



Schrödinger uncertainty relation, Wigner–Yanase–Dyson skew information and metric adjusted correlation measure

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ABSTRACT

In this paper, we give Schrödinger-type uncertainty relation using the Wigner–Yanase–Dyson skew information. In addition, we give Schrödinger-type uncertainty relation by use of a two-parameter extended correlation measure. We finally show a further generalization of Schrödinger-type uncertainty relation by use of the metric adjusted correlation measure. These results generalize our previous result in [Phys. Rev. A 82 (2010) 034101].

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1. Introduction

In quantum information theory, one of the most important results is the strong subadditivity of von Neumann entropy [22]. This important property of von Neumann entropy can be proven by the use of Lieb's theorem [16] which gave a complete solution for the conjecture of the convexity of Wigner–Yanase–Dyson skew information. In addition, the uncertainty relation has been widely studied in quantum information theory [21,31,29]. In particular, the relations between skew information and uncertainty relation have been studied in [17,4,8,9,7]. Quantum Fisher information is also called monotone metric which was introduced by Petz [23] and the Wigner–Yanase–Dyson metric is connected to quantum Fisher information (monotone metric) as a special case. Recently, Hansen gave a further development of the notion of monotone metric, so-called metric adjusted skew information [12]. The Wigner–Yanase–Dyson skew information is also connected to the metric adjusted skew information as a special case. That is, the metric adjusted skew information gave a class including the Wigner–Yanase–Dyson skew information, while the monotone metric gave a class including the Wigner–Yanase–Dyson metric. In the paper [12], the metric adjusted correlation measure was also introduced as a generalization of the quantum covariance and correlation measure defined in [17]. Therefore it is significant to give the relation between the Wigner–Yanase–Dyson skew information, metric adjusted correlation measure and uncertainty relation for the fundamental studies on quantum information theory.

We start from the Heisenberg uncertainty relation [13]:

$$V_{\rho}(A)V_{\rho}(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2 \quad (1)$$

for a quantum state (density operator) ρ and two observables (self-adjoint operators) A and B . The further stronger result was given by Schrödinger in [27,28]:

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$$V_\rho(A)V_\rho(B) - |\operatorname{Re}\{\operatorname{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4}|\operatorname{Tr}[\rho[A, B]]|^2, \quad (2)$$

where the covariance is defined by $\operatorname{Cov}_\rho(A, B) \equiv \operatorname{Tr}[\rho(A - \operatorname{Tr}[\rho A]I)(B - \operatorname{Tr}[\rho B]I)]$.

The Wigner–Yanase skew information represents a measure for non-commutativity between a quantum state ρ and an observable H . Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture [18]:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (3)$$

with the Wigner–Yanase skew information [32]:

$$I_\rho(H) \equiv \frac{1}{2} \operatorname{Tr}[(i[\rho^{1/2}, H_0])^2] = \operatorname{Tr}[\rho H_0^2] - \operatorname{Tr}[\rho^{1/2} H_0 \rho^{1/2} H_0], \quad H_0 \equiv H - \operatorname{Tr}[\rho H]I,$$

and then he successfully showed a new Heisenberg-type uncertainty relation on $U_\rho(H)$ in [18]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\operatorname{Tr}[\rho[A, B]]|^2. \quad (4)$$

As stated in [18], the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows. For a mixed state ρ , the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner–Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [19,20]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so that we can regard that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity $U_\rho(H)$.

Recently, a one-parameter extension of the inequality (4) was given in [33]:

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|\operatorname{Tr}[\rho[A, B]]|^2, \quad (5)$$

where

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2},$$

with the Wigner–Yanase–Dyson skew information $I_{\rho,\alpha}(H)$ defined by

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2} \operatorname{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \operatorname{Tr}[\rho H_0^2] - \operatorname{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0].$$

It is notable that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by Lieb in [16]. The further generalization of the Heisenberg-type uncertainty relation on $U_\rho(H)$ has been given in [34] using the generalized Wigner–Yanase–Dyson skew information introduced in [3]. See also [1,5,7,8] for the recent studies on skew informations and uncertainty relations.

