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On differentiability of convex operators

Libor Veselý [a,*], Luděk Zajíček [b]

[a] Dipartimento di Matematica, Università degli Studi, Via C. Saldini 50, 20133 Milano, Italy

[b] Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic

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ABSTRACT

The main known results on differentiability of continuous convex operators f from a Banach space X to an ordered Banach space Y are due to J.M. Borwein and N.K. Kirov. Our aim is to prove some "supergeneric" results, i.e., to show that, sometimes, the set of Gâteaux or Fréchet nondifferentiability points is not only a first-category set, but also smaller in a stronger sense. For example, we prove that if Y is countably Daniell and the space $\mathcal{L}(X, Y)$ of bounded linear operators is separable, then each continuous convex operator $f : X \to Y$ is Fréchet differentiable except for a Γ-null angle-small set. Some applications of such supergeneric results are shown.

0. Introduction

By a *function* we always mean a real-valued function. Recall that a real Banach space X is called an *Asplund space* (resp., a *weak Asplund space*) if each continuous convex function on X is generically (i.e., on a complement of a first-category set) Fréchet (resp., Gâteaux) differentiable. A rich theory of such spaces can be found in, e.g., [22,4,11]. Let us only recall that each separable X is a weak Asplund space, and that a separable X is an Asplund space if and only if its dual X^* is separable.

For some spaces X, it is known that the set of Fréchet (or Gâteaux) nondifferentiability points of each continuous convex function on X is not only a first-category (=meager) set, but also small in a stronger sense. Such "supergeneric" results are contained in, e.g., [2,3,28,29,23,24,13,12].

A number of generic differentiability results for continuous convex functions were generalized to continuous convex operators $f : X \to Y$, where X is a Banach space and Y is an ordered Banach space (see [5,6,16,17,7]).

In the present article, we prove some "supergeneric" results on differentiability of convex operators. In some cases, such results can be proved by an easy combination of proofs of known results on generic differentiability of convex operators and results on "supergeneric" differentiability of convex functions. Indeed, Borwein in [5,6] observed that, in some cases, for a continuous convex operator $f : X \to Y$ there exists a continuous convex function φ on X such that f is Fréchet (resp. Gâteaux) differentiable at $x \in X$ whenever φ is. In Section 2, we present some results obtained in this easy way.

However, it seems that our main results from Sections 3 and 4 cannot be proved by such "reduction to the convex function case" and thus a more sophisticated proof is needed. Our main result (Theorem 3.5) reads as follows.

Let A be an open convex set in a Banach space X. Let Y be an ordered Banach space which is countably Daniell. Suppose that the space $\mathcal{L}(X, Y)$ of bounded linear operators is separable. Then each continuous convex operator $f : X \to Y$ is Fréchet differentiable except for a Γ-null angle-small set.

This theorem applies, for example, if (a) $X = \ell_p$, $Y = \ell_q$ $(1 < q < p < \infty)$, or (b) $X = c_0$ and Y is any countably Daniell ordered Banach space with the Radon–Nikodým property (e.g., $Y = \ell_q$ or $Y = L_q[0, 1]$ with $1 < q < \infty$). Recall that

* Corresponding author.
E-mail addresses: libor.vesely@unimi.it (L. Veselý), zajicek@karlin.mff.cuni.cz (L. Zajíček).

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each angle-small set is σ-porous (even σ-lower porous) and thus it is a first-category set, but not all first-category sets are σ-porous (see, e.g., [30]). Thus Theorem 3.5 is a "supergeneric" result.

Also the (weaker) generic version of this theorem is new. However, in all applications of Theorem 3.5 that we know of, the generic Fréchet differentiability follows (see Example 5.5) from [16, Corollary 3.12] (in which it is supposed that Y is a dual Banach lattice and $\mathcal{L}(X, Y)$ has the Radon–Nikodým property).

In Section 4, using Theorem 3.5 and the separable reduction methods from [20,9], we obtain also some supergeneric results for some rather special non-separable X (e.g., $X = c_0(\Gamma)$ or $X = C(K)$ where K is a scattered compact topological space).

In Section 1, we recall some notions and facts about Banach spaces, ordered Banach spaces and convex operators. We prove also two possibly new simple observations which could be of some independent interest (Lemma 3.2(ii), Proposition 1.10). Further, we recall definitions of some classes of small sets in Banach spaces and state related results on differentiability of convex functions.

In the last part, Section 5, we mention some applications of our results. We present also some consequences of supergeneric results, which do not follow from the known generic results.

1. Preliminaries

1.1. Basic notation

In the following, all normed linear and Banach spaces will be real spaces. By $B(x, r)$ we denote the open ball with center x and radius r. The dual space of a Banach space X is denoted by X^*, and we set $S_X := \{x \in X : \|x\| = 1\}$.

Let X, Y be normed linear spaces, $\emptyset \neq G \subset X$ an open set, and $f: G \to Y$ a mapping. Then the directional and one-sided directional derivatives of f at $x \in G$ in the direction $v \in X$ are defined respectively by

$$f'(x, v) := \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} \quad \text{and} \quad f'_+(x, v) := \lim_{t\to 0+} \frac{f(x+tv) - f(x)}{t}.$$

Recall that $f'(x, v)$ exists if and only if $f'_+(x, v) = -f'_+(x, -v)$. The Gâteaux derivative of f at $a \in G$ will be denoted by $f'_G(a)$ and the Fréchet derivative by $f'_F(a)$.

We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y, and by $\mathcal{K}(X, Y)$ the subspace of $\mathcal{L}(X, Y)$ formed by all compact operators.

1.2. Ordered Banach spaces, and convex operators

By an *ordered normed space* we mean a real normed linear space Y with a given closed convex cone Y_+ which is pointed, i.e., $Y_+ \cap (-Y_+) = \{0\}$. In this case, Y is partially ordered by the relation

$$y_1 \le y_2 \iff y_2 - y_1 \in Y_+.$$

The dual Y^* of an ordered normed space Y is always an ordered Banach space with the *dual positive cone*

$$Y^*_+ = \{y^* \in Y^* : y^*(y) \ge 0 \text{ for each } y \in Y_+\}.$$

Definition 1.1. Let Y be an ordered normed space. We say that:

(a) Y is *normal* if there exists a constant $C > 0$ such that $x, y \in Y$ and $0 \le x \le y$ imply $\|x\| \le C\|y\|$;
(b) Y is *countably Daniell* if every decreasing sequence in Y_+ converges.

