-1} - hx) + f(x_i + hx) - f(x_{i-1} + hx)| dx \\ &\quad + \int_0^1 (1-x) |2f(1-hx) dx - [f(x_{n-1} - hx) + f(x_{n-1} + hx)]| dx \\ &\leq \int_0^1 (1-x) \left(\bigvee_0^{x_1} f + \bigvee_0^{x_2} f \right) dx + \int_0^1 (1-x) \left(\bigvee_{x_{n-2}}^1 f + \bigvee_{x_{n-1}}^1 f \right) dx + \sum_{i=2}^{n-1} \int_0^1 (1-x) \left(\bigvee_{x_{i-2}}^{x_i} f + \bigvee_{x_{i-1}}^{x_{i+1}} f \right) dx \\ &= \left[\bigvee_0^{x_1} f + \bigvee_0^{x_2} f + \sum_{i=2}^{n-1} \left(\bigvee_{x_{i-2}}^{x_i} f + \bigvee_{x_{i-1}}^{x_{i+1}} f \right) + \bigvee_{x_{n-2}}^1 f + \bigvee_{x_{n-1}}^1 f \right] \int_0^1 (1-x) dx \leq 2 \bigvee_0^1 f.\end{aligned}$$

Hence,

$$\bigvee_0^1 Q_n f \leq \frac{1}{2} \|w\|_1 \leq 6 \bigvee_0^1 f. \quad \square$$

5. The norm convergence of the method

The preliminary results in the previous two sections and our basic assumption for the Frobenius–Perron operator lead to the following convergence result. Throughout this section, we assume that the Frobenius–Perron operator P_S has a unique stationary density f^* .

Theorem 5.1. *Let $S : [0, 1] \rightarrow [0, 1]$ be such that its corresponding Frobenius–Perron operator P_S satisfies the inequality (1) with $\alpha < 1/6$. Then for any sequence $\{f_n\}$ of the least squares approximations of f^* with $\|f_n\|_1 \equiv 1$,*

$$\lim_{n \rightarrow \infty} f_n = f^*$$

under the L^1 -norm.

Proof. Since $f_n = P_n f_n = Q_n P_S f_n$, from Lemma 4.2 and (1),

$$\begin{aligned} \bigvee_0^1 f_n &= \bigvee_0^1 Q_n P_S f_n \leq 6 \bigvee_0^1 P_S f_n \\ &\leq 6 \left(\alpha \bigvee_0^1 f_n + \beta \|f_n\|_1 \right) = 6\alpha \bigvee_0^1 f_n + 6\beta. \end{aligned}$$

It follows from the assumption $6\alpha < 1$ that

$$\bigvee_0^1 f_n \leq \frac{6\beta}{1-6\alpha}, \quad \forall n,$$

hence there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\|f_{n_k} - g\|_1 \rightarrow 0$ as $k \rightarrow \infty$ for some $g \in L^1(0, 1)$ with $\|g\|_1 = 1$. From Lemma 3.2(i), since $\|P_S\|_1 = 1$, $\|P_n\|_1 \leq \|Q_n\|_1 \|P_S\|_1 \leq 6$ for all n . Thus,

$$\begin{aligned} \|g - P_S g\|_1 &\leq \|g - f_{n_k}\|_1 + \|f_{n_k} - P_{n_k} f_{n_k}\|_1 + \|P_{n_k} f_{n_k} - P_{n_k} g\|_1 + \|P_{n_k} g - P_S g\|_1 \\ &\leq \|g - f_{n_k}\|_1 + 6\|f_{n_k} - g\|_1 + \|P_{n_k} g - P_S g\|_1 \rightarrow 0 \end{aligned} \quad (16)$$

as $k \rightarrow \infty$. That is, $P_S g = g$. Since P_S is a Markov operator, the positive and negative parts of g are both fixed points of P_S [6, Proposition 4.1.3], so it follows from the uniqueness of the stationary density f^* of P_S that $g \geq 0$ or $g \leq 0$ and hence $g = f^*$ or $g = -f^*$. Since $\|f_{n_k} - g\|_1 \rightarrow 0$ as $k \rightarrow \infty$, in view of (10) we see that $g = f^*$. Therefore the theorem is proved since every convergent subsequence of $\{f_n\}$ converges to f^* . \square

Now we further strengthen the above convergence to the BV-norm. For this purpose, we need the two following lemmas.

