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Analytical properties of the Lupaş q-transform

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ABSTRACT

The *Lupaş q-transform* emerges in the study of the *limit q-Lupaş operator*. The latter comes out naturally as a limit for a sequence of the Lupaş q-analogues of the Bernstein operator. Given $q \in (0, 1)$, $f \in C[0, 1]$, the *q-Lupaş transform* of f is defined by

$$(\Lambda_q f)(z) := \frac{1}{(-z; q)_\infty} \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k)q^{k(k-1)/2}}{(q; q)_k} z^k.$$

The transform is closely related to both the q-deformed Poisson probability distribution, which is used widely in the q-boson operator calculus, and to Valiron's method of summation for divergent series. In general, $\Lambda_q f$ is a meromorphic function whose poles are contained in the set $J_q := \{-q^{-j}\}_{j=0}^{\infty}$.

In this paper, we study the connection between the behaviour of f on $[0, 1]$ and the decay of $\Lambda_q f$ as $z \to \infty$.

1. Introduction

The importance of the Bernstein polynomials paved the way to the discovery of their numerous generalizations and applications in various mathematical disciplines. As an example, recent generalizations based on the q-integers have emerged due to the fast development of the q-calculus. Lupaş was the person who pioneered the work on the q-versions of the Bernstein polynomials. In 1987, he introduced (cf. [1]) a q-analogue of the Bernstein operator investigating its approximation and shape-preserving properties. Since then, the study of the q-analogue has been in progress. See [2–6], and [7], where various convergence properties of the q-analogue have been investigated.

It should be mentioned here that, later on, towards the end of the 1990s, another generalization of Bernstein polynomials based on the q-integers, called the *q-Bernstein polynomials* appeared. The q-Bernstein polynomials have been studied by many authors, while the Lupaş q-analogues have remained out of the spotlight. This is in spite of the fact that the Lupaş operators possess an advantage to be *positive* linear operators for all $q > 0$, while the q-Bernstein polynomials give positive linear operators only for $q \in (0, 1]$.

In the sequel, we use the following standard notation (cf. [8], Ch. 10, Section 10.2):

$$(z; q)_0 := 1; \qquad (z; q)_n := \prod_{k=0}^{n-1}(1 - zq^k); \qquad (z; q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k).$$

Let $q > 0$. For any $n = 0, 1, 2, \ldots$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad (n = 1, 2, \ldots), \qquad [0]_q := 0;$$

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and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \dots [n]_q \quad (n = 1, 2, \dots), \qquad [0]_q! := 1.$$

For integers $0 \le k \le n$, the *q-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We also refer to the following particular cases of the q-binomial Theorem, which are Euler's Identities (see [8], Ch. 10, Section 10.2, Corollary 10.2.2):

$$(-z; q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k} z^k, \quad |q| < 1; \tag{1.1}$$

and

$$\frac{1}{(-z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_k} z^k, \quad |q| < 1, \ |z| < 1. \tag{1.2}$$

Definition 1.1. Let $0 < q < 1$, $f \in C[0, 1]$. The *Lupaş q-transform* of f is given by

$$(\Lambda_q f)(z) := \frac{1}{(-z; q)_\infty} \cdot \sum_{k=0}^{\infty} f(1-q^k) \frac{q^{k(k-1)/2}}{(q; q)_k} z^k, \quad z \in \mathbb{C} \setminus \{-q^k\}_{k \in \mathbb{Z}_+}.$$

Clearly, $\Lambda_q : C[0, 1] \to C[0, \infty)$ is a positive linear operator with $\|\Lambda_q\| = 1$. In general, $\Lambda_q f$ is a meromorphic function whose simple poles are contained in the set

$$J_q := \{-q^{-j}\}_{j=0}^{\infty}.$$

We notice that, in some special cases, $\Lambda_q f$ may be an entire or a rational function, the necessary and sufficient conditions for which have been supplied in [5]. In this paper, we present new results on the analytic properties of $\Lambda_q f$. More precisely, we study the connection between the behaviour of f on $[0, 1]$ and the asymptotic properties of $(\Lambda_q f)(z)$ as $z \to \infty$.