If Y is a Banach lattice, then Y is countably Daniell if and only if Y is order complete and order continuous (see [21, Proposition 1.a.8]).

Fact 1.2. *For an ordered normed space Y, the following assertions are equivalent:*

(i) *Y is normal;*
(ii) *Y admits an equivalent norm which is monotone on Y_+ (i.e., $0 \le x \le y \Rightarrow \|x\| \le \|y\|$);*
(iii) *Y^*_+ is generating, i.e., $Y^* = Y^*_+ - Y^*_+$;*
(iv) *Y^*_+ is generating in the following strong sense: there exists a constant $M_Y > 0$ such that each $y^* \in Y^*$ can be written in the form $y^* = y^*_1 - y^*_2$ with $y^*_i \in Y^*_+$ and $\|y^*_i\| \le M_Y\|y^*\|$ $(i = 1, 2)$.*

Proof. For the equivalence (i) $\Leftrightarrow$ (ii) see [1, Theorem 2.38]. For (i) $\Leftrightarrow$ (iv) see [14, Theorem 3.6.2]. For (iii) $\Leftrightarrow$ (iv) apply [1, Theorem 2.37] to Y^*. $\square$

The following lemma is an easy consequence of Fact 1.2.

Lemma 1.3. *Let Y be a normal ordered normed space. Then there exists a constant $K_Y > 0$ such that for each $y \in Y$ there exists $y^* \in Y^*_+$ such that $\|y^*\| \le K_Y$ and $|y^*(y)| = \|y\|$.*

Proof. Put $K_Y := 2M_Y$, where $M_Y > 0$ is as in Fact 1.2(iv). Let $0 \neq y \in Y$ be given. Find $z^* \in Y^*$ with $\|z^*\| = 1$, $z^*(y) = \|y\|$ and write $z^* = z_1^* - z_2^*$, where $z_i^* \in Y_+^*$ and $\|z_i^*\| \leq M_Y$, $i = 1, 2$. It follows that, for at least one $k \in \{1, 2\}$, we have $|z_k^*(y)| \geq (1/2)\|y\|$. Setting $y^* := \frac{\|y\|}{|z_k^*(y)|} z_k^*$, we have $|y^*(y)| = \|y\|$ and $\|y^*\| \leq 2\|z_k^*\| \leq 2M_Y = K_Y$. The case $y = 0$ is trivial. □

Observation 1.4. *Let Y be an ordered normed space whose norm is monotone on Y_+. If $x, u, v \in Y$ are such that $u \leq x \leq v$, then $\|x\| \leq 2(\|u\| + \|v\|)$.*

Proof. Since $0 \leq x - u \leq v - u$, we have $\|x - u\| \leq \|v - u\|$, and hence $\|x\| \leq \|x - u\| + \|u\| \leq \|v - u\| + \|u\| \leq 2\|u\| + \|v\| \leq 2(\|u\| + \|v\|)$. □

We will also use the following fact.

Fact 1.5. *For an ordered Banach space X, consider the following assertions:*

(i) *X is countably Daniell;*
(ii) *X is normal.*

Then (i) *implies* (ii), *but not vice versa. For a reflexive X,* (i) *and* (ii) *are equivalent.*

Proof. See [1, Theorem 2.45] (note that X is countably Daniell if and only if X_+ has the Levi property in the terminology of [1]). □

Definition 1.6. We say that a convex subset B of an ordered normed space Y is a base for Y_+ if, for each $y \in Y_+ \setminus \{0\}$, there exists a unique $\lambda > 0$ such that $\lambda y \in B$. Following [14, p. 120], we say that Y_+ is *well-based* if it has a bounded base B such that $0 \notin \overline{B}$.

Definition 1.7. Let X be a Banach space, $A \subset X$ an open convex set and Y an ordered normed linear space. A mapping $f : A \to Y$ is called a *convex operator* if

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

whenever $x_1, x_2 \in A$ and λ_1, λ_2 are positive reals with $\lambda_1 + \lambda_2 = 1$.

In the same way as for real-valued convex functions it is easy to see that, for each $x \in A$ and each $v \in X$,

$$\text{the function } t \mapsto \frac{f(x + tv) - f(x)}{t} \text{ is nondecreasing} \tag{1}$$

on the set $\{t \in \mathbb{R} \setminus \{0\} : x + tv \in A\}$. This readily implies the following fact (cf. [5, Proposition 3.7]).

Fact 1.8. *Let X, Y, A, f be as in Definition 1.7. If Y is countably Daniell, $a \in A$ and $v \in X$, then $f'_+(a, v)$ exists and $f'_+(a, v) \leq (f(a + tv) - f(a))/t$ for each $t > 0$ with $a + tv \in A$.*

Recall also the following well-known property (see [5, Corollary 2.4.(d)]; for an alternative proof see [26, Corollary 2.3(b)]).

Fact 1.9. *Let X, Y, A, f be as in Definition 1.7. If Y is normal and f is continuous on A, then f is locally Lipschitz on A.*

The convex operator f is called *order bounded at a point $a \in A$* if there exists a neighborhood U of a and $y \in Y$ such that $f(x) \leq y$ for each $x \in U$. If f is order bounded at all points of A then f is called *order bounded*. Valadier [25] proved that *every order bounded convex operator is continuous.*

The following (possibly new) proposition is an easy generalization of the well-known fact that a continuous convex function on a Banach space is Clarke regular.

Proposition 1.10. *Let A be an open convex set in a Banach space X, let Y be a countably Daniell ordered Banach space, and let $x \in A$ and $v \in X$. Let $f : A \to Y$ be a continuous convex operator. Then*

$$\lim_{\substack{z \to x \\ t \to 0}} \frac{f(z + tv) - f(z)}{t} = f'(x, v) \tag{2}$$

whenever $f'(x, v)$ exists.