Lemma 5.1. *Suppose that $f \in C^2[0, 1]$. Let*

$$\eta^T = (\eta_0, \eta_1, \dots, \eta_n) = (f(x_0), f(x_1), \dots, f(x_n)).$$

Then

$$\|b - C\eta\|_1 = O\left(\frac{1}{n}\right).$$

Proof. We first show that

$$6 \int_0^1 (1-x) f(hx) dx = 2f(0) + f(h) + O(h^2).$$

Let

$$g(x) = (1-x)f(hx).$$

Then

$$\begin{aligned}\int_0^1 (1-x)f(hx)dx &= \int_0^1 g(x)dx = \int_0^1 \left[g(0) + g'(0)x + \frac{g''(\tilde{c})}{2}x^2 \right] dx \\ &= g(0) + \frac{g'(0)}{2} + \frac{g''(\tilde{c})}{2} \int_0^1 x^2 dx = g(0) + \frac{g'(0)}{2} + \frac{g''(\tilde{c})}{6}.\end{aligned}$$

A simple computation shows that $g(0) = f(0)$, $g'(0) = -f(0) + hf'(0)$, $g''(\tilde{c}) = -2hf'(h\tilde{c}) + h^2(1-\tilde{c})f''(h\tilde{c})$, and $-2hf'(h\tilde{c}) = -2h[f'(0) + f''(\tilde{c})h\tilde{c}] = -2hf'(0) - 2\tilde{c}f''(\tilde{c})h^2$. So we have

$$6 \int_0^1 (1-x)f(hx)dx = 3f(0) + f'(0)h + O(h^2) = 2f(0) + f(h) + O(h^2).$$

Similarly, one can show that

$$6 \int_0^1 (1-x)f(1-hx)dx = f(1-h) + 2f(1) + O(h^2).$$

Now we show that for $1 \leq i \leq n-1$,

$$6 \int_0^1 (1-x)[f(x_i - hx) + f(x_i + hx)]dx = f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) + O(h^2).$$

Let

$$g(x) = (1-x)[f(x_i - hx) + f(x_i + hx)].$$

Then as before

$$\int_0^1 (1-x)[f(x_i - hx) + f(x_i + hx)]dx = g(0) + \frac{g'(0)}{2} + \frac{g''(\tilde{c})}{6}.$$

But we have $g(0) = 2f(x_i)$, $g'(0) = -2f(x_i)$, and

$$\begin{aligned}g''(\tilde{c}) &= 2h[f'(x_i - h\tilde{c}) - f'(x_i + h\tilde{c})] + h^2(1-\tilde{c})[f''(x_i - h\tilde{c}) + f''(x_i + h\tilde{c})] \\ &= 2h\{f'(x_i) - h\tilde{c}f''(c_1) - [f'(x_i) + h\tilde{c}f''(c_2)]\} + h^2(1-\tilde{c})[f''(x_i - h\tilde{c}) + f''(x_i + h\tilde{c})] = O(h^2).\end{aligned}$$

Therefore,

$$6 \int_0^1 (1-x)[f(x_i - hx) + f(x_i + hx)]dx = 6f(x_i) + O(h^2) = f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) + O(h^2).$$

Summing up the above proves the lemma. \square

Lemma 5.2. Suppose that $f \in C^2[0, 1]$. Then

$$\bigvee_0^1 (Q_n f - f) = O\left(\frac{1}{n}\right).$$

Proof. Let $L_n f(x) = \sum_{i=0}^n f(x_i)\phi_i(x)$ be the Lagrange interpolation of f . Then it is well known (see [3]) that $\bigvee_0^1 (L_n f - f) = O(1/n)$. Hence, it is enough to show that

$$\bigvee_0^1 (Q_n f - L_n f) = O\left(\frac{1}{n}\right).$$

Since $\bigvee_0^1 \phi_i \leq 2$ for each i , using Lemma 5.1 we have

$$\begin{aligned}
\bigvee_0^1 (Q_n f - L_n f) &= \bigvee_0^1 \sum_{i=0}^n [c_i - f(x_i)] \phi_i \leq \sum_{i=0}^n |c_i - f(x_i)| \bigvee_0^1 \phi_i \\
&\leq 2 \sum_{i=0}^n |c_i - f(x_i)| = 2 \|C^{-1}b - \eta\|_1 \\
&\leq 2 \|C^{-1}\|_1 \|b - C\eta\|_1 \leq 3 \|b - C\eta\|_1 = O\left(\frac{1}{n}\right).
\end{aligned}$$