Motivated by the fact that the Schrödinger uncertainty relation is a stronger result than the Heisenberg uncertainty relation, a new Schrödinger-type uncertainty relation for mixed states using Wigner–Yanase skew information was shown in [4]. That is, for a quantum state ρ and two observables A and B , we have

$$U_\rho(A)U_\rho(B) - |\operatorname{Re}\{\operatorname{Corr}_\rho(A, B)\}|^2 \geq \frac{1}{4}|\operatorname{Tr}[\rho[A, B]]|^2, \quad (6)$$

where the correlation measure [17] is defined by

$$\operatorname{Corr}_\rho(X, Y) \equiv \operatorname{Tr}[\rho X^* Y] - \operatorname{Tr}[\rho^{1/2} X^* \rho^{1/2} Y]$$

for any operators X and Y . This result refined the Heisenberg-type uncertainty relation (4) shown in [18] for mixed states (general states). We easily find that the inequality (6) is equivalent to the following inequality:

$$U_\rho(A)U_\rho(B) \geq |\operatorname{Corr}_\rho(A, B)|^2. \quad (7)$$

The main purpose of this paper is to give some extensions of the inequality (7) by using the Wigner–Yanase–Dyson skew information $I_{\rho,\alpha}(H)$ and the metric adjusted correlation measure introduced in [12].

2. Schrödinger uncertainty relation with Wigner–Yanase–Dyson skew information

In this section, we give a generalization of the Schrödinger-type uncertainty relation (7) by the use of the quantity $U_{\rho,\alpha}(H)$ defined by the Wigner–Yanase–Dyson skew information $I_{\rho,\alpha}(H)$.

Theorem 2.1. For $\alpha \in [1/2, 1]$, a quantum state ρ and two observables A and B , we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\text{Corr}_{\rho,\alpha}(A, B)|^2, \quad (8)$$

where the generalized correlation measure [14,36] is defined by

$$\text{Corr}_{\rho,\alpha}(X, Y) \equiv \text{Tr}[\rho X^* Y] - \text{Tr}[\rho^\alpha X^* \rho^{1-\alpha} Y]$$

for any operators X and Y .

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2. (See [33].) For a spectral decomposition of $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$, putting $h_{ij} \equiv \langle\phi_i|H_0|\phi_j\rangle$, we have the following relations.

(i) For the Wigner–Yanase–Dyson skew information, we have

$$I_{\rho,\alpha}(H) = \sum_{i < j} (\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha})|h_{ij}|^2.$$

(ii) For the quantity associated to the Wigner–Yanase–Dyson skew information:

$$J_{\rho,\alpha}(H) \equiv \frac{1}{2} \text{Tr}[(\{\rho^\alpha, H_0\})(\{\rho^{1-\alpha}, H_0\})] = \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

where $\{X, Y\} \equiv XY + YX$ is an anti-commutator, we have

$$J_{\rho,\alpha}(H) \geq \sum_{i < j} (\lambda_i^\alpha + \lambda_j^\alpha)(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha})|h_{ij}|^2.$$

Lemma 2.3. (See [2,33].) For any $t > 0$ and $\alpha \in [0, 1]$, we have

$$(1 - 2\alpha)^2(t - 1)^2 \geq (t^\alpha - t^{1-\alpha})^2.$$

Proof of Theorem 2.1. We take a spectral decomposition $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$. If we put $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle\phi_j|B_0|\phi_i\rangle$, where $A_0 = A - \text{Tr}[\rho A]I$ and $B_0 = B - \text{Tr}[\rho B]I$, then we have

$$\begin{aligned} \text{Corr}_{\rho,\alpha}(A, B) &= \text{Tr}[\rho AB] - \text{Tr}[\rho^\alpha A \rho^{1-\alpha} B] \\ &= \text{Tr}[\rho A_0 B_0] - \text{Tr}[\rho^\alpha A_0 \rho^{1-\alpha} B_0] \\ &= \sum_{i,j=1}^{\infty} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} \\ &= \sum_{i \neq j} (\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} \\ &= \sum_{i < j} \{(\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}) a_{ij} b_{ji} + (\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}) a_{ji} b_{ij}\}. \end{aligned} \quad (9)$$

Thus we have

$$|\text{Corr}_{\rho,\alpha}(A, B)| \leq \sum_{i < j} \{|\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}| |a_{ij}| |b_{ji}| + |\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}| |a_{ji}| |b_{ij}|\}.$$

