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A class of dissipative wave equations with time-dependent speed and damping

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ABSTRACT

We study the long time behavior of the energy for wave-type equations with time-dependent speed and damping:

$$u_{tt} - \lambda(t)^2 \Delta u + b(t) u_t = 0.$$

We investigate the interaction between the speed of propagation $\lambda(t)$ and the damping coefficient $b(t)$, showing how to describe the dissipative effect on the energy. We study a class of dissipations for which the equation keeps its hyperbolic structure and properties.

1. Introduction

Let us consider in $[0,\infty)\times\mathbb{R}^n$, with space dimension $n \geq 1$, the Cauchy problem

$$\begin{cases} u_{tt} - \lambda(t)^2 \Delta u + b(t) u_t + \lambda(t)\tilde{b}(t)\cdot\nabla u + e(t) u = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where by $\tilde{b}(t) = (b_j(t))_{j=1,\dots,n}$ we denote the vector with components $b_j(t)$, that is,

$$\tilde{b}(t)\cdot\nabla u = \sum_{j=1}^{n} b_j(t)\, u_{x_j}.$$

It is well known that if the coefficients are sufficiently regular and the equation is strictly hyperbolic, that is, $\lambda(t) > 0$, then the Cauchy problem (1) is globally well-posed in $\mathcal{C}^\infty$ and in all Sobolev spaces with no loss of regularity. However, if we consider the energy of the solution to (1) given by

$$E_\lambda(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \lambda(t)^2 \|\nabla u(t,\cdot)\|_{L^2}^2, \tag{2}$$

then we can observe many different effects for the behavior of $E(t)$ as $t\to\infty$, according to the properties of the speed of propagation $\lambda(t)$ and of the other coefficients of the equation.

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We first consider the Cauchy problem for the homogeneous equation:

$$u_{tt} - \lambda(t)^2 \Delta u = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x). \tag{3}$$

If $0 < \lambda_0 \le \lambda(t) \le \lambda_1$ for some $\lambda_0, \lambda_1 > 0$ then the energy $E_\lambda(t)$ is equivalent to

$$E_1(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2, \tag{4}$$

but the oscillations of $\lambda = \lambda(t)$ may have a deteriorating influence [1] on the energy behavior for the solution to (3). On the other hand, if $\lambda \in \mathcal{C}^2$ and

$$|\lambda^{(k)}(t)| \le C_k (1+t)^{-k}, \quad \text{for } k = 1, 2,$$

then the so-called *generalized energy conservation* property holds [2], that is,

$$C_0 E_1(0) \le E_1(t) \le C_1 E_1(0). \tag{5}$$

If $\lambda(t) \ge \lambda_0 > 0$ and $\lambda(t) \to \infty$ as $t \to \infty$ in (3) then one can prove the estimate

$$C_0(u_0, u_1)\lambda(t) \le E_\lambda(t) \le C_1 \lambda(t)\, E(0), \quad \text{where } E(0) := \left(\|u_0\|_{H^1} + \|u_1\|_{L^2}\right), \tag{6}$$

for the solution to (3), by assuming sufficient regularity for $\lambda(t)$ and some kind of control on its oscillations [3]. Referred to this energy, an increasing speed of propagation can be considered as a dissipative effect (since $\|\nabla u(t,\cdot)\| \le C_1 \lambda(t)^{-1} E(0)$). A fundamental difference with (5) is that in the right-hand side term of (6) it appears the H^1 norm of u_0, not only the L^2 norm of its gradient.

We address the interested reader to [4–7] for other results concerning (3).

Let us consider the wave equation with time-dependent damping term $b(t)u_t$, with $b(t) > 0$:

$$u_{tt} - \Delta u + b(t) u_t = 0. \tag{7}$$

The dissipation produced by $b(t)u_t$ may be classified [8] as *non effective* if the Eq. (7) has the same asymptotic properties of the free wave equation, *effective* if the equation inherits some properties related to the parabolic equation $b(t)u_t - \Delta u = 0$. In particular, if $tb(t) < 1$ for large times [9] or in the special case $b(t) = \mu(1+t)^{-1}$ for $\mu \in (0, 2]$ (see [10]), the following estimate holds for Eq. (7):

$$E_1(t) \le C\gamma(t)\, E(0), \quad \text{where } \gamma(t) := \exp\left(-\int_0^t b(\tau)d\tau\right). \tag{8}$$

In this case, the dissipation is *non effective* for the L^2–L^2 estimates of the energy.

We will not study *effective* dissipations in this paper, but we address the interested reader to [11–14]. Neither will we study L^p–L^q estimates, with $(p, q) \neq (2, 2)$ (see, for instance, [1,2,15]).

Theorem 2 extends energy estimates (6) and (8) to a more complex situation with a unified approach. In particular, we prove the energy estimate $E_\lambda(t) \le \lambda(t)\gamma(t)E(0)$ for the solution to (1), under suitable assumptions which take into account the interaction between the speed of propagation $\lambda(t)$ and the term $b(t)u_t$. In particular, $\lambda'(t) + b(t)\lambda(t)$ is *almost-positive* (see Definition 1).

Moreover, in Theorem 2 we assume hypotheses which allow us to exclude contributions to the energy behavior coming from the other coefficients, namely $b_j(t)$ and $e(t)$. On the other hand, in Theorem 3 we also include a possible damaging contribution to the energy estimate coming from the drift terms $b_j(t)u_{x_j}$.

The class of dissipation which we study are *non effective*, in the sense that the damping term $b(t)u_t$ produces a factor $\gamma(t)$ in the L^2–L^2 estimate of the energy, with respect to the estimate (6) for (3). In [16], we show how to extend this approach to higher order equations.

2. Main results

Notation 1. Let $f, g : [0, \infty) \to (0, \infty)$ be two strictly positive functions. We use the notation $f \approx g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for all $t \ge 0$. If the inequality is one-sided, namely, if $f(t) \le Cg(t)$ (resp. $f(t) \ge Cg(t)$) for all $t \ge 0$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

In particular $f \approx 1$ means that $C_1 \le f(t) \le C_2$ for some constants C_1, C_2.

Notation 2. Through this paper, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* (resp. *strictly increasing*, *decreasing*, *strictly decreasing*) if $f(x) \le f(y)$ (resp. $f(x) < f(y)$, $f(x) \ge f(y)$, $f(x) > f(y)$) for any $x, y \in \mathbb{R}$ such that $x < y$.

2.1. The almost-positivity property

In this paper, we will deal with long time integral inequalities to derive energy estimates. In this perspective, the following definition is useful to state our assumptions.

Definition 1. Let $a : [0, \infty) \to \mathbb{R}$ be a continuous function. We say that $a(t)$ is *almost-zero*, and we denote it by $a(t) =_{(a)} 0$, if there exists a constant $C > 0$ such that

$$-C \leq \int_0^t a(\tau)\, d\tau \leq C. \tag{9}$$

We say that $a(t)$ is *almost-positive*, and we denote it by $a(t) \geq_{(a)} 0$, (or, respectively, *almost-negative*, $a(t) \leq_{(a)} 0$) if there exists an *almost-zero* function $a_1(t)$ such that $a(t) - a_1(t) \geq 0$ (or, respectively, ≤ 0).

We say that two functions $a_1, a_2 : [0, \infty) \to \mathbb{R}$ are *almost-equal* and we write $a_1(t) =_{(a)} a_2(t)$, if $a_1(t) - a_2(t)$ is *almost-zero*, whereas we say that $a_1(t)$ is *almost-greater* than $a_2(t)$ and we write $a_1(t) \geq_{(a)} a_2(t)$, if $a_1(t) - a_2(t)$ is *almost-positive*.

Remark 1. Let $a : [0, \infty) \to \mathbb{R}$ be a continuous function, and let $A : [0, \infty) \to (0, \infty)$ be defined by

$$A(t) := \exp\left(\int_0^t a(\tau)\, d\tau\right). \tag{10}$$

Trivially, $a(t) =_{(a)} 0$ if, and only if, $A \approx 1$. Moreover, $a(t)$ is *almost-positive* (respectively *almost-negative*) if, and only if, there exists an increasing (respectively decreasing) function $A_2 : [0, \infty) \to (0, \infty)$ such that $A_2 \approx A$. Indeed, $a(t) \geq_{(a)} 0$ means that there exist $a_1(t) =_{(a)} 0$ and $a_2(t) \geq 0$ such that $a(t) = a_1(t) + a_2(t)$. It is clear that $A_1(t) = \exp \int_0^t a_1(\tau)\, d\tau$ verifies $A_1 \approx 1$ and that $A_2(t) = \exp \int_0^t a_2(\tau)\, d\tau$ is increasing. Since $A(t) = A_1(t)A_2(t)$, it follows that $A \approx A_2$.

Remark 2. Let $a(t) \geq_{(a)} 0$ and let $A : [0, \infty) \to (0, \infty)$ be as in (10). Let $f : [0, \infty) \to [0, \infty)$ be a continuous function. Then, for any $s \leq t$, we can estimate

$$\int_s^t A(\tau) f(\tau)\, d\tau \lesssim A(t) \int_s^t f(\tau)\, d\tau; \qquad A(s) \int_s^t f(\tau)\, d\tau \lesssim \int_s^t A(\tau) f(\tau)\, d\tau.$$

Similarly if $a(t) \leq_{(a)} 0$ or $a(t) =_{(a)} 0$.

Example 1. Let $a_1 : [0, \infty) \to (0, \infty)$ be a continuous, strictly positive, decreasing function with $a_1 \notin L^1$ (in particular, $a_1(t)$ may be constant). Let $a_2 : [0, \infty) \to \mathbb{R}$ be a continuous p-periodic non-constant function and let

$$\overline{a}_2 := \frac{1}{p} \int_0^p a_2(\tau)\, d\tau.$$

Then $a_1(t) a_2(t) =_{(a)} a_1(t)\overline{a}_2$. Indeed $a_1(t)(a_2(t) - \overline{a_2})$ satisfies (9) for

$$C = \max_{t \in (0,p)} \left| \int_0^t a_1(\tau)(a_2(\tau) - \overline{a_2})\, d\tau \right|.$$

For instance, $a_1(t) \sin t =_{(a)} 0$ with $C \leq 2a_1(0)$.

2.2. Main theorem

To state our assumptions on the coefficients of the equation in (1) we introduce some auxiliary functions.

Definition 2. Let $\lambda \in \mathcal{C}^2$ be a strictly positive function, with $\lambda \notin L^1$ and $\lambda(0) = 1$. We define

$$\Lambda(t) := 1 + \int_0^t \lambda(\tau) d\tau, \qquad \eta(t) := \frac{\lambda(t)}{\Lambda(t)}.$$

Let $b \in \mathcal{C}^1$ be a real-valued function. We define

$$\gamma(t) := \exp\left(-\int_0^t b(\tau) d\tau\right), \qquad \Gamma(t) := 1 + \int_0^t \gamma(\tau)\, d\tau, \quad \text{and} \quad \Gamma^\sharp(s) := \int_s^\infty \gamma(\tau)\, d\tau \quad \text{if } \gamma \in L^1.$$

Remark 3. The function $\eta(t)$ has the same regularity as $\lambda(t)$, it satisfies $\eta(0) = 1$, and

$$\frac{\eta'(t)}{\eta(t)} = \frac{\lambda'(t)}{\lambda(t)} - \eta(t). \tag{11}$$

We are now ready to state our first result, for which we assume $b_j \equiv 0$ and $e \equiv 0$ in (1).