Proof. By Facts 1.5 and 1.2 (ii) we can suppose that the norm of Y is monotone on Y_+. Suppose that $f'(x,v)$ exists. Given $\varepsilon > 0$, find $r > 0$ such that

$$\left\| \frac{f(x+tv)-f(x)}{t} - f'(x,v) \right\| < \varepsilon$$

for $0 < |t| \le r$ and (using Fact 1.9) such that f is Lipschitz on $B(x, r(1+\|v\|))$ with a Lipschitz constant $K > 0$. Let $\|z-x\| < \min(r, \varepsilon r/2K)$ and $0 < t < r$. Then (1) implies

$$\frac{f(z+tv)-f(z)}{t} - f'(x,v) \le \frac{f(z+rv)-f(z)}{r} - f'(x,v) =: p,$$

and

$$\frac{f(z+tv)-f(z)}{t} - f'(x,v) \ge \frac{f(z)-f(z-rv)}{r} - f'(x,v) =: q.$$

By the Lipschitz property of f,

$$\|p\| \le \left\| \frac{f(z+rv)-f(z)}{r} - \frac{f(x+rv)-f(x)}{r} \right\| + \left\| \frac{f(x+rv)-f(x)}{r} - f'(x,v) \right\| \le \frac{2K\|z-x\|}{r} + \varepsilon \le 2\varepsilon.$$

Similarly we obtain $\|q\| \le 2\varepsilon$. Since $q \le \frac{f(z+tv)-f(z)}{t} - f'(x,v) \le p$, Observation 1.4 implies that

$$\left\| \frac{f(z+tv)-f(z)}{t} - f'(x,v) \right\| \le 2(\|p\| + \|q\|) \le 8\varepsilon.$$

So we have proved $\lim_{\substack{z\to x \\ t\to 0+}} \frac{f(z+tv)-f(z)}{t} = f'(x,v)$. The same argument with $-v$ instead of v gives

$$\lim_{\substack{z\to x \\ t\to 0-}} \frac{f(z+tv)-f(z)}{t} = \lim_{\substack{z\to x \\ \tau\to 0+}} \frac{f(z-\tau v)-f(z)}{-\tau} = -f'(x,-v) = f'(x,v).$$

This completes the proof of (2). □

1.3. *Known results on differentiability of convex functions*

We will recall a number of known results on differentiability of convex functions. In these theorems, and in our new results on differentiability of convex operators, "small sets" of several types are used.

In our main result (Theorem 3.5), the following notion of an *angle-small* set (introduced in [24]) is used.

Definition 1.11. A set M in a Banach space X is called:

(a) *α-angle porous* (where $\alpha > 0$) if for every $x \in M$ and $\varepsilon > 0$ there exist $z \in X$ and $f \in X^*$ such that $\|z-x\| < \varepsilon$ and

$$M \cap \{w \in X : f(w-z) > \alpha \, \|f\| \|w-z\|\} = \emptyset;$$

(b) *angle-small* if for each $\alpha > 0$ it can be expressed as a countable union of α-angle porous sets.

Theorem 3.5 uses also the notion of a *Γ-null set*. This interesting and important notion which in a sophisticated way "combines category and measure" was defined in [19, Definition 2.1]. We will not use the definition of Γ-null sets, but only the following result.

Theorem 1.12 (*[19, Theorem 3.10]*)**.** *Suppose that Y is a Banach space, G an open subset of a separable Banach space X, S a norm separable subspace of $\mathcal{L}(X,Y)$, and $f: G \to Y$ a locally Lipschitz mapping. Then f is Fréchet differentiable at Γ-almost every point $x \in G$ at which it is regular (see below), Gâteaux differentiable, and $f'_G(x) \in S$.*

Here "f is regular at x" means (see [19, Definition 3.1]) that, for every $v \in X$ for which $f'(x,v)$ exists,

$$\lim_{t\to 0} \frac{f(x+tu+tv)-f(x+tu)}{t} = f'(x,v) \quad \text{uniformly for } \|u\| \le 1. \tag{3}$$

(Notice that clearly (2) implies (3).)

Although [19, Theorem 3.10] was proved under the assumption that f is Lipschitz, our Theorem 1.12 clearly follows, since X is separable and Γ-null sets form a σ-ideal.

The following result was proved in [19, Corollary 3.11] with the help of Theorem 1.12.

Theorem 1.13. *If X^* is separable, $A \subset X$ is a convex open set and f is a continuous convex function on A, then the set $N_F(f)$ of all points $x \in A$ at which f is not Fréchet differentiable is Γ-null.*

Note that Γ-null sets need not be of the first category and vice versa (and that Γ-null sets in $\mathbb{R}^n$ coincide with Lebesgue null sets).

For definitions of small sets of the following types (which will be used in the sequel) we refer the reader to [30].

- The *cone-small sets* ([30, Definition 4.1]) are a natural generalization of angle-small sets (considered in separable spaces) to non-separable Banach spaces. In separable spaces, these two notions coincide.
- The notion of a *Lipschitz hypersurface* and a more restrictive notion of a *DC (or d.c.) hypersurface* ([30, Definition 4.3]) were used in a number of results in separable Banach spaces. Roughly speaking, a Lipschitz hypersurface M in X is a "graph of a Lipschitz function f defined on a closed hyperplane of X" and, if f is a difference of two convex Lipschitz functions, M is called a *DC* hypersurface. A set which can be covered by countably many DC (resp., Lipschitz) hypersurfaces is called a *DC sparse set* (resp., *sparse set*).
- A natural generalization of the sparse sets (considered in a separable X) to non-separable Banach spaces is provided by the σ*-cone-supported sets* ([30, Definition 4.4]). In separable spaces, sparse sets and σ-cone-supported sets coincide.
- Note that both cone-small sets and σ-cone-supported sets are σ*-porous* (even σ*-lower porous*) and thus they are first-category sets.

Now we are ready to recall several known "supergeneric" results on differentiability of convex functions on a Banach space X.