Since $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, taking a square of both sides and then using Schwarz inequality and Lemma 2.2, we have

$$\begin{aligned} 4\alpha(1-\alpha)|\text{Corr}_{\rho,\alpha}(A, B)|^2 &\leq 4\alpha(1-\alpha) \left\{ \sum_{i < j} \{|\lambda_i - \lambda_i^\alpha \lambda_j^{1-\alpha}| + |\lambda_j - \lambda_j^\alpha \lambda_i^{1-\alpha}|\} |a_{ij}| |b_{ji}| \right\}^2 \\ &= \left\{ \sum_{i < j} 2\sqrt{\alpha(1-\alpha)} (\lambda_i^\alpha + \lambda_j^\alpha) |\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}| |a_{ij}| |b_{ji}| \right\}^2 \\ &\leq \left\{ \sum_{i < j} 2\sqrt{\alpha(1-\alpha)} |\lambda_i - \lambda_j| |a_{ij}| |b_{ji}| \right\}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \sum_{i < j} \{ (\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha})(\lambda_i^\alpha + \lambda_j^\alpha)(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha}) \}^{1/2} |a_{ij}| |b_{ji}| \right\}^2 \\
&\leq \left\{ \sum_{i < j} (\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) |a_{ij}|^2 \right\} \left\{ \sum_{i < j} (\lambda_i^\alpha + \lambda_j^\alpha)(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha}) |b_{ij}|^2 \right\} \\
&\leq I_{\rho, \alpha}(A) J_{\rho, \alpha}(B).
\end{aligned}$$

In the above process, the inequality $(x^\alpha + y^\alpha)|x^{1-\alpha} - y^{1-\alpha}| \leq |x - y|$ for $x, y \geq 0$ and $\alpha \in [\frac{1}{2}, 1]$ and the inequality $2\sqrt{\alpha(1-\alpha)}|x - y| \leq (x^\alpha - y^\alpha)(x^{1-\alpha} - y^{1-\alpha})(x^\alpha + y^\alpha)(x^{1-\alpha} + y^{1-\alpha})$ for $x, y \geq 0$ and $\alpha \in [0, 1]$, which can be proven by Lemma 2.3, were used. In a similar way, we also have

$$4\alpha(1-\alpha)|\text{Corr}_{\rho, \alpha}(A, B)|^2 \leq I_{\rho, \alpha}(B) J_{\rho, \alpha}(A).$$

Thus for $\alpha \geq \frac{1}{2}$ we have

$$4\alpha(1-\alpha)|\text{Corr}_{\rho, \alpha}(A, B)|^2 \leq U_{\rho, \alpha}(A)U_{\rho, \alpha}(B). \quad \square \quad (10)$$

Note that Theorem 2.1 recovers the inequality (7), if we take $\alpha = \frac{1}{2}$.

Remark 2.4. We take $\alpha = 0.1$ and

$$\rho = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2-i \\ 2+i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},$$

then we have

$$U_{\rho, \alpha}(A)U_{\rho, \alpha}(B) - 4\alpha(1-\alpha)|\text{Corr}_{\rho, \alpha}(A, B)|^2 \simeq -0.28332.$$

Therefore the inequality (8) does not hold for $\alpha \in [0, 1/2)$ in general.

Corollary 2.5. Under the same assumptions with Theorem 2.1, we have the following inequality:

$$U_{\rho, \alpha}(A)U_{\rho, \alpha}(B) - 4\alpha(1-\alpha)(|\text{Re}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 - |\text{Im}\{\text{Tr}[\rho^\alpha A \rho^{1-\alpha} B]\}|^2) \geq \alpha(1-\alpha)|\text{Tr}[\rho[A, B]]|^2. \quad (11)$$

Proof. From

$$\text{Im}\{\text{Corr}_{\rho, \alpha}(A, B)\} = \frac{1}{2i} \text{Tr}[\rho[A, B]] - \text{Im}\{\text{Tr}[\rho^\alpha A \rho^{1-\alpha} B]\},$$

we have

$$\frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 \leq |\text{Im}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 + |\text{Im}\{\text{Tr}[\rho^\alpha A \rho^{1-\alpha} B]\}|^2.$$

Thus we have

$$\begin{aligned}
|\text{Corr}_{\rho, \alpha}(A, B)|^2 &= |\text{Re}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 + |\text{Im}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 \\
&\geq |\text{Re}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 + \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 - |\text{Im}\{\text{Tr}[\rho^\alpha A \rho^{1-\alpha} B]\}|^2,
\end{aligned}$$

which proves the corollary. \square

Remark 2.6. The following inequality does not hold in general for $\alpha \in [\frac{1}{2}, 1]$:

$$|\text{Re}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 \geq |\text{Im}\{\text{Tr}[\rho^\alpha A \rho^{1-\alpha} B]\}|^2. \quad (12)$$

Because we have a counter-example as follows. We take $\alpha = \frac{2}{3}$ and

$$\rho = \frac{1}{7} \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2-i \\ 2+i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},$$

then we have

$$|\text{Re}\{\text{Corr}_{\rho, \alpha}(A, B)\}|^2 - |\text{Im}\{\text{Tr}[\rho^\alpha A \rho^{1-\alpha} B]\}|^2 \simeq -0.0548142.$$

This shows that Theorem 2.1 does not refine the inequality (5) in general.

