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On the volume growth of Alexandrov spaces with nonnegative curvature☆

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ABSTRACT

Recently, Zhang and Zhu introduced a new definition for Ricci curvature bounded below on Alexandrov spaces. In this paper, we give a volume growth lower bound estimate of Alexandrov spaces with nonnegative Ricci curvature (in the sense of Zhang–Zhu), which extends Yau's result from Riemannian manifolds to Alexandrov spaces.

1. Introduction

The main object of the present paper is complete noncompact Alexandrov spaces with nonnegative curvature. An Alexandrov space with curvature bounded below (in short, we say it is an Alexandrov space) is a complete, locally compact and path connected inner metric space on which the triangle comparison theorem holds. As we know, Alexandrov spaces with curvature bounded below generalize successfully the concept of lower bounds of sectional curvature from Riemannian manifolds to singular spaces. There have been extensive studies of the geometry of Alexandrov spaces exactly from the publishing of the fundamental work of Burago, Gromov and Perelman [1], which contains plenty of fruitful ideas. Many efforts were made to extend results in Riemannian geometry to Alexandrov spaces, such as diameter sphere theorem [2], Toponogov splitting theorem [3], Cheeger–Gromoll splitting theorem [4], Bishop–Gromov relative volume comparison [1,5], etc.

For the volume growth of an Alexandrov space X, Sturm [5], Lott–Villani [6,7], and Ohta [8], independently, proved that the condition of $\mathrm{CD}(k; n)$ or $\mathrm{MCP}(k; n)$ implies Bishop–Gromov relative volume comparison theorem. Recently, Zhang–Zhu [4] utilizing a new definition of lower Ricci curvature bounds on Alexandrov spaces showed that $Ric(X) \geq (n-1)k \Rightarrow \mathrm{CD}((n-1)k; n)$, and hence Bishop–Gromov volume comparison holds for an n-dimensional Alexandrov space with Ricci curvature $\geq (n-1)k$. That is, the function

$$\frac{vol(B(x_0, r))}{vol(B_k(r))}$$

is nonincreasing with respect to $r > 0$, where $B_k(r)$ denotes a geodesic ball with radius r in the simply connected space form of constant curvature k. In particular, one can get the upper bound of volume growth of Alexandrov space with nonnegative

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Ricci curvature, i.e., $vol(B(x_0, r)) \leq Cr^n$. It is true for Riemannian manifolds with nonnegative Ricci curvature. On the other side, for Riemannian manifolds, Yau [9] proved the volume growth of complete noncompact Riemannian manifolds with nonnegative Ricci curvature satisfies

$$vol(B(x_0, r)) \geq cr.$$

This is a lower bound estimate of volume growth for Riemannian manifolds. A natural question: is there a similar lower bound estimate of volume growth for Alexandrov spaces with nonnegative Ricci curvature? The main result of the present paper will answer this question. Specifically, we will prove the following theorem.

Theorem 1.1. *Let X be an n-dimensional complete noncompact Alexandrov space without boundary. If the Ricci curvature of X is nonnegative everywhere, then the volume growth of X satisfies:*

$$vol(B(x_0, r)) \geq cr.$$

Remark 1.2. We note that, in [10], among other things, Shiohama proved the volume of a complete noncompact Alexandrov space with nonnegative curvature outside a compact set is unbounded. Theorem 1.1 improves Shiohama's result in two directions. Firstly, Ricci's curvature condition replaces sectional curvature condition. Secondly, Theorem 1.1 of the present paper gives a specific lower bound of the volume growth of geodesic ball.

We also note that we will give a different proof of Yau's original theorem in a Riemannian setting. The method used in the proof is different from Yau's which relied on the maximum principle. We use only comparison techniques which require less regularity and thus are easier to generalize to Alexandrov spaces.

Open question 1.3. In [11], Fu and the author of the present paper proved that if the Ricci curvature is nonnegative outside a compact set of a complete noncompact Riemannian manifold, then the volume growth of the manifold satisfies:

$$vol(B(x_0, r)) \geq cr.$$

In the proof of this result, we used the strict subharmonicity and exhaustion property of Busemann function on Riemannian manifolds. For the Busemann function on Alexandrov spaces, at present, we can not get the similar properties. So in Theorem 1.1, whether the condition of nonnegative Ricci curvature can everywhere be replaced by nonnegative Ricci curvature outside a compact set is an open question now.