Prior to presenting the results of the paper, it helps to provide a short historic background on the origin of the Lupaş q-transform and, as such, to establish the motivation behind this study. The remainder of the Introduction, therefore, serves solely the purpose of establishing links to previous and related researches, and will not be used as a part of the reasoning in the upcoming sections.

Following Lupaş, we denote, for $n \in \mathbb{N}$,

$$b_{nk}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+qx) \cdots (1-x+q^{n-1}x)}, \quad k = 0, \dots, n. \tag{1.3}$$

Let $q > 0$, $f : [0, 1] \to \mathbb{C}$. The linear operator

$$(R_{n,q} f)(x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{nk}(q; x), \quad n \in \mathbb{N} \tag{1.4}$$

is the *Lupaş q-analogue of the Bernstein operator*. This operator had been introduced in [1], where it was called a *q-analogue of the Bernstein operator*. We call it the *Lupaş q-analogue* to emphasize the role of Lupaş in studying the first of several currently known q-versions of the Bernstein polynomials.

Clearly, if $q = 1$, then (1.4) reduces to the classical Bernstein polynomials. In the case $q \neq 1$, operators $R_{n,q}$ are *rational functions* rather than polynomials, and they are *positive* linear operators possessing the *end-point interpolation* property:

$$(R_{n,q} f)(0) = f(0), \qquad (R_{n,q} f)(1) = f(1). \tag{1.5}$$

The convergence of $\{(R_{n,q} f)(x)\}$ for $q \neq 1$ being constant has been studied in [3,4,7]. It has been proved in [4] that, for any $q \in (0, 1)$ and $f \in C[0, 1]$, the sequence $\{(R_{n,q} f)(x)\}$ converges uniformly on $[0, 1]$ to the limit $(R_{\infty,q} f)(x) = (\Lambda_q f)(\frac{x}{1-x})$.

It should be noticed that $R_{n,q}$, despite being positive linear operators on $C[0, 1]$, do not satisfy the conditions of Korovkin's Theorem, and the limit operator $R_{\infty,q}$ is not the identity operator. However, operators $R_{n,q}$ satisfy the conditions of Wang's Korovkin-type Theorem (cf. [9], Theorem 2). A probabilistic approach to this operator has been developed in [10] implying that $R_{\infty,q}$ is closely related to the q-deformed Poisson distribution, which plays an important role in the q-boson operator calculus (cf., e.g., [11]). It is also worth mentioning that operators with a structure similar to that of Λ_q have been studied from different angles in [12,13,10,14,15]. It is a note-worthy fact that operator Λ_q can be used as a J method of summation with $J(x) = (-x; q)_\infty$, while $(\Lambda_q f)(x)$ represents $S(x)/J(x)$ for a series $\sum_{k=0}^{\infty} a_k$, where $a_0 = f(0)$, $a_k = f(1-q^k) - f(1-q^{k-1})$, $k \in \mathbb{N}$, (see [16], Section 4.12, pp. 79–81). The regularity of the method follows immediately from the end-point interpolation property (1.5). Furthermore, operator Λ_q is related to Valiron's summation method as it is mentioned in [16], Section 9.16, Remark on p. 226.

The subject of this article was inspired by the results obtained on the Lupaş q-analogue of the Bernstein operator. In distinction from the studies mentioned above, mostly carried out from the perspective of the Approximation Theory, the aim of this paper is to examine questions of an entirely different nature; specifically, it deals with some of the analytic properties of a meromorphic function $(\Lambda_q f)(z)$.

2. Statement of results

Throughout the paper, we assume $0 < q < 1$ to be fixed. In the sequel, we denote by letter C – possibly with indices – a positive constant whose value does not need to be specified. The indices on C may be either a numbering (that is, if more than one constant is involved) or an indication of the dependence on certain parameters. Having stated so, we write $f(x) \asymp g(x)$ if $C_1 f(x) \leq g(x) \leq C_2 f(x)$ for some C_1 and C_2.