Hypothesis 1. We assume that $\lambda(t)$ and $b(t)$ have *very slow oscillations* with respect to $\lambda(t)$, that is,

$$\frac{|\lambda^{(k)}(t)|}{\lambda(t)} + |b^{(k-1)}(t)| \lesssim \eta(t)^k, \quad \text{for } k = 1, 2. \tag{12}$$

Hypothesis 2. Following the notation in Definition 1, we assume that

$$0 \leq_{(a)} \frac{\lambda'(t)}{\lambda(t)} + b(t) \leq_{(a)} 2\left(\eta(t) - \frac{\gamma(t)}{\Gamma(t)}\right). \tag{13}$$

Theorem 1. *We assume Hypotheses 1 and 2. Then the solution to the Cauchy problem*

$$\begin{cases} u_{tt} - \lambda(t)^2 \Delta u + b(t) u_t = 0, \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{14}$$

satisfies the energy estimate

$$E_\lambda(t) \leq C \lambda(t) \gamma(t) E(0), \tag{15}$$

where $E_\lambda(t)$ is as in (2) and $E(0)$ is as in (6).

Remark 4. In particular, from (15) it follows that

$$\|\nabla u(t, \cdot)\|_{L^2}^2 \leq C \frac{\gamma(t)}{\lambda(t)} E(0),$$

that is, the *elastic energy* $\|\nabla u(t, \cdot)\|_{L^2}$ is bounded by a decreasing function, since $\gamma/\lambda \approx g$ for some decreasing $g(t)$.

Remark 5. The left-hand side and the right-hand side of (13) are the same inequality if, and only if,

$$\frac{\gamma(t)}{\Gamma(t)} \approx \eta(t). \tag{16}$$

Indeed, thanks to Remark 1 and to (11), it follows that $\Gamma\eta\gamma^{-1} \approx 1$ if, and only if,

$$\frac{\lambda'(t)}{\lambda(t)} + b(t) =_{(a)} \eta(t) - \frac{\gamma(t)}{\Gamma(t)}. \tag{17}$$

Remark 6. If $\gamma \in L^1$, i.e. $\Gamma(t)$ is bounded, then $\gamma(t)/\Gamma(t) =_{(a)} 0$, therefore (13) becomes

$$0 \leq \frac{\lambda'(t)}{\lambda(t)} + b(t) \leq_{(a)} 2\eta(t). \tag{18}$$

Remark 7. According to Remark 1, the left-hand side of (13) means that $\lambda(t)\gamma(t)^{-1} \approx g(t)$, for some increasing function $g(t)$. In particular, $\gamma \lesssim \lambda$. On the other hand, the right-hand side of (13) means that $\eta\Gamma(\lambda\gamma)^{-1/2} \approx g$ for some decreasing function $g(t)$. In fact, the right hand side of condition (13) can be weakened: to prove Theorem 1, it is sufficient that $\lambda'(t) + \lambda(t)b(t) \geq_{(a)} 0$ and that

$$\frac{\eta(t)\Gamma(t)}{\sqrt{\lambda(t)\gamma(t)}} \leq C. \tag{19}$$

Condition (19) is related to the request to have an estimate of the *pointwise energy* for small frequencies (i.e. *pseudo-differential* zone, see the proof of Theorem 2) which is not worse than the estimate obtained for large frequencies (i.e. *hyperbolic* zone).

2.3. Estimates from below and scattering results

Estimate (15) can be directly extended to an estimate from below if we restrict the space of initial data.

Definition 3. For any $\epsilon > 0$, we define

$$F_\epsilon := \left\{(u_0, u_1) \in H^1 \times L^2,\ \widehat{u_0}(\xi) = \widehat{u_1}(\xi) = 0 \text{ for any } |\xi| \leq \epsilon\right\}.$$

For any $\epsilon > 0$, F_ϵ is a closed subspace of the energy space $H^1 \times L^2$.

We remark that $\cup_{\epsilon>0} F_\epsilon$ is a dense subset of $H^1 \times L^2$.

Corollary 1. *We assume Hypotheses 1 and 2. Then the solution to the Cauchy problem (14) satisfies the energy estimate from below*

$$E_\lambda(t) \geq C(u_0, u_1)\, \lambda(t)\gamma(t), \tag{20}$$

where $C(u_0, u_1) > 0$ for nontrivial data. Moreover, if $(u_0, u_1) \in F_\epsilon$ for some $\epsilon > 0$, then there exist $C_{1,\epsilon}, C_{2,\epsilon} > 0$ such that

$$C_{1,\epsilon}\, \lambda(t)\gamma(t)\, E_1(0) \leq E_\lambda(t) \leq C_{2,\epsilon}\, \lambda(t)\gamma(t)\, E_1(0), \tag{21}$$

where $E_1(0) = \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}$ as in (4).

If the coefficient of the damping term is in L^1, then we have the following *scattering* result.

Corollary 2. *We assume Hypothesis 1. Moreover, we assume that $b \in L^1$ and that $0 \leq_{(a)} \lambda'(t)/\lambda(t) \leq_{(a)} 2\eta(t)$. Then, for any initial data $(u_0, u_1) \in H^1 \times L^2$ there exists $(v_0, v_1) \in H^1 \times L^2$ such that*

$$\lim_{t\to\infty} \frac{1}{\lambda(t)} \left\| \big(\lambda(t)\nabla u(t,\cdot) - \lambda(t)\nabla v(t,\cdot), u_t(t,\cdot) - v_t(t,\cdot)\big) \right\|_{L^2}^2 = 0, \tag{22}$$

where $u(t,x)$ is the solution to the Cauchy problem (14) and $v(t,x)$ is the solution to the Cauchy problem

$$v_{tt} - \lambda(t)^2 \Delta v = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x). \tag{23}$$

Moreover, for any $\epsilon > 0$ there exists a linear, bounded, invertible operator $W_+^\epsilon : F_\epsilon \to F_\epsilon$ such that

$$\frac{1}{\lambda(t)} \left\| \big(\lambda(t)\nabla u(t,\cdot) - \lambda(t)\nabla v(t,\cdot), u_t(t,\cdot) - v_t(t,\cdot)\big) \right\|_{L^2}^2 \leq C_\epsilon\, E_1(0) \left(\int_t^\infty |b(\tau)|\, d\tau \right)^2, \tag{24}$$

for some $C_\epsilon > 0$, where $u(t,x)$ is the solution to the Cauchy problem (14) with initial data $(u_0, u_1) \in F_\epsilon$, and $v(t,x)$ is the solution to the Cauchy problem (23) with initial data $(v_0, v_1) = W_+^\epsilon(u_0, u_1)$.

2.4. Additional terms which bring no contribution to the energy behavior

We can extend Theorem 1 to the complete equation in (1), stating sufficient conditions to exclude contributions to the energy long time behavior coming from $b_j(t)$, $e(t)$.

Hypothesis 3. Let $b_j \in \mathcal{C}^1$ and $e \in \mathcal{C}$, possibly complex-valued. Similarly to (12), we assume that

$$|b_j^{(k)}(t)| \lesssim \eta(t)^{k+1}, \quad \text{for } k = 0, 1 \text{ and } j = 1, \ldots, n, \quad \text{and} \quad |e(t)| \lesssim \eta(t)^2. \tag{25}$$

Hypothesis 4. We assume that $\Re\, b_j(t) =_{(a)} 0$ for any $j = 1, \ldots, n$ (see Definition 1).

Hypothesis 5. We assume that there exist two functions $g(t)$ and $h(s)$ such that

$$\int_s^t \gamma(\tau)\, d\tau \leq g(t)h(s), \quad \text{with } g(t)h(t) \leq \Gamma(t) \text{ and } g(t) \geq \Gamma(t), \tag{26}$$

where $g(t)$ is an increasing function with $g(0) = 1$ and $h(s)$ is a positive decreasing function, such that

$$\frac{\gamma}{gh} \approx \eta, \tag{27}$$

$$\frac{\eta(t)g(t)}{\sqrt{\lambda(t)\gamma(t)}} \exp\left(\int_0^t \frac{g(\tau)h(\tau)}{\gamma(\tau)}\, |e(\tau)| d\tau \right) \leq C. \tag{28}$$

In particular, condition (28) implies (19).

Hypothesis 6. We assume that

$$\sqrt{\frac{\gamma(t)}{\lambda(t)}} \int_0^t \sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\, \frac{|e(\sigma)|}{\eta(\sigma)} d\sigma \leq C. \tag{29}$$

Theorem 2. *We assume Hypotheses 1–6. Then the solution to (1) satisfies the energy estimate (15).*

2.5. Perturbation to the energy behavior coming from drift terms

The real part of the lower order terms $b_j(t)$ may bring contribution to the energy behavior if Hypothesis 4 does not hold. In this case, we replace it by the following.

Hypothesis 7. For any $j = 1, \dots, n$, we assume that $\Re b_j(t) \geq_{(a)} 0$ or $\Re b_j(t) \leq_{(a)} 0$.

Definition 4. From Hypothesis 7 and from Definition 1, it follows that for any $j = 1, \dots, n$, there exist two real-valued functions $b_{j,s}(t)$ and $b_{j,w}(t)$ such that $\Re b_j(t) = b_{j,s}(t) + b_{j,w}(t)$, where $b_{j,s}(t)$ has a constant sign and $b_{j,w}(t) =_{(a)} 0$. We define

$$\gamma_s(t) = \exp\left(\int_0^t b_s(\tau)\,d\tau\right), \qquad \text{where } b_s(t) := \max_{|\xi|=1} \sum_{j=1}^n |b_{j,s}(t)|\,|\xi_j|,$$

and we put

$$d(t) := \lambda(t)\gamma(t)\gamma_s(t). \tag{30}$$

In general, the contribution from $b_s(t)$ produces an energy estimate for the high frequencies of the *pointwise energy* worse than (15), since $\gamma_s(t) \geq 1$. Indeed, the energy might blow up as $t \to \infty$ (see Example 11).

We need to rewrite our conditions by means of $d(t)$ given in (30), rather than by means of $d(t) = \lambda(t)\gamma(t)$ as in Theorem 2. However, since we now expect an estimate worse than (15), we can relax some assumptions on the coefficients. In particular, the left-hand side of condition (13) is weakened to the following (31).

