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Existence of a class of rotopulsators

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ABSTRACT

We prove the existence of a class of rotopulsators for the n-body problem in spaces of constant Gaussian curvature of dimension $k \geq 2$.

1. Introduction

By n-body problems, we mean problems where we want to find the dynamics of n point particles. If the space in which such a problem is defined is a space of zero Gaussian curvature, then we call any solution to such a problem for which the point particles describe the vertices of a polytope that retains its shape over time (but not necessarily its size) a homographic orbit.

A rotopulsator, also known as a rotopulsating orbit, is a type of solution to an n-body problem for spaces of constant Gaussian curvature $\kappa \neq 0$ that extends the definition of homographic orbits to spaces of constant Gaussian curvature (see [7]).

Homographic orbits (and therefore rotopulsators) can be used to determine the geometry of the universe locally (see for example [4,7]).

In this paper, we will prove the existence of a subclass of rotopulsators that form a natural generalization of orbits found in [4,6].

While the orbits in [4,6] are referred to as homographic, it has been argued in [7] that using the term 'homographic orbit' for spaces of constant Gaussian curvature makes little sense. We will therefore, following [7], speak of rotopulsating orbits instead.

While this paper mainly builds on results obtained in [4,6,22], research on n-body problems for spaces of constant Gaussian curvature goes back to Bolyai [1] and Lobachevsky [19], who independently proposed a curved 2-body problem in hyperbolic space $\mathbb{H}^3$ in the 1830s. In later years, n-body problems for spaces of constant Gaussian curvature have been studied by mathematicians such as Dirichlet, Schering [20,21], Killing [12–14] and Liebmann [16–18]. More recent results were obtained by Kozlov and Harin [15], but the study of n-body problems in spaces of constant Gaussian curvature for the case that $n \geq 2$ started with [9–11] by Diacu, Pérez-Chavela, and Santoprete. Further results for the $n \geq 2$ case were then

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obtained by Cariñena, Rañada, and Santander [2], Diacu [3,4,6], Diacu and Kordlou [7], and Diacu and Pérez-Chavela [8]. For a more detailed historical overview, please see [4,6,5,7], or [9].

In this paper, we will prove the following two theorems:

Theorem 1.1. *For any rotopulsating solution of (2.2) formed by vectors $\{\mathbf{q}_i\}_{i=1}^n$ as defined in (2.3), the vectors $\{\mathbf{Q}_i\}_{i=1}^n$ have to form a regular polygon if ρ is non-constant.*

Theorem 1.2. *Rotopulsating orbits formed by vectors $\{\mathbf{q}_i\}_{i=1}^n$ as defined in (2.3) exist if the vectors $\{\mathbf{Q}_i\}_{i=1}^n$ form a regular polygon.*

To prove these theorems, we will use a method strongly inspired by [4,6,22]. Specifically, we will first deduce a necessary and sufficient criterion for the existence of rotopulsators. This will be done in Section 2. We will then prove Theorems 1.1 and 1.2 in Sections 3 and 4 respectively.

2. A criterion for the existence of rotopulsators

In this section, we will formulate a necessary and sufficient criterion for the existence of rotopulsating orbits of the type described in (2.3).

Consider the n-body problem in spaces of constant Gaussian curvature $\kappa \neq 0$.

As has been shown in [5], we may assume that κ equals either -1, or 1.

We will denote the masses of its n point particles to be $m_1, m_2, \ldots, m_n > 0$ and their positions by the k-dimensional vectors

$$\mathbf{q}_i^T = (q_{i1}, q_{i2}, \ldots, q_{ik}) \in \mathbf{M}_\kappa^{k-1}, \quad i = \overline{1, n}$$

where

$$\mathbf{M}_\kappa^{k-1} = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k | \kappa(x_1^2 + x_2^2 + \cdots + x_{k-1}^2 + \sigma x_k^2) = 1\}, \quad k \in \mathbb{N}$$

and

$$\sigma = \begin{cases} 1 & \text{for } \kappa > 0 \\ -1 & \text{for } \kappa < 0. \end{cases}$$

Furthermore, consider for m-dimensional vectors $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ the inner product

$$\mathbf{a} \odot_m \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_{m-1} b_{m-1} + \sigma a_m b_m. \tag{2.1}$$

Then, following [3,4,6,9–11] and the assumption that $\kappa = \pm 1$ from [5], we define the equations of motion for the curved n-body problem as the dynamical system described by

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j - (\sigma \mathbf{q}_i \odot_k \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - (\mathbf{q}_i \odot_k \mathbf{q}_j)^2]^{\frac{3}{2}}} - (\sigma \dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad i = \overline{1, n}. \tag{2.2}$$

Let

$$T(t) = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}$$

be a 2×2 rotation matrix, where $\theta(t)$ is some real valued, twice continuously differentiable, scalar function, for which $\theta(0) = 0$.

We will consider rotopulsating orbit solutions of (2.2) of the form

$$\mathbf{q}_i(t) = \begin{pmatrix} \rho(t) T(t) \mathbf{Q}_i \\ Z(t) \end{pmatrix} \tag{2.3}$$

where $\mathbf{Q}_i \in \mathbb{R}^2$ is a constant vector and $Z(t) \in \mathbb{R}^{k-2}$ is a twice differentiable, vector valued function.