Theorem 1.14 (*[24,22]*)**.** *If X^* is separable, $A \subset X$ is a convex open set and f is a continuous convex function on A, then the set $N_F(f)$ of all points $x \in A$ at which f is not Fréchet differentiable is angle-small.*

Theorem 1.15 (*[29]*)**.** *Let X be an Asplund space, $A \subset X$ a convex open set and f a continuous convex function on A. Then there exists a σ-cone supported set $M_1 \subset A$ and a cone-small set $M_2 \subset A$ such that f is Fréchet differentiable at all points of $A \setminus (M_1 \cup M_2)$.*

Theorem 1.16 (*[28,4]*)**.** *If X is separable, $A \subset X$ is a convex open set and f is a continuous convex function on A, then the set $N_G(f)$ of all points $x \in A$ at which f is not Gâteaux differentiable can be covered by countably many DC hypersurfaces.*

Theorem 1.17 (*[13,12]*)**.** *Let X be either a Gâteaux smooth Banach space or a subspace of an Asplund generated (i.e., a GSG) space. Let $A \subset X$ be a convex open set and f a continuous convex function on A. Then the set $N_G(f)$ of all points $x \in A$ at which f is not Gâteaux differentiable is σ-cone supported.*

(We will not recall the definition [13] of an Asplund generated space; note only that all Asplund spaces and all weakly compactly generated (WCG) spaces are Asplund generated.)

2. Results obtained by a reduction to the convex function case

Borwein proved in [5,6] a number of results on differentiability of convex operators by a reduction to "the convex function case". Using Borwein's proofs together with supergeneric results on differentiability of convex functions, we obtain a number of supergeneric results on differentiability of convex operators.

For example, in the proof of [5, Theorem 5.1(b)], Borwein in fact proved the following obvious consequence of [5, Proposition 3.7(b),(c)] and [5, Proposition 4.2]. Recall that $y^* \in Y^*$ is a *strictly positive functional* (called a "positive functional" in [5]) on an ordered normed space Y if $y^*(y) > 0$ whenever $y \in Y_+ \setminus \{0\}$.

Lemma 2.1. *Let X be a Banach space and Y an ordered Banach space which is normal and countably Daniell. Let y^* be a strictly positive functional on Y. Let $A \subset X$ be an open convex set, $x \in A$, and $f : A \to Y$ a continuous convex operator. Then f is Gâteaux differentiable at x whenever $y^* \circ f$ is Gâteaux differentiable at x.*

(By Fact 1.5, the assumption that Y is normal can be omitted, but this was not noticed in [5].) Using this lemma, we easily obtain the following two theorems.

Theorem 2.2. *Let A be an open convex set in a separable Banach space X, Y an ordered Banach space which is countably Daniell, and $f : A \to Y$ a continuous convex operator. Then:*

(i) *f is Gâteaux differentiable at all points of A except a DC-sparse set;*
(ii) *if X^* is separable, then f is Gâteaux differentiable at all points of A except an angle-small Γ-null set.*

Proof. Set $Z := \overline{\operatorname{span}}\, f(A)$. Then Z is clearly a closed subspace of Y which is separable and countably Daniell. By Fact 1.5, Z is normal. By [5, Propositions 2.7 and 2.8], there exists a strictly positive functional $z^* \in Z_+^*$. To prove (i), observe that by Theorem 1.16 the continuous real-valued convex function $\varphi := z^* \circ f$ is Gâteaux differentiable at all points of A except a DC-sparse set. Thus by Lemma 2.1, $f : A \to Z$, and therefore also $f : A \to Y$, is Gâteaux differentiable at all points of A except a DC-sparse set. To prove (ii), we can proceed as above, using now Theorems 1.14 and 1.13 instead of Theorem 1.16. $\square$

In the same way, Theorem 1.17 implies the following "non-separable" result.

Theorem 2.3. *Assume that either X is a Gâteaux smooth Banach space or X is a subspace of an Asplund generated space. Let A be an open convex set in X and let Y be a separable ordered Banach space which is countably Daniell. Then each continuous convex operator $f : A \to Y$ is Gâteaux differentiable at all points of A except a σ-cone supported set.*

The following "supergeneric" result provides an improvement of the corresponding "generic" result in [5, Theorem 5.2].

Theorem 2.4. *Let A be an open convex set in an Asplund Banach space X, let Y be an ordered Banach space which is countably Daniell and for which Y_+ is well-based, and let $f : A \to Y$ be a continuous convex operator. Then f is Fréchet differentiable at all points of A except a union of a σ-cone supported set and an angle-small set. If X is moreover separable, then f is Fréchet differentiable at all points of A except an angle-small Γ-null set.*

Proof. We can proceed as in the proof of [5, Theorem 5.2] using Theorems 1.15, 1.14 and 1.13. An alternative possibility is to use [26, Proposition 4.1] which gives that f is a delta-convex (d.c., DC) mapping. Since each delta-convex mapping is Fréchet differentiable at all points where its "control function" (which is a continuous convex function) is Fréchet differentiable [27, Proposition 3.9], the assertions of the present theorem follow from Theorems 1.15, 1.14 and 1.13. □

Also the main result Theorem 2.1 of [6] on differentiability of order bounded convex operators is proved by reduction to "the convex function case". So, using the proof of [6, Theorem 2.1] together with theorems of Section 1.3, we obtain a number of corresponding "supergeneric" results. We state explicitly only the following theorem, which follows from the proof of [6, Theorem 2.1], [6, Proposition 3.1] and Theorems 1.14, 1.13 and 1.17. Recall that an ordered Banach space Y is called *Daniell* if each decreasing net in Y_+ is convergent.

Theorem 2.5. *Let A be an open convex set in a Banach space X. Let Y be a Banach lattice which is Daniell, and $f : A \to Y$ an order bounded convex operator. Then the following assertions hold.*

(i) *If X is a separable Asplund space, then f is Fréchet differentiable except for an angle-small Γ-null set.*
(ii) *If X is Gâteaux smooth or X is a subspace of an Asplund generated space, then f is Gâteaux differentiable except for a σ-cone supported set.*

3. The main results

To prove our main results on differentiability of convex operators, we will work with generalized monotone mappings, following [16].

Definition 3.1. Let X be a Banach space, Y an ordered normed space and $T : X \rightrightarrows \mathcal{L}(X, Y)$ a multivalued mapping. We will say that T is a *generalized monotone mapping* if

$$(A_1 - A_2)(x_1 - x_2) \geq 0 \quad \text{whenever } x_i \in X,\ A_i \in T(x_i),\ i = 1, 2.$$

We set $D(T) := \{x \in X : T(x) \neq \emptyset\}$.