2. Preliminaries

In this section we shall recall some notions and preliminary results on Alexandrov space. In the sequel, we always denote by $(X, |\cdot\cdot|)$ a metric space and $\mathbb{M}_k^n$ the n-dimensional complete simply connected space form of constant sectional curvature k. We call $\mathbb{M}_k^2$ the *k-plane*.

2.1. Alexandrov space

A metric space $(X, |\cdot\cdot|)$ is called a *length space* if and only if for any two points $x, y \in X$, the distance between x and y is given by

$$|xy| = \inf_{\gamma} Length(\gamma),$$

where the infimum is taken over all curves γ in X which connect x and y. A length space X is called a *geodesic space* if and only if, for each pair of points $x, y \in X$, the distance $|xy|$ is realized as the length of a rectifiable curve connecting x and y. Such a distance-realizing curve, parameterized by arc-length, is called minimal geodesic. (This geodesic is not required to be unique.)

Given any $k \in \mathbb{R}$ we say that a length space X *locally has curvature* $\geq k$ if and only if each point $p \in X$ has a neighborhood $U(p) \subset X$ such that for each quadruple of points $\{z, x_1, x_2, x_3\} \subset U(p)$,

$$\widetilde{\angle}_k x_1 z x_2 + \widetilde{\angle}_k x_2 z x_3 + \widetilde{\angle}_k x_3 z x_1 \leq 2\pi \tag{2.1}$$

where $\widetilde{\angle}_k x_1 z x_2$, $\widetilde{\angle}_k x_2 z x_3$ and $\widetilde{\angle}_k x_3 z x_1$ are the comparison angles in the k-plane. That is, $\widetilde{\angle}_k xzy$ is the angle at $\bar{z}$ of a triangle $\triangle\bar{x}\bar{z}\bar{y}$ with side lengths $|\bar{x}\bar{z}| = |xz|$, $|\bar{z}\bar{y}| = |zy|$ and $|\bar{x}\bar{y}| = |xy|$ in the k-plane. We say that a length space X *globally has curvature* $\geq k$ if the inequality (2.1) holds for any quadruple of points $\{z, x_1, x_2, x_3\} \subset X$.

There is an exceptional definition for 1-dimensional spaces: we say that X *(locally/globally) has curvature* $\geq k$ if and only if $k \leq (\pi/L)^2$, where $L \leq +\infty$ denotes the diameter of X.

Proposition 2.1.1. *We note that, for a complete length space, X locally has curvature $\geq k$, is equivalent to X globally has curvature $\geq k$. For more details we refer the reader to* [1,5].

Definition 2.1.2. A complete length space X locally (or globally) has curvature $\geq k$ called an *Alexandrov space with curvature* $\geq k$ (for short, we say X to be an *Alexandrov space*), if it is locally compact.

Remark 2.1.3. The basic example of Alexandrov space with curvature $\geq k$ is a Riemannian manifold without boundary or with locally convex boundary, whose sectional curvature is not less than k.

Remark 2.1.4. Since a complete and locally compact length space is a geodesic space (refer to [5]), we know that an Alexandrov space is, of course, a geodesic space.

2.2. Ricci curvature

Next we shall introduce some notions for the Ricci curvature bounded below on Alexandrov spaces. For an Alexandrov space, several different definitions of Ricci curvature having lower bounds by k have been given (refer to [4–8] for the details). Here, let us recall the definition of lower bounds of Ricci curvature on Alexandrov space introduced by Zhang–Zhu in [4]. Let X be an n-dimensional Alexandrov space and $x \in X$, and let T_x and Σ_x be the tangent cone and the space of directions respectively. It is well known in [12] or [13] that, for any $x \in X$ and $\xi \in \Sigma_x$, there exists a quasi-geodesic starting at x along direction ξ. The exponential map $\exp_x : T_x \to X$ is defined as follows. For any $v \in T_x$, $\exp_x(v)$ is a point on some quasi-geodesic of length $|v|$ starting point x along $v/|v| \in \Sigma_x$. Denote by $\log_x := \exp_x^{-1}$.