Hypothesis 8. We assume that

$$\frac{\lambda'(t)}{\lambda(t)} + b(t) + b_s(t) \geq_{(a)} 0, \tag{31}$$

and that there exist $g(t)$, $h(s)$ as in (26) which satisfy (27), and such that

$$\frac{\eta(t)g(t)}{\sqrt{d(t)}} \exp\left(\int_0^t \frac{g(\tau)h(\tau)}{\gamma(\tau)}\,|e(\tau)|d\tau\right) \leq C, \tag{32}$$

$$\frac{\gamma(t)}{\sqrt{d(t)}} \int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)} \frac{|e(\sigma)|}{\eta(\sigma)} d\sigma \leq C. \tag{33}$$

Theorem 3. *We assume Hypotheses 1 and 3 together with Hypotheses 7 and 8. Then the solution to (1) satisfies the following energy estimate:*

$$E_\lambda(t) \leq C\lambda(t)\gamma(t)\gamma_s(t)E(0). \tag{34}$$

Remark 8. In some cases, it is sufficient to take either $g(t) = \Gamma(t)$ and $h(s) = 1$, or $g(t) = 1$ and $h(s) = \Gamma^\sharp(s)$ if $\gamma \in L^1$ in (26), provided that (27) holds (see Remark 5). Indeed, in this case, (26) is satisfied thanks to (19). On the other hand, in some cases a different choice for (g, h) may be necessary, or it may be convenient to make condition (28) less restrictive (see Example 13).

2.6. The general second order equation

Theorem 3 can be applied for studying second-order equations which contain a term u_{tx}. For the ease of reading, we assume $n = 1$. Let us consider the equation

$$u_{tt} + 2\lambda_1(t)a_1(t)u_{tx} + \lambda_1(t)^2 a_2(t)u_{xx} + b(t)u_t = 0, \tag{35}$$

assuming that $a_1(t)^2 - a_2(t) > 0$. Let $a(t) := \sqrt{a_1(t)^2 - a_2(t)}$ and $\lambda(t) := \lambda_1(t)a(t)$. Then we can perform the change of variable $y = x - \int_0^t \lambda_1(\tau)a_1(\tau)\,d\tau$ obtaining

$$v_{tt} - \lambda(t)^2 v_{yy} + b(t)v_t + \lambda(t)b_1(t)v_y = 0, \tag{36}$$

where we put

$$b_1(t) = -\frac{a_1(t)}{a(t)}\left(b(t) + \frac{\lambda_1'(t)}{\lambda_1(t)} + \frac{a_1'(t)}{a_1(t)}\right).$$

In general, we cannot expect Hypothesis 4 to be satisfied, but we can use Theorem 3. In such a way, if the assumptions of Theorem 3 hold, from (34) one can obtain an estimate on the energy $E_\lambda(t)$ for $v(t,y)$:

$$E_\lambda(t) \lesssim \lambda(t)\gamma(t)\gamma_s(t)E(0) \lesssim \lambda(t)\gamma(t)\gamma_s(t)\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big), \tag{37}$$

since $v_0 = u_0$ and $v_1 = u_1 + \lambda_1(0)a_1(0)u_0$. We remark that

$$E_\lambda(t) \equiv \|v_t(t,\cdot)\|_{L^2}^2 + \lambda(t)^2\|v_y(t,\cdot)\|_{L^2}^2 = \|u_t(t,\cdot) + \lambda_1(t)a_1(t)u_x(t,\cdot)\|_{L^2}^2 + \lambda(t)^2\|u_x(t,\cdot)\|_{L^2}^2,$$

therefore from (37) we immediately get an estimate for $\lambda(t)^2\|u_x(t,\cdot)\|_{L^2}^2$. Moreover, by triangle inequality,

$$\|u_t(t,\cdot)\|_{L^2}^2 \le \|v_t(t,\cdot)\|_{L^2}^2 + \lambda_1(t)^2a_1(t)^2\|u_x(t,\cdot)\|_{L^2}^2 \equiv \|v_t(t,\cdot)\|_{L^2}^2 + \lambda(t)^2(a_1(t)/a(t))^2\|v_y(t,\cdot)\|_{L^2}^2,$$

thus we can roughly estimate

$$\|u_t(t,\cdot)\|_{L^2}^2 + \lambda(t)^2\|u_x(t,\cdot)\|_{L^2}^2 \lesssim \lambda(t)\gamma(t)\gamma_s(t)\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big), \tag{38}$$

provided that $|a_1(t)| \lesssim a(t)$, the so-called Colombini–Orrú condition [17] (see also [18]).

Example 2. Let us assume that

$$a_1(t) = \alpha a(t) \equiv \alpha\sqrt{a_1(t)^2 - a_2(t)}, \quad \text{that is, } a_2(t) = \frac{\alpha^2-1}{\alpha^2}\,a_1(t)^2,$$

for some $\alpha \neq 0$, with $a \approx 1$, together with $\lambda'(t) + b(t)\lambda(t) \ge 0$. Then $a_1'(t)/a_1(t) =_{(a)} 0$, hence it follows that

$$b_s(t) =_{(a)} |\alpha|\,\frac{\lambda'(\tau) + \lambda(t)b(t)}{\lambda(\tau)}, \qquad \gamma_s(t) = \left(\frac{\lambda(t)}{\gamma(t)}\right)^{|\alpha|}.$$

If the assumptions of Theorem 3 hold, from (38) we derive for the solution $u(t,x)$ to (35) that:

$$\|u_t(t,\cdot)\|_{L^2}^2 + \lambda(t)^2\|u_x(t,\cdot)\|_{L^2}^2 \le C\lambda(t)^{1+|\alpha|}\gamma(t)^{1-|\alpha|}\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big). \tag{39}$$

From (39), we obtain the following estimate on the *elastic energy* of the solution $u(t,x)$ to (35):

$$\|u_x(t,\cdot)\|_{L^2}^2 \le C\left(\frac{\gamma(t)}{\lambda(t)}\right)^{1-|\alpha|}\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big).$$

In particular, $\|u_x(t,\cdot)\|_{L^2}$ is bounded by a decreasing function if $|\alpha| \le 1$, that is, if $a_2(t) \le 0$.

3. Examples

We will present several examples for Theorem 1 and two special examples for Theorem 3. In Appendix we will show how to apply Theorem 2 if we add a coefficient $e(t)$ in Example 3.

Example 3. Let

$$b(t) := (1-\kappa)\frac{\lambda(t)}{\Lambda(t)} - \frac{\lambda'(t)}{\lambda(t)}, \tag{40}$$

for some $\kappa \in [-1,1]$. It follows that

$$\gamma(t) = \lambda(t)/\Lambda(t)^{1-\kappa}.$$

We distinguish three cases. If $\kappa \in (0,1]$ then

$$\Gamma(t) = 1 + \left(\Lambda(t)^\kappa - 1\right)/\kappa.$$

Condition (16) holds, hence Remark 6 is applicable and (13) is satisfied since from $\kappa \le 1$ it follows that

$$\frac{\lambda'(t)}{\lambda(t)} + b(t) = (1-\kappa)\eta(t) \ge 0.$$

If $\kappa \in [-1,0)$ then $\gamma \in L^1$ and condition (18) immediately follows, since $\kappa \ge -1$. We notice that $\Gamma^\sharp(t) = \Lambda(t)^\kappa/(-\kappa)$. Now let $\kappa = 0$, that is, $b(t) = -\eta'(t)/\eta(t)$ and $\gamma = \eta(t)$. Since $\Gamma(t) = 1 + \log\Lambda(t)$, it is easy to check that condition (13) holds. We notice that (16) does not hold in this case.

Summarizing, if (40) holds for some $\kappa \in [-1,1]$, Theorem 1 is applicable. We remark that $\lambda(t)\gamma(t) = \lambda(t)\eta(t)\Lambda(t)^\kappa$ in (15). In particular, $E_\lambda(t)$ vanishes as $t \to \infty$ if $\lambda(t)\eta(t)\Lambda(t)^\kappa \to 0$ as $t \to \infty$.

Remark 9. More in general, the reasoning in Example 3 holds if

$$b(t) =_{(a)} (1-\kappa)\frac{\lambda(t)}{\Lambda(t)} - \frac{\lambda'(t)}{\lambda(t)}. \tag{41}$$

Remark 10. We remark that $b(t)$ as in Example 3 means that the equation in (14) may be written in *scale-invariant* form, that is, if we put $w(\tau) \equiv w\big(\Lambda(t)\,|\xi|\big) := \widehat{u}(t,\xi)$, then for any $\xi \neq 0$ we obtain

$$w'' + w + \frac{1-\kappa}{\tau}\, w' = 0.$$

Example 4 (*The Case* $b \equiv 0$)**.** If $b \equiv 0$, then $\gamma(t) = 1$ and $\Gamma(t) = 1 + t$, therefore (13) becomes

$$0 \leq_{(a)} \frac{\lambda'(t)}{\lambda(t)} \leq_{(a)} 2\left(\eta(t) - \frac{1}{1+t}\right).$$

More in general, for any $b(t) =_{(a)} 0$, it holds $\gamma \approx 1$ and condition (19) holds if, and only if,

$$(1+t)\sqrt{\lambda(t)} \lesssim \Lambda(t). \tag{42}$$

We remark that (42) is a very natural condition (see, for instance, [7]) and it holds for a large class of functions $\lambda(t)$. Together with $\lambda'(t)/\lambda(t) \geq_{(a)} 0$, condition (42) is sufficient to apply Theorem 1. It is clear that $\lambda(t)\gamma(t) \approx \lambda(t)$ in (15).

Remark 11. Let us assume that $0 \leq_{(a)} b(t) \leq_{(a)} q\eta(t)$ for some $q \in (0,1)$. In this case, we can estimate

$$\Lambda(t)^{-q} \lesssim \gamma(t) \lesssim 1, \quad \text{hence } 1 + t\Lambda(t)^{-q} \lesssim \Gamma(t) \lesssim 1 + t.$$

Then we can be sure that (19) holds if we assume a condition stronger than (42):

$$(1+t)\sqrt{\lambda(t)} \lesssim \Lambda(t)^{1-q}. \tag{43}$$

Example 5. Let

$$b(t) = \frac{\mu\lambda(t)}{(e-1+\Lambda(t))(\log(e-1+\Lambda(t)))^{\kappa}},$$

for some $\mu > 0$ and $\kappa > 0$; then

$$\int_0^t b(\tau)d\tau = \begin{cases} \dfrac{\mu}{\kappa-1}\left(1 - \left(\dfrac{1}{\log(e-1+\Lambda(t))}\right)^{\kappa-1}\right) & \text{if } \kappa > 1, \\ \mu \log\log(e-1+\Lambda(t)) & \text{if } \kappa = 1, \\ \dfrac{\mu}{1-\kappa}\left((\log(e-1+\Lambda(t)))^{1-\kappa} - 1\right) & \text{if } 0 < \kappa < 1. \end{cases}$$

If $\kappa > 1$ then $b \in L^1$, that is, it does not influence the energy behavior (see also Corollary 2). If $\kappa \leq 1$ then we can expect some influence. For $\kappa = 1$, we can easily calculate

$$\gamma(t) = (\log(e-1+\Lambda(t)))^{-\mu},$$

whereas for $k \in (0,1)$, we obtain

$$\gamma(t) = e^{C_1}\exp\left(-\log(e-1+\Lambda(t))^{C_2}\right)^{1-\kappa}, \quad \text{where } C_1 = \mu/(1-k) \text{ and } C_2 = C_1^{1/(1-k)}.$$

In particular, in both cases we can follow Remark 11 for $q = \epsilon$ for any $\epsilon > 0$. Then (19) holds if

$$(1+t)\sqrt{\lambda(t)} \lesssim \Lambda(t)^{1-\epsilon}, \quad \text{for some } \epsilon > 0. \tag{44}$$

We may consider many different behaviors for the speed of propagation $\lambda(t)$. For the sake of brevity we only study polynomial and exponential growth and we briefly present some other examples.