Since $\Lambda_q f$ has poles at $\{-q^{-j}\}_{j=0}^{\infty}$, we provide estimates for $(\Lambda_q f)(z)$ for z being outside "small" discs surrounding the poles. More precisely, for $\delta > 0$, we define the set A_δ as follows:

$$A_\delta := \mathbb{C} \setminus \cup_{j=0}^{\infty}\{z : |1 + q^j z| \leq \delta\}. \tag{2.1}$$

It is easy to see that, for $x > 0$, $(\Lambda_q f)(x) \leq \|f\|_{[0,1]}$, that is $\Lambda_q f$ is bounded on $[0, \infty)$. This result is extended by showing that $(\Lambda_q f)(z)$ is bounded in A_δ for any $\delta > 0$, and the condition for the possible rate of decay is established. We start with the following assertion showing an impact of the Lipschitz-type condition at 1 on the behaviour of $(\Lambda_q f)(z)$.

Theorem 2.1. (i) *Suppose that, for some $\alpha \geq 0$, one has:*

$$f(1 - q^k) = O(q^{\alpha k}), \quad k \to \infty. \tag{2.2}$$

Then, for any $\delta > 0$, the following estimate holds:

$$(\Lambda_q f)(z) = O(|z|^{-\alpha}), \quad z \in A_\delta,\ z \to \infty. \tag{2.3}$$

(ii) *If, for some $\alpha \geq 0$,*

$$(\Lambda_q f)(x) = O(|x|^{-\alpha}), \quad x > 0,\ x \to +\infty, \tag{2.4}$$

then

$$f(1 - q^k) = O(q^{\alpha k}), \quad k \to \infty. \tag{2.5}$$

A slight modification of the proof shows that the statement remains true if we substitute 'O' by 'o'. In other words, the following theorem holds.

Theorem 2.2. (i) *Suppose that, for some $\alpha \geq 0$, one has:*

$$f(1 - q^k) = o(q^{\alpha k}), \quad k \to \infty. \tag{2.6}$$

Then, for any $\delta > 0$, the following estimate holds:

$$(\Lambda_q f)(z) = o(|z|^{-\alpha}), \quad z \in A_\delta,\ z \to \infty. \tag{2.7}$$

(ii) *If, for some $\alpha \geq 0$,*

$$(\Lambda_q f)(x) = o(|x|^{-\alpha}), \quad x > 0,\ x \to +\infty, \tag{2.8}$$

then

$$f(1 - q^k) = o(q^{\alpha k}), \quad k \to \infty. \tag{2.9}$$

Remark 2.1. We notice that A_δ can be extended so as to include all removable poles.

Corollary 2.3. *For $\Lambda_q f$, conditions (2.3) and (2.4) are equivalent. And so are conditions (2.7) and (2.8).*

As an immediate application of Theorem 2.2, we derive the following result.

Theorem 2.4. (i) *If f has $m \geq 0$ derivatives along $\{1 - q^k\}_{k=0}^{\infty}$ at 1, then, for any $\delta > 0$, the following relation holds:*

$$(\Lambda_q f)(z) = \sum_{k=0}^{m} \frac{(-1)^k f^{(k)}(1)}{k!(-z; q)_k} + o(|z|^{-m}), \quad z \to \infty,\ z \in A_\delta.$$

(ii) *If, for $x > 0$,*

$$(\Lambda_q f)(x) = \sum_{k=0}^{m} \frac{a_k}{(-z;q)_k} + o(x^{-m}), \quad x \to +\infty, \tag{2.10}$$

then f has m derivatives along $\{1-q^k\}_{k=0}^{\infty}$ at 1, and

$$a_k = \frac{(-1)^k f^{(k)}(1)}{k!}, \quad k = 0, 1, \ldots, m.$$

Corollary 2.5. *A function f is infinitely differentiable along $\{1-q^k\}_{k=0}^{\infty}$ at 1 if an only if $(\Lambda_q f)(x)$ admits the representation as the asymptotic series below:*

$$(\Lambda_q f)(x) \approx \sum_{k=0}^{\infty} \frac{a_k}{(-z;q)_k}, \quad x > 0,\ x \to +\infty.$$