We will need the following easy fact. Its first part goes back to [18] and the second one is an analogue of [16, Theorem 3.1]. However, we are not going to use the notion of the subdifferential of a convex operator; for our purposes it is sufficient (and more convenient) to work only with Gâteaux derivatives.

Lemma 3.2. *Let A be an open convex set in a Banach space X. Let Y be an ordered Banach space which is countably Daniell, and $f : A \to Y$ a continuous convex operator. Let f be Gâteaux differentiable at each point of a set $D \subset A$. Set $T(x) := \{f'_G(x)\}$ for $x \in D$ and $T(x) := \emptyset$ for $x \in X \setminus D$. Then the following hold.*

(i) *$T : X \rightrightarrows \mathcal{L}(X, Y)$ is a generalized (multivalued) monotone mapping.*
(ii) *If D is dense in A, and T is upper-semicontinuous at a point $a \in D$ (i.e., $f'_G : D \to \mathcal{L}(X, Y)$ is continuous at a), then f is Fréchet differentiable at a.*

Proof. (i) Let $x_1, x_2 \in D$, $x_1 \neq x_2$. Using Fact 1.8 (with $a = x_1$, $v = x_2 - x_1$, $t = 1$), we obtain

$$f(x_2) - f(x_1) \geq f'_+(x_1, x_2 - x_1) = f'_G(x_1)(x_2 - x_1) = -f'_G(x_1)(x_1 - x_2).$$

Similarly we obtain $f(x_1) - f(x_2) \geq f'_G(x_2)(x_1 - x_2)$. Summing these two inequalities, we obtain $0 \geq -(f'_G(x_1) - f'_G(x_2))(x_1 - x_2)$.

(ii) Let $K_Y > 0$ be the number from Lemma 1.3. Choose an arbitrary $\varepsilon > 0$. Find $\delta > 0$ such that $B(a, \delta) \subset A$, and $\|f'_G(x) - f'_G(a)\| < \varepsilon$ for each $x \in B(a, \delta) \cap D$. Set $g(x) := f(x) - f(a) - f'_G(a)(x - a)$. Then g is clearly a continuous convex operator on A, $g(a) = 0$ and $g'_G(x) = f'_G(x) - f'_G(a)$ for $x \in D$. Thus $g'_G(a) = 0$ and $\|g'_G(x)\| < \varepsilon$ for each $x \in B(a, \delta) \cap D$. Now consider an arbitrary $x \in B(a, \delta)$. By Lemma 1.3 we can choose $y^* \in Y^*_+$ such that $|y^*(g(x))| = \|g(x)\|$ and $\|y^*\| \leq K_Y$. Set $h(x) := y^*(g(x))$ for $x \in A$. Then h is a continuous convex function, $h(a) = 0$ and $h'_G(x) = y^* \circ g'_G(x)$ for $x \in D$.

So $h'_G(a) = 0$ and $\|h'_G(x)\| \le \|y^*\|\|g'_G(x)\| \le K_Y\varepsilon$ for each $x \in B(a,\delta)\cap D$. Choose a sequence $\{x_n\}$ in $B(a,\delta)\cap D$ which converges to x. Then $h(x_n) = h(x_n) - h(a) \ge h'_G(a)(x_n - a) = 0$. Further $h(a) - h(x_n) \ge h'_G(x_n)(a - x_n)$, and so $|h(x_n)| = h(x_n) - h(a) \le h'_G(x_n)(x_n - a) \le K_Y\varepsilon\|x_n - a\|$. By continuity of h we obtain $|h(x)| \le K_Y\varepsilon\|x-a\|$. So

$$\|f(x) - f(a) - f'_G(a)(x-a)\| = \|g(x)\| = |h(x)| \le K_Y\varepsilon\|x-a\| \quad \text{for } x \in B(a,\delta).$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows. □

The following "supergeneric" theorem is an analogue of the "generic" result [16, Theorem 3.11] (in which Y is a conjugate lattice, S does not occur, and $\mathcal{L}(X,Y)$ has the Radon–Nikodým property). Our proof is a refinement of the proof of [24, Theorem 1]. Note that, in the most interesting cases in which [16, Theorem 3.11] works, our result also works (with $S = \mathcal{L}(X,Y)$) and gives a stronger conclusion. Theorem 3.3 will be used to prove our main result, Theorem 3.5.

Theorem 3.3. *Let X be a Banach space and Y a normal ordered normed space. Let S be a separable subset of $\mathcal{L}(X,Y)$. Suppose that $T: X \rightrightarrows \mathcal{L}(X,Y)$ is a (multivalued) generalized monotone mapping with an arbitrary domain $D(T) = \{x \in X : T(x) \ne \emptyset\}$ such that $T(x)\cap S \ne \emptyset$ for each $x \in D(T)$. Then there exists an angle-small set $M \subset D(T)$ such that T is single-valued and upper-semicontinuous at every $x \in D(T)\setminus M$.*

Proof. Let $K_Y > 0$ be as in Lemma 1.3, and let $C \subset S$ be a countable dense set in S. We need to show that the set

$$M := \left\{x \in D(T) : \lim_{\delta\to 0+} \operatorname{diam} T[B(x,\delta)] > 0\right\}$$

is angle-small. So fix an arbitrary $\alpha > 0$. For each $x \in M$ there exist $n_x \in \mathbb{N}$ and $c_x \in C$ such that

$$\lim_{\delta\to 0+} \operatorname{diam} T[B(x,\delta)] > \frac{1}{n_x} \quad \text{and} \quad \operatorname{dist}(c_x, T(x)) < \frac{\alpha}{2n_xK_Y}.$$

Since M is the (countable) union of the sets

$$M_{n,c} := \{x \in M : n_x = n, \ c_x = c\} \quad (n \in \mathbb{N}, \ c \in C),$$

it is sufficient to prove that each $M_{n,c}$ is α-angle porous.