Example 6. Let $\lambda(t) = (1+t)^{p-1}$ for some $p > 0$; then the function $\Lambda(t)$ has a *polynomial* growth. We obtain:

$$\Lambda(t) = \frac{(1+t)^p + (p-1)}{p}, \qquad \frac{\lambda'(t)}{\lambda(t)} = \frac{p-1}{1+t},$$
$$\eta(t) = \frac{p}{(1+t)(1+(p-1)(1+t)^{-p})} =_{(a)} \frac{p}{1+t}.$$

Let $b(t) = \mu/(1+t)$, for some $\mu \in \mathbb{R}$; we remark that the case $b \equiv 0$ (Example 4) is included. Then $b(t)$ satisfies (41) for $\kappa = (1-\mu)/p$, provided that $-p+1 \leq \mu \leq p+1$, hence Theorem 1 is applicable. We remark that $\lambda(t)\gamma(t) = (1+t)^{p-1-\mu}$ in (15). In particular, $E_\lambda(t)$ vanishes as $t \to \infty$ if $\mu \in (p-1, p+1]$.

Similarly to Example 5, now let

$$b(t) = \frac{\mu}{(1+t)(\log(e+t))^\kappa},$$

for some $\mu > 0$ and $\kappa \in (0,1]$. It is easy to check that the left-hand side of (13) holds if, and only if, $p \geq 1$. On the other hand, condition (44) is satisfied for any $p > 1$.

Example 7. Let $\lambda(t) = e^{pt}$ for some $p > 0$; then

$$\Lambda(t) = \frac{p-1+e^{pt}}{p}, \qquad \frac{\lambda'(t)}{\lambda(t)} = p, \qquad \eta(t) = \frac{pe^{pt}}{p-1+e^{pt}} =_{(a)} p.$$

Conditions (12)–(25) are satisfied if b, b', b_1, b_1', e are bounded. Let $b(t) = \mu$ for some $\mu \neq 0$. Then $b(t)$ satisfies (41) for $\kappa = (1-\mu)/p$, provided that $0 < |\mu| \leq p$. Therefore Theorem 1 is applicable, as in Example 3. We remark that $\lambda(t)\gamma(t) = e^{(p-\mu)t}$ in (15). In particular, $E_\lambda(t)$ is bounded if $\mu = p$.

If $b \equiv 0$, as in Example 4, the left-hand side of condition (13) and condition (42) hold for any $p > 0$. Now let $b(t) = \mu(1+t)^{-\kappa}$ for some $\mu > 0$ and $\kappa \in (0,1]$, as in Example 5. The left-hand side of (13) and (19) hold for any $p > 0$ (see also (44)).

Example 8. Let $\Lambda(t) = e^{e^t - 1}$; then

$$\lambda(t) = e^t e^{e^t-1}, \qquad \frac{\lambda'(t)}{\lambda(t)} = e^t + 1, \qquad \eta(t) = e^t.$$

If $b(t) =_{(a)} \mu e^t$ for some $\mu \in (0,1)$ then $\Gamma(t)$ is bounded and condition (18) holds.

Let $\lambda(t) = qt^{q-1}e^{t^q} + 1$ for some $q \in (1,\infty)$; then

$$\Lambda(t) = e^{t^q} + t, \qquad \frac{\lambda'(t)}{\lambda(t)} =_{(a)} qt^{q-1} + (q-1)t^{q-2}, \qquad \eta(t) =_{(a)} qt^{q-1}.$$

If $b(t) =_{(a)} \mu t^{q-1}$ for some $\mu \in (0,q)$ then $\Gamma(t)$ is bounded and condition (18) holds.

Let $\lambda(t) = r(1+t)^{-(1-r)}e^{(1+t)^r-1} + (1-r)$ for some $r \in (0,1)$; then

$$\Lambda(t) = e^{(1+t)^r-1} + (1-r)t, \qquad \frac{\lambda'(t)}{\lambda(t)} =_{(a)} \frac{r}{(1+t)^{1-r}} - \frac{1-r}{1+t}, \qquad \eta(t) =_{(a)} \frac{r}{(1+t)^{1-r}}.$$

If $b(t) =_{(a)} \mu(1+t)^{-(1-r)}$ for some $\mu \in (0,r]$ then $\Gamma(t)$ is bounded and condition (18) holds.

Example 9. Let $\eta \gtrsim 1$ as in Examples 7 and 8. Let us assume that Hypotheses 1 and 2 hold for some $b(t)$. Let $a : [0,\infty) \to \mathbb{R}$ be a $\mathcal{C}^1$ p-periodic non-constant function, with $\int_0^p a(\tau)\,d\tau = 0$. Then Hypotheses 1 and 2 also hold if we replace $b(t)$ by $b(t) + a(t)$. Indeed, $a(t) =_{(a)} 0$ (see Example 1) and it satisfies (12), since $|a'(t)|$ is bounded.

If we do not exclude the chance that the real parts of $b_j(t)$ bring a contribution to the energy behavior, then we may apply Theorem 3 and additional effects may appear.

Example 10. We follow Example 3, but now we take $\Re b_j(t) =_{(a)} \ell_j\,\eta(t)$ for some $\ell_j \in \mathbb{R}$. Then we obtain

$$d(t) = \lambda(t)\eta(t)\Lambda(t)^{\kappa+\ell}, \quad \text{where } \ell := \max_{|\xi|=1} \sum_{j=1}^n |\ell_j|\,|\xi_j|,$$

since $\gamma_s(t) = \Lambda(t)^\ell$. Condition (31) is satisfied if, and only if, $|\kappa| \leq \ell + 1$. Let us take $(g,h) = (\Gamma, 1)$ if $\kappa \in (0,1]$, $(g,h) = (1, \Gamma^\sharp)$ if $\kappa \in [-1,0)$, or $g(t) = \Lambda(t)^\epsilon/\epsilon$ and $h(s) = \Lambda(s)^{-\epsilon}$ for some $\epsilon \in (0,1]$ if $\kappa = 0$. Then condition (27) holds.

In particular, let $n = 1$. If $b_1(t) =_{(a)} \mu_1/(1+t)$ for $\mu_1 \neq 0$ in Example 6, then (31) gives $1-(p+|\mu_1|) \leq \mu \leq (p+|\mu_1|)+1$ and $d(t) = (1+t)^{p-1+|\mu_1|-\mu}$ in (34). If $b_1(t) =_{(a)} \mu_1$ for $\mu_1 \neq 0$ in Example 7 it gives $0 < |\mu| \leq p + |\mu_1|$ and $d(t) = e^{(p+|\mu_1|-\mu)t}$ in (34).

The following example shows that the energy may blow up as $t \to \infty$ if Hypothesis 4 does not hold.

Example 11. Let $n = 1$. We consider the Cauchy problem

$$u_{tt} - (1+t)^2 u_{xx} - d_0 u_x = 0, \qquad u(0,x) = u_0(x) \in C_0^\infty, \qquad \partial_t u(0,x) \equiv 0,$$

that is, $\lambda(t) = 1+t$, $b \equiv 0$, $b_1(t) = -d_0/(1+t)$. Here $\gamma(t) \equiv 1$ and $\gamma_s(t) = (1+t)^{d_0}$. Theorem 3 gives

$$E_\lambda(t) \le C(1+t)^{d_0+1} E(0),$$

in particular $\|u_x(t,\cdot)\|_{L^2}^2 \le C(1+t)^{d_0-1}E(0)$. The same long time asymptotic behavior appears for

$$v_{tt} - t^2 v_{xx} - d_0 v_x = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) \equiv 0, \tag{45}$$

where $d_0 = 4k+1$, with $k \in \mathbb{N}$. The unique solution has the form

$$v(t,x) = \sum_{j=0}^{k} \frac{\sqrt{\pi}\, t^{2j}}{j!(n-j)!\Gamma\left(j+\frac{1}{2}\right)} (\partial_x^j v_0)(x + t^2/2),$$

where Γ is the Euler *Gamma function* (see [19]). Therefore $\|v_x(t,\cdot)\|_{L^2}^2 \approx t^{4k} = t^{d_0-1}$ as $t \to \infty$ for a suitable choice of initial data. We remark that in the Example above, loss of regularity of the solution also appears, since the equation in (45) is *weakly hyperbolic* at $t = 0$. Since we are interested in long-time behavior for strictly hyperbolic equations, we replace t^2 with $(1+t)^2$.

4. Proofs

In the following we will use a *sharp* Gronwall-type integral inequality which plays a fundamental role in our energy estimates. It follows as corollary of Theorem 1.5 in [20].

Lemma 1. *Let $u(t)$ and $b(t)$ be continuous functions in $J = [\alpha, \beta]$, and let $a(t)$ be a Riemann-integrable function in J. Suppose that $a(t)$ and $b(t)$ are non negative in J.*

$$\text{If } u(t) \le a(t) + a(t)\int_\alpha^t b(\sigma)u(\sigma)\, d\sigma, \tag{46}$$

$$\text{then } u(t) \le a(t) \exp\left(\int_\alpha^t a(\tau)b(\tau)\, d\tau\right). \tag{47}$$

Moreover if we replace $\le$ with $=$ (or, resp., with $\ge$) in (46) then (47) still holds by replacing $\le$ with $=$ (or, resp., with $\ge$), that is, estimate (47) is sharp.

In the following we will prove at the same time Theorems 1 and 2, since only minor changes appear. Later we will prove Theorem 3.

Notation 3. If $v = (v_1, \dots, v_m)$ is a vector in $\mathbb{C}^m$, then we denote by diag$v$ or diag$(v_1, \dots, v_m)$ the $m \times m$ diagonal matrix $M = (M_{ij})$ with entries $M_{ii} = v_i$ and $M_{ij} = 0$ for any $i \ne j$. On the other hand, if $M = (M_{ij})$ is a square matrix, then we denote the diagonal part of M by Diag$M$, that is, $(\text{Diag}M)_{ii} = M_{ii}$, and $(\text{Diag}M)_{ij} = 0$ if $i \ne j$.