3. A two-parameter extension

In this section, we introduce the parametric extended correlation measure $\text{Corr}_{\rho,\alpha,\gamma}(X, Y)$ by the convex combination between $\text{Corr}_{\rho,\alpha}(X, Y)$ and $\text{Corr}_{\rho,1-\alpha}(X, Y)$. Then we establish the parametric extended Schrödinger-type uncertainty relation applying the parametric extended correlation measure $\text{Corr}_{\rho,\alpha,\gamma}(X, Y)$.

Definition 3.1. We define the parametric extended correlation measure $\text{Corr}_{\rho,\alpha,\gamma}(X, Y)$ for two parameters $\alpha, \gamma \in [0, 1]$ by

$$\text{Corr}_{\rho,\alpha,\gamma}(X, Y) \equiv (1 - \gamma)\text{Corr}_{\rho,\alpha}(X, Y) + \gamma\text{Corr}_{\rho,1-\alpha}(X, Y) \quad (13)$$

for any operators X and Y .

Note that we have $\text{Corr}_{\rho,\alpha,\gamma}(H, H) = I_{\rho,\alpha}(H)$ for any observable H . Then we can prove the following inequality.

Theorem 3.2. If $0 \leq \alpha, \gamma \leq \frac{1}{2}$ or $\frac{1}{2} \leq \alpha, \gamma \leq 1$, then we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1 - \alpha)|\text{Corr}_{\rho,\alpha,\gamma}(A, B)|^2$$

for two observables A, B and a quantum state ρ .

Proof. By a similar way to the proof of Theorem 2.1, we have Eq. (9) and we also have

$$\begin{aligned} \text{Corr}_{\rho,1-\alpha}(A, B) &= \text{Tr}[\rho AB] - \text{Tr}[\rho^{1-\alpha} A \rho^\alpha B] \\ &= \sum_{i < j} \{ (\lambda_i - \lambda_i^{1-\alpha} \lambda_j^\alpha) a_{ij} b_{ji} + (\lambda_j - \lambda_j^{1-\alpha} \lambda_i^\alpha) a_{ji} b_{ij} \}. \end{aligned} \quad (14)$$

Thus we have

$$\begin{aligned} \text{Corr}_{\rho,\alpha,\gamma}(A, B) &= (1 - \gamma)\text{Corr}_{\rho,\alpha}(A, B) + \gamma\text{Corr}_{\rho,1-\alpha}(A, B) \\ &= \sum_{i < j} \{ (1 - \gamma)\lambda_i^\alpha (\lambda_i^{1-\alpha} - \lambda_j^{1-\alpha}) + \gamma\lambda_i^{1-\alpha} (\lambda_i^\alpha - \lambda_j^\alpha) a_{ij} b_{ji} \} \\ &\quad + \sum_{i < j} \{ (1 - \gamma)\lambda_j^\alpha (\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}) + \gamma\lambda_j^{1-\alpha} (\lambda_j^\alpha - \lambda_i^\alpha) a_{ji} b_{ij} \}. \end{aligned}$$

Since $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, we then have

$$\begin{aligned} |\text{Corr}_{\rho,\alpha,\gamma}(A, B)| &\leq \sum_{i < j} \{ (1 - \gamma)(\lambda_i^\alpha + \lambda_j^\alpha) |\lambda_i^\alpha - \lambda_j^\alpha| + \gamma(\lambda_i^{1-\alpha} + \lambda_j^{1-\alpha}) |\lambda_i^\alpha - \lambda_j^\alpha| \} |a_{ij}| |b_{ji}| \\ &\leq \sum_{i < j} |\lambda_i - \lambda_j| |a_{ij}| |b_{ji}|, \end{aligned}$$

thanks to the inequality

$$(1 - \gamma)(x^\alpha + y^\alpha) |x^{1-\alpha} - y^{1-\alpha}| + \gamma(x^{1-\alpha} + y^{1-\alpha}) |x^\alpha - y^\alpha| \leq |x - y| \quad (15)$$

for $0 \leq \alpha, \gamma \leq \frac{1}{2}$ or $\frac{1}{2} \leq \alpha, \gamma \leq 1$, and $x, y \geq 0$. The rest of the proof goes a similar way to that of Theorem 2.1. \square

Corollary 3.3. For any $\alpha \in [0, 1]$, two observables A, B and a quantum state ρ , we have

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1 - \alpha)|\text{Corr}_{\rho,\alpha,\frac{1}{2}}(A, B)|^2,$$

where we call $\text{Corr}_{\rho,\alpha,\frac{1}{2}}(A, B)$ a symmetrized correlation measure.