In this case,

$$a_k = \frac{(-1)^k f^{(k)}(1)}{k!}, \quad k = 0, 1, \ldots$$

Corollary 2.6. *The following relation is true:*

$$(\Lambda_q f)(z) = o(|z|^{-m}), \quad z \in A_\delta,\ z \to \infty \Leftrightarrow f(1) = f'(1) = \cdots = f^{(m)}(1) = 0.$$

Furthermore, if

$$(\Lambda_q f)(z) = o(|z|^{-m}), \quad z \in A_\delta,\ z \to \infty \text{ for all } m = 0, 1, \ldots$$

then $f^{(m)}(1) = \cdots = 0$ for all $m = 0, 1, \ldots$

Using the fact that, for any $\delta > 0$,

$$\frac{1}{(-z;q)_k}$$

are uniformly bounded in A_δ, we derive the following assertion.

Corollary 2.7. *If f is analytic in $\{z : |z-1| \le 1\}$, then, for any $\delta > 0$, we have:*

$$(\Lambda_q f)(z) = \sum_{k=0}^{\infty} \frac{(-1)^k f^{(k)}(1)}{k!(-z;q)_k}, \quad a \in A_\delta,$$

where the series converges in A_δ uniformly.

Remark 2.2. The latter corollary implies that if f is analytic in $\{z : |z-1| \le 1\}$, then, for some $m \ge 0$, $(\Lambda_q f)(z) \ge C_{q,\delta}|z|^{-m}$ in $A_\delta,\ \delta > 0$. In this case $f^{(m)}(1) \ne 0$.

Generally seen, Theorems 2.1 and 2.2 show that the faster $f(1-q^k)$ tends to 0 as $k \to \infty$, the faster $(\Lambda_q f)(z)$ decreases as $z \to \infty,\ z \in A_\delta$. The opposite is also valid. Theorem 2.8 below demonstrates that the rate of decay of $\Lambda_q f$ at infinity reaches the saturation if $f(1-q^k) = 0$ for $k \ge 1$ and $f(0) \ne 0$, when

$$|(\Lambda_q f)(z)| = \frac{|f(0)|}{(-z;q)_\infty} \asymp \exp\left\{-\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\}, \quad z \in A_\delta.$$

Theorem 2.8. *If*

$$|(\Lambda_q f)(x)| = o\left(\exp\left\{-\frac{\ln^2(x/\sqrt{q})}{2\ln(1/q)}\right\}\right), \quad x \to +\infty,$$

then $f(1-q^k) = 0$ for all $k = 0, 1, \ldots$

Remark 2.3. The case $f(1-q^k) = 0,\ k \ge k_0$, with $f(1-q^{k_0}) \ne 0$ is characterized by the following property:

$$|(\Lambda_q f)(x)| \asymp x^{k_0} \exp\left\{-\frac{\ln^2(x/\sqrt{q})}{2\ln(1/q)}\right\}, \quad x \to +\infty.$$

Remark 2.4. The statements of Theorem 2.1(i), 2.2(i), and 2.8 together with their corollaries remain true if we substitute $x(x > 0,\ x \to +\infty)$ by $re^{i\varphi}$ $(r > 0, r \to +\infty, -\pi < \varphi < \pi)$.

3. Some auxiliary results

For $f \in C[0, 1]$, we denote:

$$g_f(z) := \sum_{k=0}^{\infty} \frac{f(1-q^k)q^{k(k-1)/2}}{(q;q)_k}. \tag{3.1}$$

Then, for any $f \in C[0, 1]$, g_f is an entire function satisfying

$$|g_f(z)| \le \|f\| \cdot (-|z|;q)_\infty.$$

The growth estimates for $(-|z|;q)_\infty$ can be found, for example, in [17], formula (2.6):

$$(-r;q)_\infty \asymp \exp\left\{\frac{\ln^2(r/\sqrt{q})}{2\ln(1/q)}\right\}, \quad r > 0, \tag{3.2}$$

whence

$$|g_f(z)| \le C_1\|f\| \cdot \exp\left\{\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\}. \tag{3.3}$$

The proofs of Theorem 2.1 through 2.8 are based on the lemmas below.