Proof of Theorems 1 and 2. We perform the Fourier transform of (1) with respect to x obtaining

$$\begin{cases} \widehat{u}_{tt} + |\xi|^2\lambda(t)^2\widehat{u} + b(t)\widehat{u}_t + i|\xi|\lambda(t)b^\sharp(t,\xi)\widehat{u} + e(t)\widehat{u} = 0, \\ \left(\widehat{u}(0,\xi), \widehat{u}_t(0,\xi)\right) = \left(\widehat{u_0}(\xi), \widehat{u_1}(\xi)\right), \end{cases} \tag{48}$$

where we put

$$b^\sharp(t,\xi) := \frac{1}{|\xi|}\sum_{j=1}^n b_j(t)\xi_j.$$

We claim that

$$\mathcal{E}(t,\xi) \le C\lambda(t)\gamma(t)\mathcal{E}_0(\xi), \tag{49}$$

uniformly with respect to $\xi \in \mathbb{R}^n$, where $\mathcal{E}(t,\xi)$ and $\mathcal{E}_0(\xi)$ are the *pointwise energies* given by

$$\mathcal{E}(t,\xi) := |\widehat{u}_t(t,\xi)|^2 + |\xi|^2\lambda(t)^2|\widehat{u}(t,\xi)|^2, \tag{50}$$

$$\mathcal{E}_0(\xi) := |\widehat{u_1}(\xi)|^2 + (1+|\xi|^2)|\widehat{u_0}(\xi)|^2. \tag{51}$$

Indeed, by integrating this inequality with respect to ξ and by Plancherel's Theorem, estimate (15) will follow from (49). In order to prove (49), for some constant $N > 0$, we divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into the *pseudo-differential zone* $Z_{\mathrm{pd}}(N)$ and into the *hyperbolic zone* $Z_{\mathrm{hyp}}(N)$, defined by

$$Z_{\mathrm{pd}}(N) = \{(t,\xi)\in[0,\infty)\times\mathbb{R} : \Lambda(t)|\xi| \le N\},$$
$$Z_{\mathrm{hyp}}(N) = \{(t,\xi)\in[0,\infty)\times\mathbb{R} : \Lambda(t)|\xi| \ge N\}.$$

Since $\Lambda : [0,\infty)\to[1,\infty)$ is strictly increasing and surjective, the function which describes the zone boundaries is given by

$$\theta : (0,N]\to[0,\infty), \qquad \theta_{|\xi|} = \Lambda^{-1}(N/|\xi|).$$

We put also $\theta_0=\infty$, and $\theta_{|\xi|}=0$ for any $|\xi|>N$. The pair (t,ξ) in the extended phase space is in $Z_{\mathrm{pd}}(N)$ (resp. in $Z_{\mathrm{hyp}}(N)$) if, and only if, $t\le\theta_{|\xi|}$ (resp. $t\ge\theta_{|\xi|}$). In $Z_{\mathrm{hyp}}(N)$ we put

$$U = (i|\xi|\lambda(t)\widehat{u}, \widehat{u}_t), \qquad U_0(\xi) := \left(i|\xi|\lambda(\theta_{|\xi|})\widehat{u}(\theta_{|\xi|},\xi), \widehat{u}_t(\theta_{|\xi|},\xi)\right),$$

so that from (48) we derive the system

$$\partial_t U = \begin{pmatrix}0&1\\1&0\end{pmatrix} i|\xi|\lambda(t)U + \begin{pmatrix}\dfrac{\lambda'(t)}{\lambda(t)} & 0\\ -b^\sharp(t,\xi) & -b(t)\end{pmatrix}U + \begin{pmatrix}0&0\\-e(t)&0\end{pmatrix}(i|\xi|\lambda(t))^{-1}U, \tag{52}$$

for $t\ge\theta_{|\xi|}$, with initial datum $U(\theta_{|\xi|},\xi)=U_0(\xi)$. We remark that the *pointwise energy* $\mathcal{E}(t,\xi)$ in (50) is equivalent to $|U(t,\xi)|^2$ (in particular, $\mathcal{E}(\theta_{|\xi|},\xi)\approx|U_0(\xi)|^2$). Moreover, for $|\xi|\ge N$, that is, $\theta_{|\xi|}=0$, it holds

$$|U_0(\xi)|^2 = |\xi|^2\,|\widehat{u_0}(\xi)|^2 + |\widehat{u_1}(\xi)|^2 \le C\,\mathcal{E}_0(\xi).$$

Let P be the (constant, unitary) diagonalizer of the principal part of (52), given by

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}, \qquad P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix},$$

that is, if we put $V(t,\xi)=P^{-1}U(t,\xi)$, then we obtain

$$\partial_t V = \begin{pmatrix}-1&0\\0&1\end{pmatrix} i\,|\xi|\,\lambda(t)V + B_0(t,\xi)\,V + B_1(t)(i\,|\xi|\,\lambda(t))^{-1}V, \tag{53}$$

where

$$B_0(t,\xi) = \frac12\frac{\lambda'(t)}{\lambda(t)}\begin{pmatrix}1&1\\1&1\end{pmatrix} - \frac12 b^\sharp(t,\xi)\begin{pmatrix}-1&-1\\1&1\end{pmatrix} - \frac12 b(t)\begin{pmatrix}1&-1\\-1&1\end{pmatrix}, \qquad B_1(t) = -\frac{e(t)}{2}\begin{pmatrix}-1&-1\\1&1\end{pmatrix}.$$

We define the *refined diagonalizer* which depends on the not diagonal entries of $B_0(t,\xi)$:

$$K(t,\xi) := \begin{pmatrix}1 & \dfrac{h_+(t,\xi)}{2i\,|\xi|\,\lambda(t)}\\ -\dfrac{h_-(t,\xi)}{2i\,|\xi|\,\lambda(t)} & 1\end{pmatrix}, \qquad h_\pm := \frac12\left(\frac{\lambda'}{\lambda}+b\pm b^\sharp\right). \tag{54}$$

Thanks to Hypotheses 1 and 3, in $Z_{\mathrm{hyp}}(N)$ we have

$$\frac{|h_\pm(t,\xi)|}{|\xi|\,\lambda(t)} \le \frac{C\,\eta(t)}{|\xi|\,\lambda(t)} = \frac{C}{|\xi|\,\Lambda(t)} \le \frac{C}{N}, \tag{55}$$

hence $|\det K|\ge 1-C^2/N^2$. Therefore, $K(t,\xi)$, $K^{-1}(t,\xi)$ are bounded for a sufficiently large N, which depends only on the constants in Hypotheses 1 and 3. We replace $V(t,\xi)=K(t,\xi)W(t,\xi)$ and we get

$$\partial_t W = \begin{pmatrix}-1&0\\0&1\end{pmatrix}\varphi(t,\xi)W + f(t)W + J(t,\xi)W, \tag{56}$$

for $t\ge\theta_{|\xi|}$, where $\varphi(t,\xi)$ and $f(t)$ are scalar functions given by

$$\varphi(t,\xi) = i\,|\xi|\,\lambda(t) - \frac12 b^\sharp(t,\xi), \qquad f(t) = \frac12\left(\frac{\lambda'(t)}{\lambda(t)} - b(t)\right),$$

and the matrix $J(t,\xi)=K^{-1}(t,\xi)R(t,\xi)$ is derived (see Notation 3) by

$$\begin{aligned}R&=(i\,|\xi|\,\lambda)(D_0K-KD_0)+B_0K-K_t-K\mathrm{Diag}B_0+(i\,|\xi|\,\lambda)^{-1}B_1K\\&=-H\mathrm{Diag}B_0-H_t+B_0H+(i\,|\xi|\,\lambda)^{-1}B_1K,\end{aligned}$$

where we put $D_0=\mathrm{diag}(-1,1)$ and $H(t,\xi)=K(t,\xi)-\mathrm{I}_2$. Thanks to Hypotheses 1 and 3, the matrices $R(t,\xi)$ and $J(t,\xi)$, satisfy the following estimate in $Z_{\mathrm{hyp}}(N)$:

$$\|R(t,\xi)\|,\ \|J(t,\xi)\|\lesssim\frac{\eta(t)^2}{|\xi|\,\lambda(t)}.\tag{57}$$

Now let

$$D(t,\xi):=\mathrm{diag}\left(\exp\left(\int_{\theta_{|\xi|}}^t\varphi(\tau,\xi)\,d\tau\right),\ \exp\left(-\int_{\theta_{|\xi|}}^t\varphi(\tau,\xi)\,d\tau\right)\right).\tag{58}$$

Since $\lambda(t)$ is real-valued and $\Re b_j(t)=_{(\mathrm{a})}0$ for any $j=1,\dots,n$, it holds $\|D(t,\xi)\|,\ \|D^{-1}(t,\xi)\|\le C$. On the other hand, the term $f(t)$ brings a scalar contribution, that is,

$$\sqrt{d(t,\xi)}:=\exp\left(\int_{\theta_{|\xi|}}^t f(\tau)\,d\tau\right)=\sqrt{\frac{\lambda(t)\gamma(t)}{\lambda(\theta_{|\xi|})\gamma(\theta_{|\xi|})}}.$$

We put $W(t,\xi)=\sqrt{d(t,\xi)}\,D(t,\xi)Z(t,\xi)$ and we obtain in $Z_{\mathrm{hyp}}(N)$,

$$\begin{cases}\partial_tZ=\widetilde{J}(t,\xi)\,Z, & t\ge\theta_{|\xi|},\\ Z(\theta_{|\xi|},\xi)=K^{-1}(\theta_{|\xi|},\xi)P^{-1}\,U(\theta_{|\xi|},\xi),\end{cases}\tag{59}$$

where the matrix $\widetilde{J}(t,\xi)=D^{-1}(t,\xi)J(t,\xi)D(t,\xi)$ satisfies again (57). For any $s,t\ge\theta_{|\xi|}$, we have

$$\int_s^t\|\widetilde{J}(\tau,\xi)\|\,d\tau\le C\int_{\theta_{|\xi|}}^\infty\frac{\lambda(\tau)}{|\xi|\ \Lambda(\tau)^2}\,d\tau\le\frac{C'}{|\xi|\ \Lambda(\theta_{|\xi|})}=\frac{C'}{N},$$

hence $|Z(t,\xi)|\le C\|Z(\theta_{|\xi|},\xi)\|$ and, by using Liouville's formula, $|Z(t,\xi)|\ge C'\|Z(\theta_{|\xi|},\xi)\|$. We have proved that in $Z_{\mathrm{hyp}}(N)$ it holds

$$C_1\,d(t,\xi)\,|U(\theta_{|\xi|},\xi)|^2\le|U(t,\xi)|^2\le C_2\,d(t,\xi)\,|U(\theta_{|\xi|},\xi)|^2.\tag{60}$$