Proof. If $\gamma = \frac{1}{2}$, then the equality of the inequality (15) holds for any $\alpha \in [0, 1]$ and $x, y \geq 0$. Therefore we have the present corollary from Theorem 3.2. \square

4. A further generalization by metric adjusted correlation measure

Inspired by the recent results in [10] and the concept of metric adjusted skew information introduced by Hansen in [12], we here give a further generalization for Schrödinger-type uncertainty relation applying metric adjusted correlation measure introduced in [12]. We firstly give some notations according to those in [10]. Let $M_n(\mathbb{C})$ and $M_{n,sa}(\mathbb{C})$ be the sets of all $n \times n$ complex matrices and all $n \times n$ self-adjoint matrices, equipped with the Hilbert–Schmidt scalar product $\langle A, B \rangle = \text{Tr}[A^*B]$, respectively. Let $M_{n,+}(\mathbb{C})$ be the set of all positive definite matrices of $M_{n,sa}(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of all density matrices, that is

$$M_{n,+,1}(\mathbb{C}) \equiv \{\rho \in M_{n,sa}(\mathbb{C}) \mid \text{Tr } \rho = 1, \rho > 0\} \subset M_{n,+}(\mathbb{C}).$$

Here $X \in M_{n,+}(\mathbb{C})$ means that $\langle \phi | X | \phi \rangle \geq 0$ for any vector $|\phi\rangle \in \mathbb{C}^n$. In the study of quantum physics, we usually use a positive semidefinite matrix with a unit trace as a density operator ρ . In this section, we assume the invertibility of ρ .

A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is said to be operator monotone if the inequalities $0 \leq f(A) \leq f(B)$ hold for any $A, B \in M_{n,sa}(\mathbb{C})$ such that $0 \leq A \leq B$. An operator monotone function $f : (0, +\infty) \rightarrow (0, +\infty)$ is said to be symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$. We represent the set of all symmetric normalized operator monotone functions by \mathcal{F}_{op} . We have the following examples as elements of \mathcal{F}_{op} :

Example 4.1. (See [12,10,6,25].)

$$\begin{aligned} f_{RLD}(x) &= \frac{2x}{x+1}, & f_{SLD}(x) &= \frac{x+1}{2}, & f_{BKM}(x) &= \frac{x-1}{\log x}, \\ f_{WY}(x) &= \left(\frac{\sqrt{x}+1}{2} \right)^2, & f_{WYD}(x) &= \alpha(1-\alpha) \frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}, & \alpha &\in (0, 1). \end{aligned}$$

The functions $f_{BKM}(x)$ and $f_{WYD}(x)$ are normalized in the sense that $\lim_{x \rightarrow 1} f_{BKM}(x) = 1$ and $\lim_{x \rightarrow 1} f_{WYD}(x) = 1$. Note that a simple proof of the operator monotonicity of $f_{WYD}(x)$ was given in [6]. See also [30] for the proof of the operator monotonicity of $f_{WYD}(x)$ by use of majorization.

Remark 4.2. (See [10,15,24,25].) For any $f \in \mathcal{F}_{op}$, we have the following inequalities:

$$\frac{2x}{x+1} \leq f(x) \leq \frac{x+1}{2}, \quad x > 0.$$

That is, all $f \in \mathcal{F}_{op}$ lie in between the harmonic mean and the arithmetic mean.

For $f \in \mathcal{F}_{op}$ we define $f(0) = \lim_{x \rightarrow 0} f(x)$. We also denote the sets of regular and non-regular functions by

$$\mathcal{F}_{op}^r = \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\} \quad \text{and} \quad \mathcal{F}_{op}^n = \{f \in \mathcal{F}_{op} \mid f(0) = 0\}.$$

Definition 4.3. (See [8,10].) For $f \in \mathcal{F}_{op}^r$, we define the function \tilde{f} by

$$\tilde{f}(x) = \frac{1}{2} \left\{ (x+1) - (x-1)^2 \frac{f(0)}{f(x)} \right\} \quad (x > 0).$$

Then we have the following theorem.

Theorem 4.4. (See [8,6,26].) The correspondence $f \rightarrow \tilde{f}$ is a bijection between \mathcal{F}_{op}^r and \mathcal{F}_{op}^n .