Lemma 3.1 *([5]). For all $j \in \mathbb{Z}_+$, the following equalities hold:*

$$(\Lambda_q(1-t)^j)(z) = \frac{1}{(-z;q)_j}. \tag{3.4}$$

Proof. Applying the definition, we write:

$$(\Lambda_q(1-t)^j)(z) = \frac{1}{(-z;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^j z)^k}{(q;q)_k}.$$

By virtue of Euler's Identity (3.6), we have:

$$\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^j z)^k}{(q;q)_k} = (-q^j z;q)_\infty = \frac{(-z;q)_\infty}{(-z;q)_j}.$$

The statement now follows. □

Lemma 3.2. *For any $\delta > 0$, the following estimate holds uniformly with respect to* $\arg z$*:*

$$\frac{(-|z|;q)_\infty}{|(-z;q)_\infty|} \asymp 1, \quad z \in A_\delta.$$

Proof. Obviously, $|(-z;q)_\infty| \le (-|z|;q)_\infty$ and

$$\frac{(-|z|;q)_\infty}{|(-z;q)_\infty|} \ge 1$$

is true. We have to prove, therefore, that

$$\frac{(-|z|;q)_\infty}{|(-z;q)_\infty|} \le C_{q,\delta} \quad \text{for } A_\delta.$$

Step 1. First, we prove that for $|1+z| \ge \delta$, the following estimate holds:

$$\frac{1+|z|}{|1+z|} \le \exp\left\{\frac{2}{\delta^2}\min\left\{|z|, \frac{1}{|z|}\right\}\right\}. \tag{3.5}$$

We set $z = re^{i\varphi}$, where $r = |z|$, $\varphi \in (-\pi, \pi]$. Then we have:

$$\left(\frac{1+|z|}{|1+z|}\right)^2 = \frac{1+2r+r^2}{1+2r\cos\varphi+r^2} = 1 + 2r\frac{1-\cos\varphi}{1+2r\cos\varphi+r^2}$$
$$\le 1 + \frac{4r\sin^2(\varphi/2)}{\delta^2} \le 1 + \frac{4r}{\delta^2}.$$

Observing that the left-hand side is invariant under the transformation $z \mapsto 1/z$, we conclude that:

$$\left(\frac{1+|z|}{|1+z|}\right)^2 \le 1 + \frac{4\sin^2(\varphi/2)}{r\delta^2} \le 1 + \frac{4}{r\delta^2}.$$

Thus,

$$\left(\frac{1+|z|}{|1+z|}\right)^2 \le 1 + \frac{4}{\delta^2}\min\left\{r, \frac{1}{r}\right\} \le \exp\left\{\frac{4}{\delta^2}\min\left\{r, \frac{1}{r}\right\}\right\}$$

and (3.5) is proved.

Step 2. Now, we show that, for $z \in A_\delta$, the following inequality holds:

$$\frac{(-|z|; q)_\infty}{|(-z; q)_\infty|} \le \exp\left\{\frac{4}{\delta^2}\sum_{j=0}^{\infty}\min\left\{q^j|z|, \frac{1}{q^j|z|}\right\}\right\}. \tag{3.6}$$

Indeed, we derive from (3.5) that, for $z \in A_\delta$, there holds:

$$\frac{1+q^j|z|}{|1+q^jz|} \le \exp\left\{\frac{2}{\delta^2}\min\left\{q^j|z|, \frac{1}{q^j|z|}\right\}\right\}$$

and (3.6) follows.