Let $\gamma : [0, l) \to X$ be a geodesic. For any $t \in (0, l)$, $\gamma(t)$ is an interior of γ then the tangent cone $T_{\gamma(t)}$ can be isometrically split into a direct metric product. We denote

$$\Lambda_{\gamma(t)} = \{\xi \in \Sigma_{\gamma(t)} : |\xi| = 1 \text{ and } \langle \xi, \gamma^+(t)\rangle = \langle \xi, \gamma^-(t)\rangle = \pi/2\},$$

where $\gamma^\pm(t) := \lim_{h\to 0^+} \frac{1}{h} \log_{\gamma(t)} \gamma(t \pm h)$. Based on the second variation formula of arc-length, Zhang–Zhu propose the following condition which is similar to the lower bounds for the radial curvature along the geodesic γ in a Riemannian manifolds setting.

Definition 2.2.1. Let $\sigma(t) : (-l, l) \to X$ be a geodesic and $\{g_{\sigma(t)}(\xi)\}_{-l<t<l}$ be a family of functions on $\Lambda_{\sigma(t)}$ such that $g_{\sigma(t)}$ is continuous on $\Lambda_{\sigma(t)}$ for each $t \in (-l, l)$. We say that the family $\{g_{\sigma(t)}(\xi)\}_{-l<t<l}$ satisfies *Condition (RC)* on σ if for any two points $x, y \in \sigma$ and any sequence $\{\theta_j\}_{j=1}^{\infty}$ with $\theta_j \to 0$ as $j \to \infty$, there exists an isometry $T : \Lambda_x \to \Lambda_y$ and a subsequence $\{\delta_j\}$ of $\{\theta_j\}$ such that

$$|\exp_x(\delta_j l_1 \xi), \exp_y(\delta_j l_2 T\xi)| \leq |xy| + \frac{(l_1 - l_2)^2}{2|xy|} \cdot \delta_j^2 - \frac{g_x(\xi) \cdot |xy|}{6} \cdot (l_1^2 + l_1 \cdot l_2 + l_2^2)\delta_j^2 + o(\delta_j^2),$$

for any $l_1, l_2 \geq 0$ and any $\xi \in \Lambda_x$.

If X has curvature bounded below by k (for some $k \in \mathbb{R}$), then by Petrunin [14], it is easy to show that the family $\{g_{\sigma(t)}(\xi) = k\}_{-l<t<l}$ satisfies Condition (RC) on σ. In particular, if a family $\{g_{\sigma(t)}(\xi)\}_{-l<t<l}$ satisfies Condition (RC), then the family $\{g_{\sigma(t)}(\xi) \vee k\}_{-l<t<l}$ satisfies Condition (RC) too.

Definition 2.2.2. Let $\gamma : [0, a) \to X$ be a geodesic. We say that X has *Ricci curvature bounded below by* $(n-1)k$ *along* γ, if for any $\epsilon > 0$ and any $0 < t_0 < a$, there exists $l = l(t_0, \epsilon) > 0$ and a family of continuous functions $\{g_{\gamma(t)}(\xi)\}_{t_0-l<t<t_0+l}$ on $\Lambda_{\gamma(t)}$ such that the family of functions satisfy Condition (RC) on $\gamma|_{(t_0-l,t_0+l)}$ and

$$\frac{1}{Vol(\Lambda_{\gamma(t)})} \int_{\Lambda_{\gamma(t)}} g_{\gamma(t)}(\xi) d\xi \geq k - \epsilon, \quad \forall t \in (t_0 - l, t_0 + l).$$

We say that X *locally has Ricci curvature bounded below by* $(n-1)k$, if each point $x \in X$ has a neighborhood $U(x)$ such that X has Ricci curvature bounded below by k along every geodesic γ in $U(x)$. We say that X *globally has Ricci curvature bounded below by* $(n-1)k$ if the previous is true with $U(x) = X$, and denoted by $Ric(X) \geq (n-1)k$. When $k = 0$, i.e., $Ric(X) \geq 0$, we say X has *nonnegative Ricci curvature*. According to the second variation formula of arc-length proved by Petrunin [14], it is easy to show that an n-dimensional Alexandrov space with curvature $> k$ must have Ricci curvature bounded from below by $(n-1)k$. By Proposition 2.1.1 we can get the following result.