We can use matrix mean theory introduced by Kubo and Ando in [15]. Then a mean m_f corresponds to each operator monotone function $f \in \mathcal{F}_{op}$ by the following formula

$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

for $A, B \in M_{n,+}(\mathbb{C})$. By the notion of matrix mean, we may define the set of the monotone metrics [23] by the following formula

$$\langle A, B \rangle_{\rho, f} = \text{Tr}[A m_f(L_\rho, R_\rho)^{-1}(B)],$$

where $L_\rho(A) = \rho A$ and $R_\rho(A) = A \rho$.

Definition 4.5. (See [12,8].) For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+1}(\mathbb{C})$ and $f \in \mathcal{F}_{op}^r$, we define the following quantities:

$$\begin{aligned} \text{Corr}_\rho^f(A, B) &\equiv \frac{f(0)}{2} \langle i[\rho, A], i[\rho, B] \rangle_{\rho, f}, & I_\rho^f(A) &\equiv \text{Corr}_\rho^f(A, A), \\ C_\rho^f(A, B) &\equiv \text{Tr}[m_f(L_\rho, R_\rho)(A)B], & C_\rho^f(A) &\equiv C_\rho^f(A, A), \\ U_\rho^f(A) &\equiv \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho^f(A))^2}. \end{aligned}$$

The quantity $I_\rho^f(A)$ is known as metric adjusted skew information [12]. It is notable that the metric adjusted correlation measure $\text{Corr}_\rho^c(A, B)$ was firstly introduced in [12] for a regular Morozova–Chentsov function c . Recently the notation $I_\rho^c(A, B)$ in [1] and the notation $I_\rho^f(A, B)$ in [11] were used. In addition, it is useful for the readers to be noted that the correlation $I_\rho^f(A, B)$ can be expressed as a difference of covariances [11]. Throughout the present paper, we use the notation $\text{Corr}_\rho^f(A, B)$ as the metric adjusted correlation measure, to avoid the confusion of the readers. (In the previous sections, we have already used $\text{Corr}_\rho(A, B)$, $\text{Corr}_{\rho,\alpha}(A, B)$ and $\text{Corr}_{\rho,\alpha,\gamma}(A, B)$ as correlation measures and done $I_\rho(H)$ and $I_{\rho,\alpha}(H)$ as skew informations.) Then we have the following proposition.

Proposition 4.6. (See [8,10].) For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+1}(\mathbb{C})$ and $f \in \mathcal{F}_{op}^r$, we have the following relations, where we put $A_0 \equiv A - \text{Tr}[\rho A]I$ and $B_0 \equiv B - \text{Tr}[\rho B]I$.

- (1) $I_\rho^f(A) = \text{Tr}[\rho A_0^2] - \text{Tr}[m_{\tilde{f}}(L_\rho, R_\rho)(A_0)A_0] = V_\rho(A) - C_\rho^{\tilde{f}}(A_0)$.
- (2) $J_\rho^f(A) = \text{Tr}[\rho A_0^2] + \text{Tr}[m_{\tilde{f}}(L_\rho, R_\rho)(A_0)A_0] = V_\rho(A) + C_\rho^{\tilde{f}}(A_0)$.
- (3) $0 \leq I_\rho^f(A) \leq U_\rho^f(A) \leq V_\rho(A)$.
- (4) $U_\rho^f(A) = \sqrt{I_\rho^f(A)J_\rho^f(A)}$.
- (5) $\text{Corr}_\rho^f(A, B) = \frac{1}{2} \text{Tr}[\rho A_0 B_0] + \frac{1}{2} \text{Tr}[\rho B_0 A_0] - \text{Tr}[m_{\tilde{f}}(L_\rho, R_\rho)(A_0)B_0] = \frac{1}{2} \text{Tr}[\rho A_0 B_0] + \frac{1}{2} \text{Tr}[\rho B_0 A_0] - C_\rho^{\tilde{f}}(A_0, B_0)$.

The following inequality is the further generalization of Corollary 3.3 by the use of the metric adjusted correlation measure.

Theorem 4.7. For $f \in \mathcal{F}_{op}^r$, if

$$\frac{x+1}{2} + \tilde{f}(x) \geq 2f(x), \quad (16)$$

then we have

$$U_\rho^f(A)U_\rho^f(B) \geq 4f(0)|\text{Corr}_\rho^f(A, B)|^2, \quad (17)$$

for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+1}(\mathbb{C})$.

In order to prove Theorem 4.7, we use the following two lemmas.