Step 3. Finally, we prove that, for $r \ge 0$,

$$\sum_{j=0}^{\infty}\min\left\{q^jr, \frac{1}{q^jr}\right\} \le \frac{1+q}{q(1-q)}. \tag{3.7}$$

For $0 \le r \le 1$, the inequality is obvious, because in this case $\min\{q^jr, \frac{1}{q^jr}\} = q^jr$ and

$$\sum_{j=0}^{\infty} q^jr = \frac{r}{1-q} \le \frac{1}{1-q} \le \frac{1}{1-q}\cdot\frac{1+q}{q}.$$

For $r > 1$, we choose $N = N(r)$ in such a way that

$$q^Nr \le 1 < q^{N-1}r$$

or

$$\frac{\ln r}{\ln(1/q)} \le N < \frac{\ln r}{\ln(1/q)} + 1.$$

Now, we write:

$$\begin{aligned}
\sum_{j=0}^{\infty}\min\left\{q^jr, \frac{1}{q^jr}\right\} &= \sum_{j=0}^{n}\frac{1}{q^jr} + \sum_{j=N+1}^{\infty} q^jr \\
&= \frac{1}{r}\cdot\frac{(1/q)^{N+1}-1}{(1/q)-1} + r\cdot\frac{q^{N+1}}{1-q} \le \frac{1}{r}\cdot\frac{(1/q)^{N+1}}{1-q}\cdot q + r\cdot\frac{q^N}{1-q} \\
&= \frac{1}{1-q}\left(\frac{1}{q^Nr} + q^Nr\right) \le \frac{1}{1-q}\left(\frac{1}{q}+1\right) = \frac{1+q}{q(1-q)},
\end{aligned}$$

because $q \le q^Nr \le 1$. Thus, inequality (3.7) is proved. The statement of the lemma is a consequence of (3.6) and (3.7). $\square$

Lemma 3.3. *For any $a \ge 0$ and $\delta > 0$, the following estimate holds:*

$$\left|\frac{(-q^az; q)_\infty}{(-z; q)_\infty}\right| = O(|z|^{-a}), \quad z \to \infty,\ z \in A_\delta.$$

Proof. It can be easily derived from (3.2) that, for $x > 0$, the following relation is true:

$$\frac{(-q^ax; q)_\infty}{(-x; q)_\infty} = O(x^{-a}).$$

Further more, for $z \in A_\delta$, one has:

$$\left|\frac{(-q^a z;q)_\infty}{(-z;q)_\infty}\right| \le \frac{(-q^a|z|;q)_\infty}{|(-z;q)_\infty|} = \frac{(-q^a|z|;q)_\infty}{(-|z|;q)_\infty}\cdot\frac{(-|z|;q)_\infty}{|(-z;q)_\infty|} \le C_{q,\delta}|z|^{-a}$$

by virtue of Lemma 3.2. □

Remark 3.1. For $a \in \mathbb{N}$, the statement is obvious, because

$$\frac{(-q^a z;q)_\infty}{(-z;q)_\infty} = \frac{1}{(-z;q)_a}.$$

4. Proofs of the theorems

Proof of Theorem 2.1. (i) Assume that $f(1-q^k) \le Cq^{\alpha k},\ k = 0, 1, \ldots$ Then

$$\begin{aligned}|(\Lambda_q f)(z)| &\le \frac{C}{|(-z;q)_\infty|}\cdot\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^\alpha|z|)^k}{(q;q)_k}\\ &= C\frac{-(q^\alpha|z|;q)_\infty}{|(-z;q)_\infty|} = O(|z|^{-\alpha}),\quad z\in A_\delta,\ z\to\infty\end{aligned}$$

by virtue of Lemma 3.3.

(ii) Suppose that (2.4) holds. Consider the entire function $g_f(z)$ given by (3.1). Since $g_f(z) = (\Lambda_q f)(z)\cdot(-z;q)_\infty$, we obtain, using (3.2):

$$|g_f(x)| \le C_1 x^{-\alpha}\exp\left\{\frac{\ln^2(x/\sqrt{q})}{2\ln(1/q)}\right\}\quad \text{for } x\in(0,\infty). \tag{4.1}$$