This proves the lemma. \square

Theorem 5.2. Under the condition of Theorem 5.1, if $f^* \in C^2[0, 1]$, then

$$\lim_{n \rightarrow \infty} \|f_n - f^*\|_{BV} = 0.$$

Proof. Using the decomposition

$$f_n - f^* = P_n(f_n - f^*) + Q_n f^* - f^*, \quad (17)$$

Lemma 4.2, inequality (1), and the fact $\|P_n\|_1 \leq 6$, we obtain

$$\begin{aligned}
\|f_n - f^*\|_{BV} &\leq \|P_n(f_n - f^*)\|_{BV} + \|Q_n f^* - f^*\|_{BV} \\
&= \bigvee_0^1 Q_n P_S(f_n - f^*) + \|P_n(f_n - f^*)\|_1 + \|Q_n f^* - f^*\|_{BV} \\
&\leq 6 \bigvee_0^1 P_S(f_n - f^*) + 6 \|(f_n - f^*)\|_1 + \|Q_n f^* - f^*\|_{BV} \\
&\leq 6\alpha \bigvee_0^1 (f_n - f^*) + 6(\beta + 1) \|f_n - f^*\|_1 + \|Q_n f^* - f^*\|_{BV} \\
&\leq 6\alpha \|f_n - f^*\|_{BV} + 6(\beta + 1) \|f_n - f^*\|_1 + \|Q_n f^* - f^*\|_{BV}.
\end{aligned}$$

Since $6\alpha < 1$ by assumption,

$$\|f_n - f^*\|_{BV} \leq \frac{1}{1 - 6\alpha} [6(\beta + 1) \|f_n - f^*\|_1 + \|Q_n f^* - f^*\|_{BV}] \rightarrow 0$$

as $n \rightarrow \infty$ because of Theorem 5.1, Lemma 3.2(iii), and Lemma 5.2. \square

6. Convergence rate analysis

We further provide a convergence rate analysis for the BV -norm convergence of the least squares method, based on the idea and technique developed in [2] and [3]. First, we note that the Lasota–Yorke inequality (1) ensures that $P_S : BV(0, 1) \rightarrow BV(0, 1)$ is quasi-compact [7], so its eigenvalue 1 is isolated. We list some useful lemmas, whose proofs are basically the same as in [3]. Lemmas 6.1 and 6.2 are needed in the proof of Theorem 6.1 and Lemma 6.3, respectively.

Lemma 6.1. $\dim N(P_n - I) = 1$ for n large enough. Furthermore, the generalized eigenspace of P_n is also one-dimensional.

Lemma 6.2. If λ is not equal to 1 and f belongs to either $N(P_n - \lambda I)$ or $N(P_S - \lambda I)$, then $\int_0^1 f(x) dx = 0$.

Lemma 6.3. Let Γ be the set of eigenvalues $\lambda \neq 1$ of P_n for all n and P_S (here P_S is viewed as a mapping from $BV(0, 1)$ into itself). Then 1 is not a limit point of Γ . Moreover, any nonzero spectral point of P_n is an eigenvalue of P_n .

Let

$$\delta = \frac{1}{3} \min\{d(1, \Gamma), 1 - 6\alpha\} > 0$$

and denote

$$\Omega = \left\{ z : \frac{\delta}{2} \leq |z - 1| \leq \delta \right\}$$

the ring region of the complex plane with radii $\delta/2$ and δ . Then $P_n - zI$ is one-to-one from $L^1(0, 1)$ onto itself for all n and $z \in \Omega$.

Lemma 6.4. *The operators $(P_n - zI)^{-1} : BV(0, 1) \rightarrow L^1(0, 1)$ are uniformly bounded with respect to n and $z \in \Omega$, that is, there is a constant C such that*

$$\|(P_n - zI)^{-1}f\|_1 \leq C\|f\|_{BV}, \quad \forall z \in \Omega, f \in BV(0, 1), n = 1, 2, \dots$$

Proof. Suppose the conclusion is not true. Then there is a sequence of functions $f_n \in BV(0, 1)$ and a sequence $\{z_n\}$ in Ω such that

$$\lim_{n \rightarrow \infty} \frac{\|(P_n - z_n I)^{-1}f_n\|_1}{\|f_n\|_{BV}} = \infty.$$