Lemma 4.8. (See [35].) If Eq. (16) is satisfied, then we have the following inequality:

$$\left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 \geq f(0)(x-y)^2.$$

Proof. By Eq. (16), we have

$$\frac{x+y}{2} + m_{\tilde{f}}(x, y) \geq 2m_f(x, y).$$

We also have

$$\begin{aligned} m_{\tilde{f}}(x, y) &= y\tilde{f}\left(\frac{x}{y}\right) \\ &= \frac{y}{2} \left\{ \frac{x}{y} + 1 - \left(\frac{x}{y} - 1\right)^2 \frac{f(0)}{f(x/y)} \right\} \\ &= \frac{x+y}{2} - \frac{f(0)(x-y)^2}{2m_f(x, y)}. \end{aligned}$$

Therefore

$$\begin{aligned} \left(\frac{x+y}{2}\right)^2 - m_{\tilde{f}}(x, y)^2 &= \left\{ \frac{x+y}{2} - m_{\tilde{f}}(x, y) \right\} \left\{ \frac{x+y}{2} + m_{\tilde{f}}(x, y) \right\} \\ &\geq \frac{f(0)(x-y)^2}{2m_f(x, y)} 2m_f(x, y) = f(0)(x-y)^2. \quad \square \end{aligned}$$

We have the following expressions for the quantities $I_\rho^f(A)$, $J_\rho^f(A)$, $U_\rho^f(A)$ and $\text{Corr}_\rho^f(A, B)$ by using Proposition 4.6 and a mean $m_{\tilde{f}}$.

Lemma 4.9. (See [10].) Let $\{|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle\}$ be a basis of eigenvectors of ρ , corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. We put $a_{jk} = \langle \phi_j | A_0 | \phi_k \rangle$, $b_{jk} = \langle \phi_j | B_0 | \phi_k \rangle$, where $A_0 \equiv A - \text{Tr}[\rho A]I$ and $B_0 \equiv B - \text{Tr}[\rho B]I$ for $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+1}(\mathbb{C})$. Then we have

$$\begin{aligned} I_\rho^f(A) &= \frac{1}{2} \sum_{j,k} (\lambda_j + \lambda_k) a_{jk} a_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k) a_{jk} a_{kj} = 2 \sum_{j < k} \left\{ \frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k) \right\} |a_{jk}|^2, \\ J_\rho^f(A) &= \frac{1}{2} \sum_{j,k} (\lambda_j + \lambda_k) a_{jk} a_{kj} + \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k) a_{jk} a_{kj} \geq 2 \sum_{j < k} \left\{ \frac{\lambda_j + \lambda_k}{2} + m_{\tilde{f}}(\lambda_j, \lambda_k) \right\} |a_{jk}|^2, \\ U_\rho^f(A)^2 &= \frac{1}{4} \left(\sum_{j,k} (\lambda_j + \lambda_k) |a_{jk}|^2 \right)^2 - \left(\sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k) |a_{jk}|^2 \right)^2 \end{aligned}$$

and

$$\begin{aligned} \text{Corr}_\rho^f(A, B) &= \frac{1}{2} \sum_{j,k} \lambda_j a_{jk} b_{kj} + \frac{1}{2} \sum_{j,k} \lambda_k a_{jk} b_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k) a_{jk} b_{kj} \\ &= \sum_{j < k} \left(\frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k) \right) a_{jk} b_{kj} + \sum_{j < k} \left(\frac{\lambda_k + \lambda_j}{2} - m_{\tilde{f}}(\lambda_k, \lambda_j) \right) a_{kj} b_{jk}. \end{aligned} \quad (18)$$

We are now in a position to prove Theorem 4.7.

Proof of Theorem 4.7. From Eq. (18), we have

$$\begin{aligned} |\text{Corr}_\rho^f(A, B)| &\leq \sum_{j < k} \left| \left(\frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k) \right) a_{jk} b_{kj} \right| + \sum_{j < k} \left| \left(\frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_k, \lambda_j) \right) a_{kj} b_{jk} \right| \\ &\leq \sum_{j < k} \left| \frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k) \right| |a_{jk}| |b_{kj}| + \sum_{j < k} \left| \frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_k, \lambda_j) \right| |a_{kj}| |b_{jk}| \\ &= 2 \sum_{j < k} \left| \frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k) \right| |a_{jk}| |b_{kj}| \\ &\leq \sum_{j < k} |\lambda_j - \lambda_k| |a_{jk}| |b_{kj}|. \end{aligned}$$