To extend this estimate – possibly with a different constant – into the complex plane, we introduce an auxiliary function $h(z)$ putting:

$$h(z) := z^\alpha g_f(z)\exp\left\{-\frac{\ln^2(z/\sqrt{q})}{2\ln(1/q)}\right\},\quad z\in D := \mathbb{C}\setminus[0,\infty). \tag{4.2}$$

Evidently, $h(z)$ is analytic in D and possesses limiting values $h_\pm(x)$ on both sides of the cut $[0,\infty)$. Estimate (4.1) implies that $|h_\pm(x)| \le C_2$ for $x\in[0,\infty)$. Let us estimate the growth of $h(z)$ in D. We obtain with the help of (3.3):

$$\begin{aligned}|h(z)| &\le |z|^\alpha\cdot|g_f(z)|\cdot\left|\exp\left\{-\frac{\ln^2(z/\sqrt{q})}{2\ln(1/q)}\right\}\right|\\ &\le C_3\|f\|\cdot|z|^\alpha \le C_{4,\rho}\exp\{|z|^\rho\}\end{aligned}$$

for any $\rho > 0$. Applying the Phragmén–Lindelöf Theorem (cf. [18], Chapter 6, Section 6.1), we conclude that $|h(z)| \le C_5$ in D. Therefore,

$$\begin{aligned}|g_f(z)| &= |h(z)|\cdot|z|^{-\alpha}\cdot\left|\exp\left\{\frac{\ln^2(z/\sqrt{q})}{2\ln(1/q)}\right\}\right|\\ &\le C_5|z|^{-\alpha}\exp\left\{\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\},\quad z\in\mathbb{C}.\end{aligned}$$

By the Cauchy estimates for the coefficients of $g_f(z)$, we obtain:

$$\begin{aligned}|f(1-q^k)|q^{k(k-1)/2} &\le C_6|z|^{-\alpha-k}\exp\left\{\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\}\\ &= C_7\exp\left\{-(\alpha+k)\ln|z| + \frac{\ln^2|z|}{2\ln(1/q)} + \frac{\ln|z|}{2}\right\}.\end{aligned}$$

The latter expression attains minimum if

$$\ln|z| = \left(\alpha + k - \frac{1}{2}\right)\ln(1/q),$$

whence

$$|f(1-q^k)|q^{k(k-1)/2} \le C_8 \exp\left\{-(\alpha+k)^2\ln(1/q) + \frac{\alpha+k}{2}\ln(1/q) + \frac{(\alpha+k)^2-(\alpha+k)}{2}\ln(1/q) + \frac{\alpha+k}{2}\ln(1/q)\right\}$$
$$= C_9 \exp\left\{-\alpha k\ln(1/q) - \frac{k(k-1)}{2}\ln(1/q)\right\} = C_{10}q^{\alpha k + k(k-1)/2}.$$

Thus, $|f(1-q^k)| \le C_9 q^{\alpha k}$ as stated. □

Proof of Theorem 2.2. (i) Let $f(1-q^k) = o(q^{\alpha k})$, $k \to \infty$. Take any $\varepsilon > 0$, then there exists a k_0 so that

$$|f(1-q^k)| < \varepsilon q^{\alpha k}, \quad k > k_0,$$

and since $f(1-q^k) \le M$ for all $k = 0, 1, \ldots$, we obtain:

$$|(\Lambda_q f)(z)| \le M\frac{P_{k_0}(z)}{|(-z;q)_\infty|} + \varepsilon\frac{(-q^\alpha|z|;q)_\infty}{|(-z;q)_\infty|},$$

where P_{k_0} is a polynomial of degree $\le k_0$ and, by Lemma 3.3, the last term is $O(|z|^{-\alpha})$. As a result, we write:

$$|(\Lambda_q f)(z)| < \varepsilon|z|^{-\alpha} + C_1\varepsilon|z|^{-\alpha} = C_2\varepsilon|z|^{-\alpha} \quad \text{for } |z| \text{ large enough.}$$

Thus, $|(\Lambda_q f)(z)| = o(|z|^{-\alpha})$, $z \in A_\delta$, $z \to \infty$.