We remark that (60) is a two-sided estimate where the same function $d(t,\xi)$ appears in both sides, that is, we have a precise description of the behavior of the energy in $Z_{\mathrm{hyp}}(N)$. In particular (60) concludes the proof of our claim (49) in $Z_{\mathrm{hyp}}(N)$, and it gives

$$\mathcal{E}(t,\xi)\approx d(t,\xi)\,\mathcal{E}(\theta_{|\xi|},\xi),\quad\text{for any } t\ge\theta_{|\xi|},\tag{61}$$

where $\mathcal{E}(t,\xi)$ is the *pointwise energy* in (50). Now we consider $Z_{\mathrm{pd}}(N)$ and we put

$$V=(i\eta(t)\widehat{u},\widehat{u}_t),\qquad V_0(\xi)=(i\widehat{u_0}(\xi),\widehat{u_1}(\xi)),\quad\text{and}\quad V=\sqrt{\lambda(t)\gamma(t)}\widetilde{V},$$

so that

$$\partial_t\widetilde{V}=\mathcal{A}(t,\xi)\widetilde{V}\equiv\begin{pmatrix}\dfrac{\eta'}{\eta}+\dfrac{b}{2}-\dfrac{\lambda'}{2\lambda} & i\eta\\ \dfrac{i\,|\xi|^2\,\lambda^2-|\xi|\,\lambda b^\sharp+ie}{\eta} & -\dfrac{b}{2}-\dfrac{\lambda'}{2\lambda}\end{pmatrix}\tilde{V},\qquad V(0,\xi)=V_0(\xi).\tag{62}$$

Since we can estimate the *pointwise energy* $\mathcal{E}(t,\xi)$ in (50) by

$$\mathcal{E}(t,\xi)\equiv|\widehat{u}_t(t,\xi)|^2+\lambda(t)^2\xi^2|\widehat{u}(t,\xi)|^2\le C_N\left(|\widehat{u}_t(t,\xi)|^2+\eta(t)^2|\widehat{u}(t,\xi)|^2\right),$$

and, on the other hand, $|V_0(\xi)|^2\equiv|\widehat{u_0}(\xi)|^2+|\widehat{u_1}(\xi)|^2\le\mathcal{E}_0(\xi)$, we have to prove that the fundamental solution $E(t,\xi)$ to (62), i.e.

$$\partial_tE=\mathcal{A}(t,\xi)E,\qquad E(0,\xi)=\mathrm{I}_2,$$

is bounded. If we put $E = (E_{ij})_{i,j=1,2}$, then we can write for $j = 1, 2$ the following integral equations:

$$E_{1j} = \eta(t)\frac{1}{\sqrt{\gamma(t)\lambda(t)}}\left(\delta_{1j} + i\int_0^t \sqrt{\gamma(\tau)\lambda(\tau)}E_{2j}(\tau,\xi)d\tau\right), \tag{63}$$

$$E_{2j} = \sqrt{\frac{\gamma(t)}{\lambda(t)}}\left(\delta_{2j} + \int_0^t \sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\frac{a(\sigma,\xi)}{\eta(\sigma)}E_{1j}(\sigma,\xi)d\sigma\right), \tag{64}$$

where we put

$$a(\sigma,\xi) := i\,|\xi|^2\,\lambda(\sigma)^2 - |\xi|\,\lambda(\sigma)b^\sharp(\sigma,\xi) + ie(\sigma). \tag{65}$$

By replacing (64) into (63) we obtain

$$E_{1j} = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) = \frac{\eta(t)}{\sqrt{\gamma(t)\lambda(t)}}\,\delta_{1j} + i\,\frac{\eta(t)(\Gamma(t)-1)}{\sqrt{\gamma(t)\lambda(t)}}\,\delta_{2j} \tag{66}$$

$$+\frac{i\eta(t)}{\sqrt{\gamma(t)\lambda(t)}}\int_0^t \gamma(\tau)\left(\int_0^\tau \sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\frac{a(\sigma,\xi)}{\eta(\sigma)}E_{1j}(\sigma,\xi)d\sigma\right)d\tau. \tag{67}$$

Thanks to (19) the terms (I) and (II) in (66) are bounded. We consider the term (III) in (67), that is, integrating by parts,

$$(\mathrm{III}) = \frac{i\eta(t)}{\sqrt{\gamma(t)\lambda(t)}}\int_0^t \sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\frac{a(\sigma,\xi)}{\eta(\sigma)}E_{1j}(\sigma,\xi)\left(\int_\sigma^t \gamma(\tau)d\tau\right)d\sigma.$$

To prove that $E_{1j}(t,\xi)$ is bounded we use Lemma 1. Using the left-hand side of (13) and $|\xi|\,\Lambda(t) \le N$, by virtue of Remark 2 we can estimate

$$|\xi|^2\,\lambda(\sigma)^2\int_\sigma^t \gamma(\tau)d\tau \le C\gamma(\sigma)\,|\xi|^2\,\lambda(\sigma)\int_\sigma^t \lambda(\tau)d\tau \le CN\gamma(\sigma)|\xi|\lambda(\sigma). \tag{68}$$

If we are proving Theorem 1, the estimate of $E_{1j}(t,\xi)$ immediately follows from Lemma 1 by virtue of (19), since

$$|E_{1j}(t,\xi)| \le \frac{\eta(t)\Gamma(t)}{\sqrt{\gamma(t)\lambda(t)}}\exp\left(CN\int_0^t |\xi|\lambda(\sigma)d\sigma\right) \le e^{CN^2}\frac{\eta(t)\Gamma(t)}{\sqrt{\gamma(t)\lambda(t)}}.$$

If we are proving Theorem 2 then we have to take into account $b^\sharp(t,\xi)$ and $e(t)$. In this case, we use (26), so that

$$|E_{1j}(t,\xi)| \le \frac{\eta(t)\Gamma(t)}{\sqrt{\gamma(t)\lambda(t)}} + CN\,\frac{\eta(t)}{\sqrt{\gamma(t)\lambda(t)}}\int_0^t \frac{\sqrt{\lambda(\sigma)\gamma(\sigma)}}{\eta(\sigma)}\,|\xi|\lambda(\sigma)\,|E_{1j}(\sigma,\xi)|d\sigma$$
$$+\frac{\eta(t)g(t)}{\sqrt{\gamma(t)\lambda(t)}}\int_0^t h(\sigma)\sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\frac{|\xi||b^\sharp(\sigma,\xi)|\lambda(\sigma)+|e(\sigma)|}{\eta(\sigma)}|E_{1j}(\sigma,\xi)|d\sigma.$$

Since $1 \le \Gamma(t) \le g(t)$, by applying Lemma 1, thanks to (26), it follows that

$$|E_{1j}(t,\xi)| \le \frac{\eta(t)g(t)}{\sqrt{\gamma(t)\lambda(t)}}\exp\left(\int_0^t f(\sigma,\xi)d\sigma\right),$$

where we take

$$f(\sigma,\xi) = CN\,|\xi|\lambda(\sigma) + \frac{g(\sigma)h(\sigma)}{\gamma(\sigma)}\left(|\xi|\lambda(\sigma)|b^\sharp(\sigma,\xi)| + |e(\sigma)|\right).$$

Since $\gamma/gh \approx \eta$ by virtue of (27) and $|b^\sharp(\sigma,\xi)| \le C'\eta(\sigma)$ thanks to (25), we can estimate

$$\int_0^t CN\,|\xi|\lambda(\sigma)\,d\sigma + \int_0^t |\xi|\lambda(\sigma)\frac{g(\sigma)h(\sigma)}{\gamma(\sigma)}|b^\sharp(\sigma,\xi)|\,d\sigma \le CN^2 + C_1N.$$

Thanks to (28), since $\eta g \lesssim \sqrt{\gamma\lambda}$ by virtue of (26), it follows that $E_{1j}(t,\xi)$ is bounded. By the boundedness of E_{1j}, using (12)–(25) and (13) (see Remark 2), together with (29), we can estimate (64) by

$$|E_{2j}| \le \sqrt{\frac{\gamma(t)}{\lambda(t)}}\left(1 + C\int_0^t \sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\frac{|a(\sigma,\xi)|}{\eta(\sigma)}d\sigma\right)$$

$$\leq C_1 + C_2 \int_0^t |\xi|^2 \lambda(\sigma)\Lambda(\sigma)d\sigma + C_3|\xi| \int_0^t \lambda(\sigma)d\sigma + C\sqrt{\frac{\gamma(t)}{\lambda(t)}} \int_0^t \sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}} \frac{|e(\sigma)|}{\eta(\sigma)} d\sigma \leq C'.$$

We proved that $E(t, \xi)$ is bounded, that is, $|\widetilde{V}(t, \xi)| \leq C|V_0(\xi)|$, therefore

$$\mathcal{E}(t, \xi) \leq C' \lambda(t)\gamma(t)\, \mathcal{E}_0(\xi), \quad \text{for any } |\xi| \leq N \text{ and } t \leq \theta_{|\xi|}. \tag{69}$$

We remark that (69) is a one-side estimate. By combining (69) with the estimate from above in (61), we conclude the proof of (49). □

Proof of Theorem 3. With the notation in Definition 4, let

$$b_{\mathrm{w}}^\sharp(t, \xi) := \frac{1}{|\xi|} \sum_{j=1}^n b_{j,\mathrm{w}}(t)\xi_j, \qquad b_{\mathrm{s}}^\sharp(t, \xi) := \frac{1}{|\xi|} \sum_{j=1}^n b_{j,\mathrm{s}}(t)\xi_j.$$

Thanks to Hypothesis 7, it is clear that

$$b_{\mathrm{s}}(t) = \max_{\xi\in\mathbb{R}^n} |b_{\mathrm{s}}^\sharp(t, \xi)|. \tag{70}$$

We follow the proof of Theorem 2 but we write (56) in the form

$$\partial_t W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \varphi_{\mathrm{w}}(t, \xi)W + f_{\mathrm{s}}(t)W + G(t)W + J(t, \xi)W, \tag{71}$$

$$\text{where } \varphi_{\mathrm{w}}(t, \xi) = i\lambda(t)\,|\xi| - \frac{1}{2}(b_{\mathrm{w}}^\sharp(t, \xi) + i\Im b^\sharp(t, \xi)) \quad \text{and} \quad f_{\mathrm{s}} = \frac{1}{2}\left(\frac{\lambda'}{\lambda} - b + b_{\mathrm{s}}\right).$$

Thanks to (70), the matrix $G(t, \xi)$ has negative diagonal entries given by

$$G(t, \xi) = \begin{pmatrix} g_+(t, \xi) & 0 \\ 0 & g_-(t, \xi) \end{pmatrix} \qquad g_\pm(t, \xi) = \frac{1}{2}\left(\pm b_{\mathrm{s}}^\sharp(t, \xi) - b_{\mathrm{s}}(t)\right) \leq 0.$$

Similarly to (58), we define

$$D_{\mathrm{w}}(t, \xi) := \begin{pmatrix} \exp\left(-\int_{\theta_{|\xi|}}^t \varphi_{\mathrm{w}}(\tau, \xi)\, d\tau\right) & 0 \\ 0 & \exp\left(\int_{\theta_{|\xi|}}^t \varphi_{\mathrm{w}}(\tau, \xi)\, d\tau\right) \end{pmatrix}. \tag{72}$$