Proposition 2.2.3. *For a complete length space, the conditions of locally and globally having Ricci curvature* $\geq (n-1)k$ *are equivalent.*

With respect to lower bounds of Ricci curvature for metric measure spaces, Sturm [5] has given the curvature-dimension conditions, denoted by $\mathrm{CD}(k; n)$ with $n \in (1, +\infty]$ and $k \in \mathbb{R}$. In [4], Zhang and Zhu proved the following proposition.

Proposition 2.2.4. *Let X be an n-dimensional Alexandrov space without boundary and $Ric(X) \geq (n-1)k$. Then X satisfies the curvature-dimension condition* $\mathrm{CD}((n-1)k; n)$.

2.3. Hausdorff measure

In this subsection we recall some notions on the Hausdorff measure of Alexandrov space which is the counterpart of volume in a Riemannian manifold setting. Refer to [1,5] for more details. Let X be an n-dimensional Alexandrov space and vol denote the n-dimensional Hausdorff measure on X. We denote the metric measure space by $(X, |\cdot\cdot|, vol)$. Let $x_0 \in X$ and $vol(\overline{B}(x_0, r))$ the Hausdorff measure of concentric geodesic balls $\overline{B}(x_0, r) \subset X$. We define

$$v(r) := vol(\overline{B}(x_0, r)),$$

and the volume of the corresponding spheres as

$$s(r) := \limsup_{\delta\to 0} \frac{1}{\delta} vol(\overline{B}(x_0, r+\delta) \setminus B(x_0, r)).$$

In [5], Sturm showed that if the metric measure space $(X, |\cdot\cdot|, vol)$ satisfies the curvature-dimension condition CD$(k; n)$, then the above defined function v is locally Lipschitz continuous on $\mathbb{R}_+$. Particularly, the function v is weakly differentiable almost everywhere on $\mathbb{R}_+$, and it coincides with the integral of its weak derivative function s.

We also note that for Alexandrov spaces it's easy to see that $s(r)$ is equal to the $(n-1)$-dimensional Hausdorff measure of the metric sphere $S(p; r)$ and the Lipschitz property of $s(r)$ is also quite easy to prove (easier than for general spaces satisfying CD$(k; n)$).

3. Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1. As preparation, we first prove the following two lemmas. The trick of Lemma 3.1 is due to Gromov [15].

Lemma 3.1. *Suppose f and g are positive integrable functions, of a real variable r, for which f/g is nonincreasing with respect to r. Then for any $S > s > 0$, $T > t > 0$, $s > t$, $S > T$, it holds that*

$$\frac{\int_s^S f(r)dr}{\int_t^T f(r)dr} \le \frac{\int_s^S g(r)dr}{\int_t^T g(r)dr}. \tag{3.1}$$

Proof. We construct the following function

$$H(x, y) = \frac{\int_x^y f(r)dr}{\int_x^y g(r)dr}.$$

It is sufficient to show that the function $H(x, y)$ satisfies $\partial H/\partial x \le 0$, and, of course, $\partial H/\partial y \le 0$. In fact, it is easy to get

$$\begin{aligned}\frac{\partial H}{\partial x} &= \frac{1}{\left(\int_x^y g(r)dr\right)^2}\left(-f(x)\int_x^y g(r)dr + g(x)\int_x^y f(r)dr\right)\\ &= \frac{g(x)\int_x^y g(r)dr}{\left(\int_x^y g(r)dr\right)^2}\left(\frac{\int_x^y f(r)dr}{\int_x^y g(r)dr} - \frac{f(x)}{g(x)}\right).\end{aligned} \tag{3.2}$$

Since the function f/g is nonincreasing, it holds that

$$\frac{f(r)}{g(r)} \le \frac{f(x)}{g(x)} \quad \text{for } x \le r \le y.$$

Then it is easy to get

$$\int_x^y f(r)dr \le \int_x^y \frac{f(x)}{g(x)} g(r)dr = \frac{f(x)}{g(x)}\int_x^y g(r)dr.$$