Fix $x \in M_{n,c}$ and $\varepsilon > 0$. Since $\lim_{\delta\to 0+}\operatorname{diam} T[B(x,\delta)] > \frac{1}{n}$, we can choose $z \in B(x,\varepsilon)$ and $T_z \in T(z)$ such that $\|T_z - c\| > \frac{1}{2n}$. Consider $\widetilde{x} \in S_X$ such that $\|(T_z - c)(\widetilde{x})\| > \frac{1}{2n}$ and, by the choice of K_Y, find $y^* \in Y^*_+$ such that $\|y^*\| \le K_Y$ and $|y^*((T_z-c)(\widetilde{x}))| = \|(T_z-c)(\widetilde{x})\| > \frac{1}{2n}$. Thus we have $\|y^*\circ(T_z-c)\| > \frac{1}{2n}$.

To prove that $M_{n,c}$ is α-angle porous, it is sufficient to prove that

$$M_{n,c} \cap \left\{w \in X : [y^*\circ(T_z-c)](w-z) > \alpha\,\|y^*\circ(T_z-c)\|\,\|w-z\|\right\} = \emptyset.$$

To this end, consider $w \in D(T)$ and $T_w \in T(w)$ such that

$$[y^*\circ(T_z-c)](w-z) > \alpha\,\|y^*\circ(T_z-c)\|\,\|w-z\|.$$

Using the inequalities $(T_w - T_z)(w-z) \ge 0$, $\|y^*\circ(T_z-c)\| > \frac{1}{2n}$, and the fact that $y^* \in Y^*_+$, we obtain

$$\begin{aligned}[y^*\circ(T_w-c)](w-z) &= [y^*\circ(T_w-T_z)](w-z) + [y^*\circ(T_z-c)](w-z)\\ &\ge [y^*\circ(T_z-c)](w-z) > \alpha\,\|y^*\circ(T_z-c)\|\,\|w-z\| > \frac{\alpha}{2n}\|w-z\|.\end{aligned}$$

Hence $\|y^*\circ(T_w-c)\| > \frac{\alpha}{2n}$, and so $K_Y\|T_w - c\| \ge \|y^*\|\|T_w-c\| \ge \frac{\alpha}{2n}$, which implies that $\|T_w - c\| \ge \frac{\alpha}{K_Y2n}$ for every $T_w \in T(w)$. So $\operatorname{dist}(c, T(w)) \ge \frac{\alpha}{2nK_Y}$ and thus $w \notin M_{n,c}$. □

We will deduce our main result, Theorem 3.5, from Theorems 3.3 and 1.12 via the following proposition.

Proposition 3.4. *Let A be an open convex set in a Banach space X. Let Y be a countably Daniell ordered Banach space. Let S be a norm separable subset of $\mathcal{L}(X,Y)$. Let $f: A \to Y$ be a continuous convex operator. Denote by D the set of all $x \in A$ at which f is Gâteaux differentiable and $f'_G(x) \in S$. Then the following hold.*

(i) *If D is dense in A, then f is Fréchet differentiable at all points of D except a set which is angle-small.*
(ii) *If X is separable, then f is Fréchet differentiable at all points of D except a Γ-null set.*

Proof. To prove (i), set $T(x) := \{f'_G(x)\}$ for $x \in D$ and $T(x) := \emptyset$ for $x \in X\setminus D$. Then T is a (multivalued) generalized monotone mapping by Lemma 3.2(i). Since $D = D(T)$, by Theorem 3.3 there exists an angle-small set $M \subset D$ such that T is single-valued and upper-semicontinuous at every $x \in D\setminus M$. Now, Lemma 3.2(ii) gives that f is Fréchet differentiable at every $x \in D\setminus M$.

To prove (ii), observe that f is locally Lipschitz by Facts 1.9 and 1.5. Further, Proposition 1.10 readily implies that f is regular, in the sense of Theorem 1.12, at each point of M. So (ii) follows from Theorem 1.12. □

Theorem 3.5. *Let A be an open convex set in a Banach space X. Let Y be a countably Daniell ordered Banach space. Suppose that the space $\mathcal{L}(X, Y)$ of all bounded linear operators is separable. Then each continuous convex operator $f: A \to Y$ is Fréchet differentiable except for a Γ-null angle-small set.*

Proof. First observe that the separability of $\mathcal{L}(X, Y)$ clearly implies that both X^* and Y are separable. Let D be the set of all Gâteaux differentiability points of f. By Theorem 2.2, the set $A \setminus D$ is angle-small and Γ-null. In particular, D is dense in A. So, using Proposition 3.4 with $S = \mathcal{L}(X, Y)$, we obtain the assertion of our theorem. □

Remark 3.6. The assertion on angle-smallness in Theorem 3.5 remains valid under the assumption that Y is normal and has the "monotone sequence property" (MSP, saying that every decreasing sequence in Y_+ has an infimum), which is weaker than the assumption that Y is countably Daniell. The proof can be done in a similar way by using "order Gâteaux derivatives" instead of Gâteaux derivatives.

However, we do not know whether $\mathcal{L}(X, Y)$ can be separable for some Y which is normal, has the MSP property but fails to be countably Daniell. If Y is a Banach lattice, this is impossible (this easily follows from [21, Proposition 1.a.7]).

Remark 3.7. Let us compare Theorem 3.5 with the previous results Theorems 2.2(ii), 2.4 (last part) and 2.5(i). These theorems yield similar conclusions under the assumption that X^* is separable (i.e., X is a separable Asplund space), which is weaker than the assumption that $\mathcal{L}(X, Y)$ is separable in Theorem 3.5. However:

(a) Theorem 2.2(ii) concerns only Gâteaux differentiability;
(b) Theorem 2.4 (last part) uses the strong assumption that Y_+ is well-based;
(c) Theorem 2.5(i) assumes that Y is a Banach lattice that is Daniell, and f is order bounded.

4. An application of the method of separable reduction

In this section, using the separable reduction method, we will show that Theorem 3.5 implies some "supergeneric" results on differentiability of convex operators with non-separable domains. Let us start with the following auxiliary fact (for the proof see [20, pp. 114–115]).