Since $\lambda(t)$ is real-valued and $b_{j,\mathrm{w}}(t) =_{(a)} 0$, it holds $\|D_{\mathrm{w}}(t, \xi)\|, \|D_{\mathrm{w}}^{-1}(t, \xi)\| \leq C$. Since $D_{\mathrm{w}}(t, \xi)$ is diagonal, it follows $G(t, \xi) \equiv D_{\mathrm{w}}^{-1}(t, \xi)G(t, \xi)D_{\mathrm{w}}(t, \xi)$. The contribution coming from the term $f_{\mathrm{s}}(t)$ is now

$$\sqrt{d_{\mathrm{s}}(t, \xi)} = \exp\left(\int_{\theta_{|\xi|}}^t f_{\mathrm{s}}(\tau)\, d\tau\right) = \sqrt{\frac{\lambda(t)\gamma(t)\gamma_{\mathrm{s}}(t)}{\lambda(\theta_{|\xi|})\gamma(\theta_{|\xi|})\gamma_{\mathrm{s}}(\theta_{|\xi|})}}.$$

If we put $W(t, \xi) = \sqrt{d_{\mathrm{s}}(t, \xi)}\, D_{\mathrm{w}}(t, \xi)Z(t, \xi)$, the equation in (59) becomes

$$\partial_t Z = G(t, \xi)\, Z + \widetilde{J}(t, \xi)\, Z, \quad t \geq \theta_{|\xi|}, \tag{73}$$

where $\widetilde{J}(t, \xi) = D_{\mathrm{w}}^{-1}(t, \xi)J(t, \xi)D_{\mathrm{w}}(t, \xi)$ satisfies again (57). Since $G(t, \xi)$ is a diagonal matrix with negative entries, it is easy to prove that $|Z(t, \xi)| \leq C\,|Z(\theta_{|\xi|}, \xi)|$, thus

$$|U(t, \xi)|^2 \leq C\, d_{\mathrm{s}}(t, \xi)\, |U(\theta_{|\xi|}, \xi)|^2. \tag{74}$$

In $Z_{\mathrm{pd}}(N)$ we will prove the boundedness of

$$E_{1j} = \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{1j} + i\int_0^t \sqrt{d(\tau)} E_{2j}(\tau, \xi)d\tau\right), \tag{75}$$

$$E_{2j} = \frac{\gamma(t)}{\sqrt{d(t)}}\left(\delta_{2j} + \int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)} \frac{a(\sigma, \xi)}{\eta(\sigma)} E_{1j}(\sigma, \xi)d\sigma\right) \tag{76}$$

where $a(\sigma, \xi)$ is as in (65), but the proof of Theorem 2 can be followed replacing $\sqrt{\lambda(t)\gamma(t)}$ with the function $\sqrt{d(t)} = \sqrt{\lambda(t)\gamma(t)\gamma_{\mathrm{s}}(t)}$, which appears in Hypothesis 8, where needed. We just need to pay attention to the part which contains

$\lambda(t)^2\,|\xi|^2$ in the integral in (67), since (68) is no longer applicable if (13) does not hold. However, thanks to (26), we can estimate $E_{1j}(t,\xi)$ in (75) by

$$|E_{1j}(t,\xi)| \le C\frac{\eta(t)\Gamma(t)}{\sqrt{d(t)}} + C\frac{\eta(t)g(t)}{\sqrt{d(t)}}\int_0^t h(\sigma)\frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\frac{|a(\sigma,\xi)|}{\eta(\sigma)}|E_{1j}(\sigma,\xi)|d\sigma.$$

Since $1 \le \Gamma(t) \le g(t)$, by applying Lemma 1, thanks to (27) and (32), we are able to prove that $E_{1j}(t,\xi)$ is bounded. In particular, we used (27) to estimate $\lambda^2\xi^2\,gh/\gamma \lesssim \lambda\Lambda\xi^2 \le N\lambda\,|\xi|$. By the boundedness of E_{1j}, using (12)–(25), (31) and (33), we can estimate (76) by

$$\begin{aligned}|E_{2j}| &\le \frac{\gamma(t)}{\sqrt{d(t)}}\left(1 + C\int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\frac{|a(\sigma,\xi)|}{\eta(\sigma)}d\sigma\right)\\ &\le C_1 + C_2\int_0^t |\xi|^2\,\lambda(\sigma)\Lambda(\sigma)d\sigma + C_3|\xi|\int_0^t \lambda(\sigma)d\sigma + C\frac{\gamma(t)}{\sqrt{d(t)}}\int_0^t\frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\frac{|e(\sigma)|}{\eta(\sigma)}d\sigma \le C'.\end{aligned}$$

This concludes the proof. □

Proof of Corollary 1. Following the proof of Theorem 1, let $E(t,s,\xi)$ be the fundamental solution to (52) with initial time $s \ge 0$, that is,

$$\partial_t E = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} i|\xi|\lambda(t)E + \begin{pmatrix}\dfrac{\lambda'(t)}{\lambda(t)} & 0\\ 0 & -b(t)\end{pmatrix}E, \qquad E(s,s,\xi) = \mathrm{I}_2. \tag{77}$$

In particular, for any $\xi \ne 0$ it holds

$$U(t,\xi) = E(t,\theta_\xi,\xi)\,U(\theta_\xi,\xi), \quad \text{for } t \ge \theta_{|\xi|}.$$

From (60) it follows that

$$\|E(t,\theta_{|\xi|},\xi)\| \le C\frac{\sqrt{\lambda(t)\,\gamma(t)}}{\sqrt{\lambda(\theta_{|\xi|})\,\gamma(\theta_{|\xi|})}}, \qquad \|E^{-1}(t,\theta_{|\xi|},\xi)\| \le C\frac{\sqrt{\lambda(\theta_{|\xi|})\,\gamma(\theta_{|\xi|})}}{\sqrt{\lambda(t)\,\gamma(t)}},$$

for any $t \ge \theta_{|\xi|}$. Now let $|\xi| \ge \epsilon$ for some $\epsilon > 0$. Then

$$0 < C_{1,\epsilon} \le \lambda(\theta_{|\xi|})\,\gamma(\theta_{|\xi|}) \le C_{2,\epsilon},$$

since $\theta_{|\xi|} \in [0,\theta_\epsilon]$ for $|\xi| \ge \epsilon$. On the other hand, we can roughly estimate

$$\|E(\theta_{|\xi|},0,\xi)\|,\ \|E^{-1}(\theta_{|\xi|},0,\xi)\| \le \exp\Big(\Lambda(\theta_{|\xi|})\,|\xi| + \int_0^{\theta_\epsilon}\big(|b(\tau)| + |\lambda'(\tau)/\lambda(\tau)|\big)d\tau\Big) \le C_{\epsilon,N},$$

for any $|\xi| \ge \epsilon$, since $\theta_{|\xi|} \le \theta_\epsilon$. Combining the two estimates, we obtain

$$\|E(t,0,\xi)\| \le C_{1,\epsilon}\sqrt{\lambda(t)\,\gamma(t)}, \qquad \|E^{-1}(t,0,\xi)\| \le C_{2,\epsilon}\frac{1}{\sqrt{\lambda(t)\,\gamma(t)}}, \quad \text{for any } |\xi| \ge \epsilon. \tag{78}$$

If $(u_0,u_1) \in F_\epsilon$, then $\widehat{u}(t,\xi) \equiv 0$ for any $|\xi| \le \epsilon$ and the proof of (21) follows from (78). If $(u_0,u_1) \in H^1 \times L^2$, then there exists $\epsilon > 0$ such that

$$\int_{\mathbb{R}^n}(1+|\xi|^2)\,|\widehat{u_0}(\xi)|^2\,d\xi \le C_\epsilon\int_{|\xi|\ge\epsilon}|\xi|^2\,|\widehat{u_0}(\xi)|^2\,d\xi, \qquad \int_{\mathbb{R}^n}|\widehat{u_1}(\xi)|^2\,d\xi \le 2\int_{|\xi|\ge\epsilon}|\widehat{u_1}(\xi)|^2\,d\xi.$$

Let us define $(u_0^\epsilon,u_1^\epsilon) \in F_\epsilon$ such that

$$\widehat{u_j^\epsilon}(\xi) := \begin{cases}\widehat{u_j}(\xi) & \text{if } |\xi| \ge \epsilon,\\ 0 & \text{if } |\xi| < \epsilon,\end{cases}$$

and let $u^\epsilon(t,x)$ be the solution to (14) with initial data $(u_0^\epsilon,u_1^\epsilon)$. The proof of (20) follows from the left-hand side of (21) thanks to the inequalities:

$$\begin{aligned}&\lambda(t)\,\|\xi\,\widehat{u}(t,\cdot)\|_{L^2} \ge \lambda(t)\,\|\xi\,\widehat{u^\epsilon}(t,\cdot)\|_{L^2} \ge C_\epsilon\sqrt{\lambda(t)\,\gamma(t)}\|\xi\,\widehat{u_0^\epsilon}\|_{L^2} \ge C'_\epsilon\sqrt{\lambda(t)\,\gamma(t)}\|(1+|\xi|^2)^{1/2}\,\widehat{u_0}\|_{L^2},\\ &\|\partial_t\widehat{u}(t,\cdot)\|_{L^2} \ge \|\partial_t\widehat{u^\epsilon}(t,\cdot)\|_{L^2} \ge C_\epsilon\sqrt{\lambda(t)\,\gamma(t)}\|\widehat{u_1^\epsilon}\|_{L^2} \ge 2C_\epsilon\sqrt{\lambda(t)\,\gamma(t)}\|\widehat{u_1}\|_{L^2}.\end{aligned}$$

We address the interested reader to [21], where a similar technique is used to prove estimates from below for two by two systems. □

Proof of Corollary 2. Let $\widetilde{E}(t,s,\xi)$ be the fundamental solution to (77) with $b \equiv 0$. We claim that, for any $\xi \neq 0$, there exists

$$W_+(\xi) := \lim_{t\to\infty} \widetilde{E}^{-1}(t,0,\xi)E(t,0,\xi).$$

As in [22], we look for $\overline{E}(t,s,\xi)$ such that

$$E(t,s,\xi) = \widetilde{E}(t,s,\xi)\,\overline{E}(t,s,\xi).$$

For any $t \geq s \geq 0$ and for any $\xi \neq 0$, the matrix $\overline{E}(t,s,\xi)$ is the solution to

$$\partial_t \overline{E} = R(t,s,\xi)\,\overline{E}, \qquad \overline{E}(s,s,\xi) = \mathrm{I}_2, \tag{79}$$

where

$$R(t,s,\xi) := \widetilde{E}^{-1}(t,s,\xi) \begin{pmatrix} 0 & 0 \\ 0 & -b(t) \end{pmatrix} \widetilde{E}(t,s,\xi).$$

Now let $|\xi| \geq \epsilon$ for some $\epsilon > 0$. Using (78), we obtain $\|R(t,0,\xi)\| \leq C_\epsilon\,|b(t)|$. Since $b \in L^1$, for any $\xi \neq 0$ it holds

$$W_+(\xi) = \lim_{t\to\infty} \overline{E}(t,0,\xi) = \mathrm{I}_2 + \int_0^\infty R(\tau,0,\xi)\,d\tau + \int_0^\infty R(\tau,0,\xi) \int_0^\tau R(\sigma,0,\xi)\,d\sigma\,d\tau + \cdots,$$

which satisfies $\|W_+(\xi)\|, \|W_+^{-1}(\xi)\| \leq C'_\epsilon$ for $|\xi| \geq \epsilon$. More precisely,

$$\|W_+(\xi) - \overline{E}(t,0,\xi)\| \leq \int_t^\infty \|R(\tau,0,\xi)\| \exp\left(\int_0^\tau \|R(\sigma,0,\xi)\|\,d\tau\right) d\tau \leq C_\epsilon \int_t^\infty |b(\tau)|\,d\tau.$$