Denote $\hat{g}_n = (P_n - z_n I)^{-1}f_n / \|(P_n - z_n I)^{-1}f_n\|_1$ and $\hat{f}_n = f_n / \|(P_n - z_n I)^{-1}f_n\|_1$. Then $\hat{f}_n = (P_n - z_n I)\hat{g}_n$, $\|\hat{g}_n\|_1 \equiv 1$, and

$$\lim_{n \rightarrow \infty} \|\hat{f}_n\|_{BV} = \lim_{n \rightarrow \infty} \frac{\|f_n\|_{BV}}{\|(P_n - z_n I)^{-1}f_n\|_1} = 0.$$

Without loss of generality, we assume $z_n \rightarrow z \in \Omega$, so by (1) and Lemma 4.2,

$$\begin{aligned} \bigvee_0^1 \hat{g}_n &= \bigvee_0^1 \frac{1}{z_n} (P_n \hat{g}_n - \hat{f}_n) \leq \frac{1}{|z_n|} \left(\bigvee_0^1 P_n \hat{g}_n + \bigvee_0^1 \hat{f}_n \right) \\ &\leq \frac{1}{1-\delta} \left(6\alpha \bigvee_0^1 \hat{g}_n + 6\beta + \bigvee_0^1 \hat{f}_n \right). \end{aligned}$$

It follows that

$$\bigvee_0^1 \hat{g}_n \leq \frac{1}{1-6\alpha-\delta} \left(6\beta + \bigvee_0^1 \hat{f}_n \right).$$

Since $\bigvee_0^1 \hat{f}_n \leq \|\hat{f}_n\|_{BV} \rightarrow 0$, by Helly's lemma, there is a subsequence $\{\hat{g}_{n_k}\}$ of $\{\hat{g}_n\}$ which converges to some $g \in L^1(0, 1)$ with $\|g\|_1 = 1$. The same technique as (16) in the proof of Theorem 5.1 ensures that $P_S g = zg$, which implies that $z \in \Gamma$, a contradiction to the fact $\Gamma \cap \Omega = \emptyset$. \square

Now we can prove the following error estimates result. Assume n is large enough so that conditions of the above lemmas are satisfied.

Theorem 6.1. *Under the same condition as Theorem 5.2,*

$$\|f_n - f^*\|_{BV} = O\left(\frac{1}{n}\right).$$

Proof. Lemma 3.2(iii) and Lemma 5.2 ensure that

$$\|f^* - Q_n f^*\|_{BV} = O\left(\frac{1}{n}\right).$$

Hence, by the proof of Theorem 5.2, it is enough to show that

$$\|f_n - f^*\|_1 = O(\|f^* - Q_n f^*\|_{BV}).$$

Let $\gamma \subset \Omega$ be a fixed circle centered at 1 with radius ϵ . The decomposition (17) implies

$$\frac{1}{z-1}(f^* - f_n) = (zI - P_n)^{-1}(f^* - f_n) + \frac{1}{z-1}(zI - P_n)^{-1}(f^* - Q_n f^*)$$

for all $z \in \gamma$. By Lemma 6.1, $\dim N(P_n - I) = 1$ for n large enough. Then, for these large n , as proved in [3,6],

$$\frac{1}{2\pi i} \int_{\gamma} (zI - P_n)^{-1}(f^* - f_n) dz = 0,$$

Table 1
 L^1 -norm errors comparison for S_1 .

Number of subintervals	Ulam	Markov	LSM
4	5.3×10^{-2}	1.1×10^{-2}	2.7×10^{-3}
8	2.4×10^{-2}	4.3×10^{-3}	6.5×10^{-4}
16	1.2×10^{-2}	1.6×10^{-3}	1.7×10^{-4}
32	5.5×10^{-3}	5.3×10^{-4}	4.3×10^{-5}
64	2.7×10^{-3}	1.7×10^{-4}	1.1×10^{-5}
128	1.3×10^{-3}	5.0×10^{-5}	2.7×10^{-6}
256	6.6×10^{-4}	1.5×10^{-5}	6.4×10^{-7}

from which we obtain

$$f^* - f_n = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-1} (zI - P_n)^{-1} (f^* - Q_n f^*) dz.$$

Thus, it follows from Lemma 6.4 that

$$\begin{aligned} \|f^* - f_n\|_1 &\leq \frac{1}{2\pi} \frac{2\pi\epsilon}{\epsilon} \max_{z \in \gamma} \|(zI - P_n)^{-1} (f^* - Q_n f^*)\|_1 \\ &\leq C \|f^* - Q_n f^*\|_{BV}. \end{aligned}$$

This proves the theorem. \square

7. Numerical results

Now we give several numerical examples to compare the performance of the Ulam's method (Ulam) [11], the improved piecewise linear Markov method (Markov) [4], and the new piecewise linear least squares method (LSM) proposed in this paper, when they are applied to computing stationary densities of Frobenius–Perron operators associated with piecewise monotonic interval mappings from $[0, 1]$ into itself.