Fact 4.1. *Let X be either $C(K)$ for countable compact K or a closed subspace of c_0, and let Y be separable and have the Radon–Nikodým property. Then $\mathcal{L}(X, Y)$ is separable.*

Proposition 4.2. *Let Y be a countably Daniell ordered Banach space with the Radon–Nikodým property. Assume that*

(a) *either X is a closed subspace of $c_0(\Delta)$, where Δ is an uncountable set,*
(b) *or $X = C(K)$, where K is a scattered compact topological space.*

Let $A \subset X$ be an open convex set and $f: A \to Y$ a continuous convex operator. Then f is Fréchet differentiable on A except for a σ-lower porous Γ-null set.

Proof. The statement on Γ-nullness is essentially proved in [8, Theorem 6.16] (cf. also [20, (4), p.45] for the case (a)), where it is shown that the statement holds if f is an arbitrary Lipschitz mapping. In our case we know only that f is locally Lipschitz (see Facts 1.9 and 1.5). But we can easily repeat the proof of [8, Theorem 6.16] for locally Lipschitz f (using the easy fact that a locally Γ-null set in a separable Banach space is clearly Γ-null). An alternative possibility is to show by a standard argument that a locally Γ-null set in every Banach space is Γ-null.

The statement on σ-lower porosity in the case (a) easily follows from Theorem 3.5 and the following result.

Proposition CR. *Let X, Y be Banach spaces, $G \subset X$ an open set and $f: G \to Y$ an arbitrary mapping. Let, for each closed separable subspace V of X, the restriction $f \restriction_{V \cap G}$ be Fréchet differentiable on $V \cap G$ except for a σ-lower porous set (in V). Then f is Fréchet differentiable on G except for a σ-lower porous set (in X).*

Indeed, it is easy to see that, in the case (a), each closed separable subspace V of X is a subspace of a space Z isomorphic to c_0 (we can choose $Z := \{x \in c_0(\Delta) : x(s) = 0 \text{ for } s \notin C\}$ for some countable $C \subset \Delta$). Define $\widetilde{f} := f \restriction_{V \cap A}$ and $\widetilde{Y} := \overline{\operatorname{span}} f(V \cap A)$. Applying Fact 4.1 and Theorem 3.5 to $\widetilde{f} : V \cap A \to \widetilde{Y}$, we obtain that $\widetilde{f}$ is Fréchet differentiable except for an angle-small set. Since each angle-small set is σ-lower porous, Proposition CR implies our statement.

Proposition CR is essentially proved (but not stated) in [9]. Indeed, if we write in Proposition CR "σ-upper porous" instead of "σ-lower porous", we obtain a proposition which is an immediate consequence of [9, Theorem 1.2]. This latter theorem is a consequence of [8, Theorem 5.7] and [9, Theorem 5.1]. Since only [9, Theorem 5.1] deals with σ-upper porosity and [9, Theorem 5.4] on σ-lower porosity is a precise analogue of [9, Theorem 5.1], it is easy to see that the σ-lower porosity analogue of [9, Theorem 5.1], and hence also Proposition CR, holds.

The statement on σ-lower porosity in the case (b) is more complicated. It follows easily by a combination of results from [8,9], but these results use the (set theoretical) notion of a "suitable elementary submodel" which cannot be briefly explained here. So we sketch the arguments which become clear only after reading parts of articles [8,9]. First we observe that [8, Lemmas 3.5 and 6.6] imply that, for any suitable elementary submodel M, the set $X_M := \overline{X \cap M}$ is a closed subspace of $X = C(K)$ which is linearly isometric to a space $C(L)$, where L is a countable compact (cf. the proof of [8, Theorem 6.16]). Thus, arguing as in case (a), we obtain that $f\restriction_{X_M}$ is Fréchet differentiable at all points of $A \cap X_M$ except a σ-lower porous set (in X_M) by Theorem 3.5 and Fact 4.1. Now, using [9, Theorem 5.4] and [8, Theorem 5.7], we easily obtain our assertion (like in the proof of [9, Theorem 5.6]). □

5. Examples and applications

In this last section, we will recall several known examples and results on differentiability of convex operators and present some applications of our stronger results. We will also present *several consequences of our "supergeneric" results (and "Γ-null" results) which are formulated in standard terms and do not follow from generic results — Examples 5.6 and 5.7* are of this type. Other applications are "*joint differentiability results*" in Examples 5.3 and 5.5 which use results of [19,20].

Example 5.1. Let *either* $X = \ell_p$ and $Y = \ell_q$ where $1 \leq p \leq q < \infty$, *or* $X \in \{\ell_p : 1 \leq p < \infty\} \cup \{c_0\}$ and $Y = c_0$. Then $g: X \to Y$, $g: (x_i) \mapsto (|x_i|)$, is a nowhere Fréchet differentiable Lipschitz convex operator.

For $X = Y$ this is well-known (see [16, Example 4.3], [5, Example 9.2]); the other cases can be proved in the same way.

Example 5.2. Let $X = \mathbb{R}^n$ and let Y be a normal ordered Banach space. Then any continuous convex operator $f: X \to Y$ is locally Lipschitz by Fact 1.9, and hence Gâteaux and Fréchet differentiability of f at a point are equivalent. Proofs of [16, Theorems 3.5 and 3.6] give that, *if Y is an order complete Banach lattice, then the following assertions are equivalent:*

(i) *each f as above is generically differentiable;*
(ii) *each f as above is differentiable at some point;*
(iii) *the space ℓ_∞ does not embed into Y.*

In particular, there exists a nowhere differentiable continuous convex operator $g: \mathbb{R} \to \ell_\infty$ ([16, Example 4.2(a)]). Since (for an order complete Banach lattice) (iii) implies that Y is countably Daniell (see [21, Proposition 1.a.7]), Theorem 2.2 implies that *the above conditions* (i)–(iii) *are equivalent to any of the following conditions:*

(iv) *each f as above is differentiable except for a DC-sparse set;*
(v) *each f as above is a.e. differentiable.*

Example 5.3. Theorem 2.4 (and also the corresponding original generic result of Borwein) applies only exceptionally, since there are not many ordered Banach spaces with well-based positive cone (and which are countably Daniell). The most classical infinite dimensional example is $L_1(\mu)$ (where μ is an arbitrary measure).