For any $\epsilon > 0$, let $W_+^\epsilon : F_\epsilon \to F_\epsilon$ be the linear, bounded, invertible, *scattering* operator which maps $(u_0,u_1) \in F_\epsilon$ in $(v_0,v_1) \in F_\epsilon$, defined by:

$$\left(i\xi\,\widehat{v_0}(\xi), \widehat{v_1}(\xi)\right) = W_+(\xi)\left(i\xi\,\widehat{u_0}(\xi), \widehat{u_1}(\xi)\right), \quad \text{for any } |\xi| \geq \epsilon.$$

Thanks to (78), we may conclude the proof of (24), since

$$\begin{aligned}\left\|\left(i\lambda(t)\xi\widehat{u}(t,\cdot) - i\lambda(t)\xi\widehat{v}(t,\cdot), \widehat{u}_t(t,\cdot) - \widehat{v}_t(t,\cdot)\right)\right\|_{L^2} &= \|\left(E(t,0,\xi) - \widetilde{E}(t,0,\xi)W_+(\xi)\right)(i\xi\,\widehat{u_0}, \widehat{u_1})\|_{L^2} \\ &= \|\widetilde{E}(t,0,\xi)\left(\overline{E}(t,0,\xi) - W_+(\xi)\right)(i\xi\,\widehat{u_0}, \widehat{u_1})\|_{L^2} \\ &\leq C_\epsilon \sqrt{\lambda(t)}\sqrt{E_1(0)} \int_t^\infty |b(\tau)|\,d\tau.\end{aligned}$$

By density arguments, estimate (22) holds for any $(u_0,u_1) \in H^1 \times L^2$. We address the interested reader to [22], where *scattering* results are proved for two by two systems with a similar technique. □

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Appendix. Additional remarks on the *mass* term

In this paper we have stated sufficient conditions such that the term $e(t)$ in (1) brings no contribution to the energy behavior of the equation in (1). In particular, we recall that $e(t)$ may be negative.

On the other hand, if $e(t) = m(t)^2$ is a (sufficiently large) positive *mass* term, then we may expect to gain some benefit by replacing our λ-scaled *wave type energy* with a *Klein–Gordon type* one.

Remark 12. If one considers the Cauchy problem

$$v_{tt} - \Delta v + \frac{\nu^2}{(1+t)^2}\,v = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x),$$

with $\nu^2 > 1/4$, then [23] the energy

$$E_{\mathrm{KG}}(t) := \|v_t(t,\cdot)\|_{L^2}^2 + \|\nabla v(t,\cdot)\|_{L^2}^2 + \frac{1}{1+t}\,\|v(t,\cdot)\|_{L^2}^2,$$

satisfies the estimate

$$C_1\,\frac{1}{1+t}\,E(0)\le E_{\mathrm{KG}}(t)\le C_2\,E(0) \tag{A.1}$$

with $C_1, C_2>0$. We remark that $E(0)=E_{\mathrm{KG}}(0)$.

The next example shows us which effects may have a positive or negative *mass* term on the energy behavior.

Example 12. Let us consider the Cauchy problem for the scale-invariant equation

$$u_{tt}-\Delta u+\frac{\mu}{1+t}\,u_t+\frac{m}{(1+t)^2}\,u=0,\qquad u(0,x)=u_0(x),\qquad u_t(0,x)=u_1(x), \tag{A.2}$$

for some $\mu\in\mathbb{R}$ and $m\neq 0$. We remark that if $m=0$ and $\mu\in[0,2]$ then we can apply Theorem 1 and we obtain that

$$\|\partial_t u(t,\cdot)\|_{L^2}^2+\|\nabla u(t,\cdot)\|_{L^2}^2\lesssim(1+t)^{-\mu}\,E(0). \tag{A.3}$$

If we put $u(t,x)=(1+t)^\rho v(t,x)$ for some $\rho\neq 0$ then we obtain

$$v_{tt}-\Delta v+\frac{2\rho+\mu}{1+t}\,v_t+\frac{\rho(\rho-1)+\mu\rho+m}{(1+t)^2}\,v=0,\qquad v(0,x)=u_0(x),\qquad v_t(0,x)=u_1(x)-\rho u_0(x). \tag{A.4}$$

First, let us assume that $(1-\mu)^2-4m\ge 0$. If we take $2\rho+\mu=1+\sqrt{(1-\mu)^2-4m}$, then (A.4) reduces to

$$v_{tt}-\Delta v+\frac{2\rho+\mu}{1+t}v_t=0,\qquad v(0,x)=u_0(x),\qquad v_t(0,x)=u_1(x)-\rho u_0(x). \tag{A.5}$$

Since $2\rho+\mu\ge 1$, we may now apply Theorem 3.4 of [10] to (A.5), obtaining:

$$\|v(t,\cdot)\|_{L^2}^2\lesssim E(0),$$
$$\|v_t(t,\cdot)\|_{L^2}^2+\|\nabla v(t,\cdot)\|_{L^2}^2\lesssim\begin{cases}(1+t)^{-(2\rho+\mu)}\,E(0) & \text{if } (1-\mu)^2-4m\le 1,\\ (1+t)^{-2}\,E(0) & \text{if } (1-\mu)^2-4m\ge 1.\end{cases}$$

The following estimate on the energy for $u(t,x)=(1+t)^\rho v(t,x)$ follow:

$$\|\partial_t u(t,\cdot)\|_{L^2}^2+\|\nabla u(t,\cdot)\|_{L^2}^2\lesssim\begin{cases}(1+t)^{-\mu}\,E(0) & \text{if } (1-\mu)^2-4m\le 1,\\ (1+t)^{-\mu+(\sqrt{(1-\mu)^2-4m}-1)}\,E(0) & \text{if } (1-\mu)^2-4m\ge 1.\end{cases}$$

In particular, if $\mu\in[0,2]$ and $4m<-\mu(2-\mu)$, we may expect a loss of decay with respect to (A.3).

Now let $(1-\mu)^2-4m<0$. Setting $2\rho=-\mu$ in (A.4) we get

$$v_{tt}-\Delta v+\frac{1+4m-(1-\mu)^2}{4(1+t)^2}\,v=0,$$

to which we can apply the right-hand side of (A.1), since $1+4m-(1-\mu)^2>1$. In particular, we obtain

$$\|v(t,\cdot)\|_{L^2}^2\lesssim(1+t)\,E(0),\qquad \|v_t(t,\cdot)\|_{L^2}^2+\|\nabla v(t,\cdot)\|_{L^2}^2\lesssim E(0),$$

from which estimate (A.3) follows.

Summarizing, if $m\ge -1/4$ then energy estimate (A.3) holds for any $\mu\in[1-\sqrt{1+4m},\,1+\sqrt{1+4m}]$.

The following remark shows how Hypothesis 6 is automatically verified, exception given for a few critical cases.

Remark 13. By using (25), it follows that (29) is trivially satisfied if

$$\frac{\lambda'(t)}{\lambda(t)}+b(t)\ge_{(a)}\epsilon\,\eta(t), \tag{A.6}$$

for some $\epsilon>0$. Indeed, according to Remark 2, if we put $A(t)=\lambda(t)^{1/2}\gamma(t)^{-1/2}\Lambda(t)^{-\epsilon/2}$, then

$$\sqrt{\frac{\gamma(t)}{\lambda(t)}}\int_0^t\sqrt{\frac{\lambda(\sigma)}{\gamma(\sigma)}}\,\eta(\sigma)d\sigma\le\Lambda(t)^{-\epsilon/2}\int_0^t\eta(\sigma)\Lambda(\sigma)^{\epsilon/2}d\sigma=\Lambda(t)^{-\epsilon/2}\int_0^t\frac{\lambda(\sigma)}{\Lambda(\sigma)^{1-\epsilon/2}}d\sigma\le\frac{2}{\epsilon}.$$

We remark that condition (A.6) is stronger than the left-hand side of (13).

Recalling Example 3, we look for a sufficient condition on $e(t)$ in order to satisfy Hypotheses 5 and 6.

Example 13. Let $b(t)$ be as in (40). First, we consider $\kappa \in (0, 1)$. Thanks to Remark 13, Hypothesis 6 is satisfied. Now we take $g(t) = \Gamma(t)$ and $h(s) = 1$ in (26), so that (27) immediately follows, and (28) becomes

$$\frac{1}{\Lambda(t)^{\frac{1-\kappa}{2}}} \exp\left(\int_0^t \frac{1}{\kappa\,\eta(\tau)}\,|e(\tau)|d\tau\right) \le C, \tag{A.7}$$

which is satisfied if

$$M(e) := \limsup_{t\to\infty} \frac{|e(t)|}{\eta(t)^2} < \frac{\kappa(1-\kappa)}{2}. \tag{A.8}$$

However, if $\kappa \in (0, 1/3)$, the bound in (A.8) can be improved by taking $g(t) = \Lambda(t)^{\kappa+\epsilon}/(\kappa+\epsilon)$ and $h(s) = \Lambda(s)^{-\epsilon}$ in (26), that is,

$$\int_s^t \frac{\lambda(\tau)}{\Lambda(\tau)^{1-\kappa}} \le \frac{1}{\Lambda(s)^{\epsilon}} \int_s^t \frac{\lambda(\tau)}{\Lambda(\tau)^{1-(\kappa+\epsilon)}}\,d\tau \le \frac{1}{\kappa+\epsilon}\,\frac{\Lambda(t)^{\kappa+\epsilon}}{\Lambda(s)^{\epsilon}},$$

for some $\epsilon \in (0, (1-\kappa)/2)$. In such a way (28) holds if $M(e) < (\kappa+\epsilon)(1-\kappa-2\epsilon)/2$. It is easy to check that the maximum is reached for the choice $\epsilon = (1-3\kappa)/4$.

The same conclusion follows in the case $\kappa = 0$. On the other hand, if $\kappa = 1$ then (A.7) holds if, and only if, $e\,\eta^{-1} \in L^1$. Moreover, in this case, Hypothesis 6 is also satisfied.

We may proceed similarly if $\kappa \in (-1, 0)$. Summarizing, Hypotheses 5 and 6 are satisfied if $M(e) < \overline{M}$ where

$$\overline{M} = \begin{cases} |\kappa|(1-|\kappa|)/2 & \text{if } |\kappa| \in [1/3, 1), \\ (1+|\kappa|)^2/16 & \text{if } |\kappa| \in [0, 1/3], \end{cases} \tag{A.9}$$

or $e\,\eta^{-1} \in L^1$ if $|\kappa| = 1$.

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