The first mapping S_1 is defined by

$$S_1(x) = \begin{cases} \frac{2x}{1-x^2}, & 0 \leq x \leq \sqrt{2} - 1, \\ \frac{1-x^2}{2x}, & \sqrt{2} - 1 \leq x \leq 1, \end{cases}$$

and the stationary density of the corresponding Frobenius–Perron operator P is given by

$$f_1^*(x) = \frac{4}{\pi(1+x^2)}.$$

The second mapping S_2 is defined by

$$S_2(x) = \begin{cases} \frac{2x}{1-x}, & 0 \leq x \leq \frac{1}{3}, \\ \frac{1-x}{2x}, & \frac{1}{3} \leq x \leq 1 \end{cases}$$

with the corresponding stationary density

$$f_2^*(x) = \frac{2}{(1+x)^2}.$$

The computation was done with MATLAB. In the following tables we present the L^1 and BV -norm errors of the computed densities f_n to the exact stationary density f^* for the above mentioned three methods.

It is clear from Tables 1 through 4 that the errors in the least squares method is better than both Ulam and Markov methods. Note that Tables 2 and 4 demonstrate the linear convergence of the new piecewise linear least squares method under the BV -norm. Of course, Ulam's method does not converge in BV -norm.

8. Conclusions

We have proved the L^1 norm convergence of the piecewise linear least squares method for a class of piecewise stretching interval mappings. Based on the proved L^1 -norm convergence, the Lasota–Yorke inequality and the BV -norm local convergence, we also proved the BV -norm convergence of the method. These norm convergence results much improve the weak convergence one which was proved in [5]. Furthermore we have established the convergence rate of our method under

Table 2*BV*-norm errors comparison for S_1 .

Number of subintervals	Ulam	Markov	LSM
4	1.2	1.1×10^{-1}	7.0×10^{-2}
8	1.2	6.1×10^{-2}	3.4×10^{-2}
16	1.3	3.3×10^{-2}	1.7×10^{-2}
32	1.3	1.8×10^{-2}	8.4×10^{-3}
64	1.3	9.1×10^{-3}	4.2×10^{-3}
128	1.3	4.7×10^{-3}	2.1×10^{-3}
256	1.3	2.4×10^{-3}	1.0×10^{-3}

Table 3 L^1 -norm errors comparison for S_2 .

Number of subintervals	Ulam	Markov	LSM
4	1.0×10^{-1}	4.2×10^{-2}	7.7×10^{-3}
8	5.1×10^{-2}	1.8×10^{-2}	1.9×10^{-3}
16	2.6×10^{-2}	6.8×10^{-3}	5.4×10^{-4}
32	1.3×10^{-2}	2.3×10^{-3}	1.4×10^{-4}
64	6.6×10^{-3}	7.5×10^{-4}	3.6×10^{-5}
128	3.3×10^{-3}	2.3×10^{-4}	8.5×10^{-6}
256	1.6×10^{-3}	6.9×10^{-5}	2.2×10^{-6}

Table 4*BV*-norm errors comparison for S_2 .

Number of subintervals	Ulam	Markov	LSM
4	2.6	4.0×10^{-1}	2.3×10^{-1}
8	2.7	2.4×10^{-1}	1.1×10^{-1}
16	2.8	1.3×10^{-1}	5.6×10^{-2}
32	2.9	7.0×10^{-2}	2.8×10^{-2}
64	3.0	3.7×10^{-2}	1.4×10^{-2}
128	3.0	1.9×10^{-2}	7.0×10^{-3}
256	3.0	9.7×10^{-3}	3.5×10^{-3}

the *BV*-norm. Numerical results further suggest near order 2 for the L^1 -norm convergence rate, which needs to be further investigated theoretically.

To end the paper, we mention that the idea of this paper can be extended to constructing numerical methods based on other basis functions with local supports that form the partitions of unity with higher order smoothness than the piecewise linear spline basis chosen here.

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