So we obtain that if X is an Asplund space and $f: X \to L_1(\mu)$ is a continuous convex operator, then f is Fréchet differentiable except for a σ-lower porous set.

If X is moreover separable, then f is Fréchet differentiable except for an angle-small Γ-null set. Consequently, if $g: X \to Z$, where Z has RNP, is a Lipschitz mapping, then at some points of X (in fact, at all points of X except a Γ-null set), we have both that f is Fréchet differentiable and that g is Gâteaux differentiable. Indeed, by [19, Theorem 2.5] g is Gâteaux differentiable except for a Γ-null set. (This "joint differentiability result" generalizes that of [19, p. 260] in which $Y = \mathbb{R}$.) Moreover, if h is a Lipschitz real function on X, then at some points of X both f and h are Fréchet differentiable. This clearly follows from [20, Theorem 12.1.1] which implies that h is Fréchet differentiable at all points of a set which is not σ-porous, and from the fact that each angle-small set is σ-porous.

For the next example, we shall need the following well-known fact for which we have not found an explicit reference.

Fact 5.4. *Let X, Y be Banach spaces such that both X^* and Y are separable and at least one of them has the approximation property. Then $\mathcal{K}(X, Y)$ is separable.*

This is an easy consequence of the fact that if X^* or Y has the approximation property, then the set of all finite rank operators is dense in $\mathcal{K}(X, Y)$ (see e.g. [21, Theorems 1.e.4 and 1.e.5]). Indeed, it is easy to show that the countable set of operators of the form $T(x) = \sum_{i=1}^{n} x_i^*(x) y_i$, where x_i^* and y_i belong to fixed countable dense subsets of X^* and Y, respectively, is dense in the set of all finite rank operators.

Example 5.5. Theorem 3.5 can be applied in the following cases:

(i) $X = \ell_p$, $Y = \ell_q$, $1 < q < p < \infty$,
(ii) $X = \ell_p$, $Y = L_q[0, 1]$, $\max(q, 2) < p < \infty$, $1 < q$,
(iii) $X = c_0$, $Y = \ell_q$, $1 < q < \infty$,
(iv) $X = c_0$, $Y = L_q[0, 1]$, $1 < q < \infty$,
(v) $X = L_p[0, 1]$, $Y = \ell_q$, $1 < q < \min(p, 2)$, $p < \infty$.

Indeed, in these cases $\mathcal{L}(X, Y) = \mathcal{K}(X, Y)$ (see [15, p. 310]) and so $\mathcal{L}(X, Y)$ is separable by Fact 5.4. Hence, in all cases (i)–(v), if $f: X \to Y$ is a continuous convex operator, then f is Fréchet differentiable at all points except an angle-small Γ-null set.

Note that the corresponding generic results follow from Kirov's [16, Corollary 3.12] in which it is supposed that Y is a dual Banach lattice and $\mathcal{L}(X, Y)$ has the Radon–Nikodým property. Indeed (see [10, p. 165]), Pettis's last theorem says that *if $\mathcal{L}(X, Z^*)$ is separable, then $\mathcal{L}(X, Z^*)$ has the Radon–Nikodým property*. Thus, in all cases where Theorem 3.5 applies and Y is a dual Banach lattice, the generic result follows also from Kirov's [16, Corollary 3.12].

In the same way as in Example 5.3, we obtain in all cases (i)–(v) the following "joint differentiability result".

If $f: X \to Y$ is a continuous convex operator and $g: X \to Z$, where Z has RNP, is a Lipschitz mapping, then at some points of X we have both that f is Fréchet differentiable and that g is Gâteaux differentiable. Moreover, if h is a Lipschitz real function on X, then at some points of X both f and h are Fréchet differentiable.

Example 5.6. Let Γ be an infinite (countable or uncountable) set, $p > 1$, Z an ordered Banach space which is countably Daniell, and $f: \ell_p(\Gamma) \to Z$ a continuous convex operator. Assume that either Z is separable or f is order bounded. Then the set of all $x \in \ell_1(\Gamma) \subset \ell_p(\Gamma)$ at which f is Gâteaux differentiable is uncountable and dense in $\ell_p(\Gamma)$.

Proof. Since $\ell_p(\Gamma)$ is an Asplund space, Theorems 2.3 and 2.5 imply that the set $N_G(f)$ of all $x \in \ell_p(\Gamma)$ at which f is not Gâteaux differentiable is σ-cone supported. Let $\emptyset \neq H \subset \ell_p(\Gamma)$ be an open set. Then [31, Lemma 4] (used with $Y = \ell_1(\Gamma)$, $D = G = H \cap \ell_1(\Gamma)$ and $g = id: \ell_1(\Gamma) \to \ell_p(\Gamma)$) implies that $\ell_1(\Gamma) \cap H$ is not σ-cone supported. So the set $(\ell_1(\Gamma) \setminus N_G(f)) \cap H$ is not σ-cone supported, and so it is uncountable. $\square$

Example 5.7. Let Z be an ordered Banach space which is countably Daniell, and $f: C[0, 1] \to Z$ a continuous convex operator. Then the set of all increasing real analytic functions $x \in C[0, 1]$ at which f is Gâteaux differentiable has cardinality c.

Proof. Theorem 2.2 implies that the set $N_G(f)$ of all elements of $C[0, 1]$ at which f is not Gâteaux differentiable is DC-sparse. Thus $N_G(f)$ is σ-cone supported.

Set $Y = G := \ell_2$, $D := \{(a_n) \in \ell_2 : a_n \geq 0\}$, and define

$$g: \ell_2 \to C[0, 1], \qquad g\big((a_n)\big)(t) := \sum_{n=1}^{\infty} 2^{-n} a_n t^{n-1}.$$

It is easy to see that $g: \ell_2 \to C[0, 1]$ is a continuous linear operator and $\overline{g(\ell_2)} = C[0, 1]$. Further, D is a Baire metric subspace of Y and the set $N(D)$ of all non-support points of D is residual in D (since $N(D)$ is a G_δ dense subset of D; cf. [31, (F5)–(F7)]). Clearly each element of $C := g(D)$ is an increasing real analytic function. By [31, Lemma 5] the set $C \setminus N_G(f)$ has cardinality c, which implies our assertion. $\square$

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