(ii) Assume that (2.6) holds. Consider the function $h(z)$ defined by (4.2) and analytic in $D = \mathbb{C} \setminus [0,\infty)$. Condition (2.6) implies that

$$\lim_{x\to+\infty} h_\pm(x) = 0.$$

Given $\varepsilon > 0$, we choose R so that $|h_\pm(x)| < \varepsilon$ for $x > R$, x being on the boundary of D. Let $\omega(z)$ be the harmonic measure of the part of the boundary of D satisfying $|z| < R$. Since the reasoning of Theorem 2.1(i) implies that $h(z)$ is bounded in D- say, $|h(z)| \le M$ in D — we obtain by the Two Constants Theorem (cf., e.g., [19], p. 41):

$$\ln|f(z)| \le M\omega(z) + (1-\omega(z))\ln(\varepsilon) \to \ln(\varepsilon) \quad \text{as } z \to \infty$$

As a result, $\ln|f(z)| \le (1/2)\ln(\varepsilon)$ for $|z|$ large enough, which shows that $h(z) \to 0$ as $z \to \infty$ uniformly with respect to $\arg z$.

Now, for any $\varepsilon > 0$, we choose r_0 in such a way that $|h(z)| < \varepsilon$ for $|z| \ge r_0$. Hence, for $|z| \ge r_0$,

$$|g_f(z)| \le \varepsilon|z|^{-\alpha}\exp\left\{\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\}$$

and by the Cauchy estimates, we obtain for $|z| \ge r_0$:

$$|f(1-q^k)| \le \varepsilon\exp\left\{-(\alpha+k)\ln|z| + \frac{\ln^2|z|}{2\ln(1/q)} + \frac{\ln|z|}{2}\right\}.$$

The latter expression attains minimum if

$$\ln|z| = \left(\alpha + k - \frac{1}{2}\right)\ln(1/q) > \ln r_0 \quad \text{for } k \ge k_0.$$

Therefore, for $k \ge k_0$, we have $|f(1-q^k)| < \varepsilon q^{\alpha k}$, which implies $f(1-q^k) = o(q^{\alpha k})$, $k \to \infty$. □

Proof of Theorem 2.4. (i) Let f be m times differentiable (from the left) at 1. By Taylor's formula,

$$f(x) = T_m(x) + R_m(x),$$

where

$$T_m(x) = \sum_{j=0}^{m}\frac{(-1)^j f^{(j)}(1)}{j!}(1-x)^j$$

and

$$R_m(x) = o(1-x)^m, \quad x \to 1^-.$$

With the help of Lemma 3.1, we obtain:

$$(\Lambda_q T_m)(z) = \sum_{j=0}^{m} \frac{(-1)^j f^{(j)}(1)}{j!(-z;q)_j},$$

while Theorem 2.2(i) implies

$$(\Lambda_q R_m)(z) = o(|z|^{-m}), \quad z \in A_\delta,\ z \to \infty.$$

(ii) Now, let (4.3) hold. Consider the difference:

$$R_m(x) = f(x) - \sum_{j=0}^{m} a_j (1-x)^j.$$

Condition (4.3) implies that

$$R_m(x) = o(1-x)^m, \quad x = 1-q^k,\ k \to \infty.$$

The statement now follows. □

Proof of Theorem 2.8. The condition of the theorem implies that

$$g_f(x) = o(1), \quad x > 0,\ x \to +\infty. \tag{4.3}$$

In addition, the growth of $g_f(z)$ in $\mathbb{C}$ is restricted by

$$|g_f(z)| \le C\|f\| \exp\left\{\frac{\ln^2(|z|/\sqrt{q})}{2\ln(1/q)}\right\}.$$

By the Phragmén–Lindelöf Theorem (cf. [18], Chapter 6, Section 6.1), the entire function $g_f(z)$ is bounded in $\mathbb{C}$, that is, by Liouville's Theorem, $g_f(z)$ is constant. By virtue of (3.3), we conclude that $g_f(z) \equiv 0$. Thus, $f(1-q^k) = 0$ for all $k = 0, 1, \ldots$ as required. □

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