Hence

$$\frac{\int_x^y f(r)dr}{\int_x^y g(r)dr} \le \frac{f(x)}{g(x)}.$$

According to the equality (3.2), it shows $\partial H/\partial x \le 0$. The proof is completed. □

Lemma 3.2. *Let $s, t, S, T \in \mathbb{R}$ such that $S \geq s, T \geq t, S \geq T, s \geq t$. Suppose X is an n-dimensional complete noncompact Alexandrov space with nonnegative Ricci curvature and $\partial X = \emptyset$. Then*

$$\frac{vol(A_{s,S}(x_0))}{vol(A_{t,T}(x_0))} \leq \frac{vol(B_{s,S})}{vol(B_{t,T})} \tag{3.3}$$

where $A_{s,S}(x_0) = \{x \in X : s \leq |x_0x| \leq S\}$, $B_{s,S} = \{x \in \mathbb{R}^n : s \leq |xo| \leq S\}$.

Proof. By Proposition 2.2.4, we know that nonnegative Ricci curvature condition implies that the curvature-dimension condition CD(0; n) holds for Alexandrov space. In [5] Sturm proved that the curvature-dimension condition CD(k; n) implies the generalized Brunn–Minkowski inequality and hence, the Bishop–Gromov relative volume comparison. In particular, with the curvature-dimension condition CD(0; n), Sturm showed the following inequality held for any $r < R$:

$$\frac{s(r)}{r^{n-1}} \geq \frac{s(R)}{R^{n-1}}. \tag{3.4}$$

Hence the function $s(r)/r^{n-1}$ is nonincreasing with respect to r. We thus may apply Lemma 3.1 according to which inequality (3.4) implies the conclusion of Lemma 3.2. This completes the proof. □

Proof of Theorem 1.1. Since X is complete noncompact Alexandrov space, there exists a geodesic ray denoted by γ emanating from any point $x_0 \in X$. By Lemma 3.2, for any fixed $\rho > 0$ and $t > 3\rho$ we have

$$\begin{aligned}
\frac{vol(B(\gamma(t), t-\rho))}{vol(A_{t-\rho,t+\rho}(\gamma(t)))} &\geq \frac{\omega_n(t-\rho)^n}{\omega_n(t+\rho)^n - \omega_n(t-\rho)^n} \\
&= \frac{1}{\left(1+\frac{2\rho}{t-\rho}\right)^n - 1} \\
&= \frac{1}{\sum_{k=1}^{n} C_n^k \left(\frac{2\rho}{t-\rho}\right)^k} \\
&\geq \frac{t-\rho}{2\rho \sum_{k=1}^{n} C_n^k}, \quad (\text{since } t \geq 3\rho) \\
&\geq \frac{t - t/3}{2\rho \sum_{k=1}^{n} C_n^k} \\
&= \frac{t}{3\rho \sum_{k=1}^{n} C_n^k} \\
&= c(n, \rho)t.
\end{aligned}$$

So we get

$$vol(B(\gamma(t), t-\rho)) \geq c(n, \rho) vol(A_{t-\rho,t+\rho}(\gamma(t)))t.$$

By the triangle inequality, it is easy to show $B(x_0, \rho) \subset A_{t-\rho,t+\rho}(\gamma(t))$, and hence

$$vol(B(x_0, \rho)) \leq vol(A_{t-\rho,t+\rho}(\gamma(t))).$$

Then it holds that

$$vol(B(\gamma(t), t-\rho)) \geq c(n, \rho) vol(B(x_0, \rho))t. \tag{3.5}$$

Again by the triangle inequality, we know $B(\gamma(t), t-\rho) \subset B(x_0, 2t)$, and

$$vol(B(\gamma(t), t-\rho)) \leq vol(B(x_0, 2t)). \tag{3.6}$$

Combining (3.5) and (3.6), one can get

$$vol(B(x_0, 2t)) \geq c(n, \rho) vol(B(x_0, \rho))t.$$

The proof is completed. □

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