Then we have

$$\begin{aligned} f(0) |\text{Corr}_\rho^f(A, B)|^2 &\leq \left(\sum_{j < k} f(0)^{1/2} |\lambda_j - \lambda_k| |a_{jk}| |b_{kj}| \right)^2 \\ &\leq \left(\sum_{j < k} \left\{ \left(\frac{\lambda_j + \lambda_k}{2} \right)^2 - m_{\tilde{f}}(\lambda_j, \lambda_k)^2 \right\}^{1/2} |a_{jk}| |b_{kj}| \right)^2 \\ &\leq \left(\sum_{j < k} \left\{ \frac{\lambda_j + \lambda_k}{2} - m_{\tilde{f}}(\lambda_j, \lambda_k) \right\} |a_{jk}|^2 \right) \left(\sum_{j < k} \left\{ \frac{\lambda_j + \lambda_k}{2} + m_{\tilde{f}}(\lambda_j, \lambda_k) \right\} |b_{kj}|^2 \right) \\ &\leq \frac{1}{4} I_\rho^f(A) J_\rho^f(B). \end{aligned}$$

In a similar way, we also have

$$I_{\rho}^f(B) J_{\rho}^f(A) \geq 4f(0) |\text{Corr}_{\rho}^f(A, B)|^2.$$

Hence we have the desired inequality (17). \square

Remark 4.10. Under the same assumptions of Theorem 4.7, we have the following Heisenberg-type uncertainty relation [35]:

$$U_{\rho}^f(A) U_{\rho}^f(B) \geq f(0) |\text{Tr}[\rho[A, B]]|^2 \quad (19)$$

by a similar way to the proof of Theorem 4.7, since we have

$$|\text{Tr}[\rho[A, B]]| \leq 2 \sum_{j < k} |\lambda_j - \lambda_k| |a_{jk}| |b_{kj}|.$$

As stated in Remark 2.6, there is no ordering between the right hand side of the inequality (17) and that of the inequality (19), in general.

If we use the function

$$f_{WYD}(x) = \alpha(1 - \alpha) \frac{(x - 1)^2}{(x^{\alpha} - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0, 1),$$

then we obtain the following uncertainty relation.

Corollary 4.11. For $A, B \in M_{n,sa}(\mathbb{C})$ and $\rho \in M_{n,+1}(\mathbb{C})$, we have

$$U_{\rho}^{f_{WYD}}(A) U_{\rho}^{f_{WYD}}(B) \geq 4\alpha(1 - \alpha) |\text{Corr}_{\rho, \alpha, \frac{1}{2}}(A, B)|^2.$$

Proof. From the definition

$$f_{WYD}(x) = \alpha(1 - \alpha) \frac{(x - 1)^2}{(x^{\alpha} - 1)(x^{1-\alpha} - 1)},$$

it is clear that

$$\tilde{f}_{WYD}(x) = \frac{1}{2} \{x + 1 - (x^{\alpha} - 1)(x^{1-\alpha} - 1)\}.$$

By Lemma 2.3, we have for $0 \leq \alpha \leq 1$ and $x > 0$,

$$(1 - 2\alpha)^2(x - 1)^2 - (x^{\alpha} - x^{1-\alpha})^2 \geq 0.$$

This inequality can be rewritten as

$$(x^{2\alpha} - 1)(x^{2(1-\alpha)} - 1) \geq 4\alpha(1 - \alpha)(x - 1)^2.$$

Thus we have

$$\begin{aligned} \frac{x+1}{2} + \tilde{f}_{WYD}(x) &= x + 1 - \frac{1}{2}(x^{\alpha} - 1)(x^{1-\alpha} - 1) \\ &= \frac{1}{2}(x^{\alpha} + 1)(x^{1-\alpha} + 1) \\ &\geq 2\alpha(1 - \alpha) \frac{(x - 1)^2}{(x^{\alpha} - 1)(x^{1-\alpha} - 1)} \\ &= 2f_{WYD}(x). \end{aligned}$$

It follows from Theorem 4.7 that we obtain the aimed result, since the function f_{WYD} corresponds to a symmetrized correlation measure, that is, we have $\text{Corr}_{\rho}^{f_{WYD}}(A, B) = \text{Corr}_{\rho, \alpha, \frac{1}{2}}(A, B)$ by Eq. (18). \square

Note that Corollary 3.3 coincides with Corollary 4.11, since we have $U_{\rho, \alpha}(A) = U_{\rho}^{f_{WYD}}(A)$ which is obtained by the fact that the function $f_{WYD}(x)$ corresponds to the Wigner–Yanase–Dyson skew information.

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