Finally, before formulating our criterion, we need to introduce some notation and a lemma:

Let $m \in \mathbb{N}$. Let $\langle \cdot, \cdot \rangle_m$ be the Euclidean inner product on $\mathbb{R}^m$ and let $\| \cdot \|_m$ be the Euclidean norm on $\mathbb{R}^m$. Let $i, j \in \{1, \ldots, n\}$. By construction $\|\mathbf{Q}_i\|_2 = \|\mathbf{Q}_j\|_2$ for all $i, j \in \{1, \ldots, n\}$ and we will assume that $\|\mathbf{Q}_i\|_2 = 1$. Let β_i be the angle between $\mathbf{Q}_i$ and the first coordinate axis. The lemma we will need to prove our criterion is:

Lemma 2.1. *The functions ρ and θ, are related through the following formula: $\rho^2(t)\dot{\theta}(t) = \rho^2(0)\dot{\theta}(0)$.*

Proof. In [6], using the wedge product, Diacu proved that

$$\sum_{i=1}^{n} m_i \dot{\mathbf{q}}_i \wedge \mathbf{q}_i = \mathbf{c}$$

where $\mathbf{c}$ is a constant bivector.

If $\{\mathbf{e}_i\}_{i=1}^k$ are the standard base vectors in $\mathbb{R}^k$, then we can write $\mathbf{c}$ as

$$\mathbf{c} = \sum_{i=1}^{k}\sum_{j=1}^{k} c_{ij}\mathbf{e}_i \wedge \mathbf{e}_j \tag{2.4}$$

where $\{c_{ij}\}_{i=1,j=1}^k$ are constants. As $\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i$ and $\mathbf{e}_i \wedge \mathbf{e}_i = 0$ (see [6]), for $i, j \in \{1, \ldots, n\}$, we can rewrite (2.4) as

$$\mathbf{c} = \sum_{i=1}^{k}\sum_{j=i+1}^{k} C_{ij}\mathbf{e}_i \wedge \mathbf{e}_j \tag{2.5}$$

where $C_{ij} = c_{ij} - c_{ji}$.

Calculating C_{12} will give us our result:

Note that

$$T^T = T^{-1} \quad \text{and} \quad \dot{T} = \dot{\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \tag{2.6}$$

and

$$\begin{aligned} C_{12} &= \sum_{i=1}^{n} m_i \left(q_{i1}\dot{q}_{i2} - q_{i2}\dot{q}_{i1}\right) \\ &= \sum_{i=1}^{n} m_i \left(q_{i1}, q_{i2}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \dot{q}_{i1} \\ \dot{q}_{i2} \end{pmatrix}. \end{aligned} \tag{2.7}$$

Using (2.3) with (2.7) gives

$$C_{12} = \sum_{i=1}^{n} m_i\rho^2 \left(Q_{i1}, Q_{i2}\right) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{T}\begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} + \sum_{i=1}^{n} m_i\rho\dot{\rho}\left(Q_{i1}, Q_{i2}\right) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T\begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix}. \tag{2.8}$$

Note that

$$\rho\dot{\rho}\left(Q_{i1}, Q_{i2}\right) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T\begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} = \frac{\dot{\rho}}{\rho}\left(q_{i1}, q_{i2}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q_{i1} \\ q_{i2} \end{pmatrix} = 0.$$

So, using (2.6) repeatedly, we get that

$$\begin{aligned} C_{12} &= \sum_{i=1}^{n} m_i\rho^2\dot{\theta}\left(Q_{i1}, Q_{i2}\right) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T\begin{pmatrix} Q_{i1} \\ Q_{i1} \end{pmatrix} + 0 \\ &= \sum_{i=1}^{n} m_i\rho^2\dot{\theta}\left(Q_{i1}, Q_{i2}\right) T^T T\begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \\ &= \sum_{i=1}^{n} m_i\rho^2\dot{\theta}\left(Q_{i1}, Q_{i2}\right)\begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \end{aligned}$$

which means that

$$C_{12} = \rho^2\dot{\theta}\sum_{i=1}^{n} m_i\left(Q_{i1}^2 + Q_{i2}^2\right). \tag{2.9}$$

As, by construction

$$\sum_{i=1}^{n} m_i\left(Q_{i1}^2 + Q_{i2}^2\right) > 0,$$

we may divide both sides of (2.9) by

$$\sum_{i=1}^{n} m_i\left(Q_{i1}^2 + Q_{i2}^2\right),$$

which gives that

$$\rho^2\dot{\theta} = \frac{C_{12}}{\sum_{i=1}^{n} m_i \left(Q_{i1}^2 + Q_{i2}^2\right)},$$

which is constant, so $\rho^2\dot{\theta} = \rho^2(0)\dot{\theta}(0)$. □

We now have the following necessary and sufficient criterion for the existence of a rotopulsating orbit, as described in (2.3):

Criterion 1. Let

$$b_i = \sum_{j=1, j\neq i}^{n} \frac{m_j(1-\cos(\beta_i-\beta_j))^{-\frac{1}{2}}}{(2-\sigma\rho^2(1-\cos(\beta_i-\beta_j)))^{\frac{3}{2}}}. \tag{2.10}$$

Then necessary and sufficient conditions for the existence of a rotopulsating orbit of non-constant size are that $b_1 = b_2 = \cdots = b_n$ and

$$0 = \sum_{j=1, j\neq i}^{n} \frac{m_j \sin(\beta_i-\beta_j)}{(1-\cos(\beta_i-\beta_j))^{\frac{3}{2}}(2-\sigma\rho^2(1-\cos(\beta_i-\beta_j)))^{\frac{3}{2}}} \tag{2.11}$$

for all $i \in \{1, \dots, n\}$.

Proof. Note that

$$\dot{T} = \dot{\theta} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.12}$$

and consequently

$$\ddot{T} = \ddot{\theta} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \dot{\theta}^2 T. \tag{2.13}$$

Inserting (2.3) into (2.2) and using (2.12) and (2.13) gives for the first and second lines of (2.2) that

$$\begin{aligned} &T\left(\ddot{\rho} I_2 + 2\dot{\rho}\dot{\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \rho\left(\ddot{\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \dot{\theta}^2 I_2\right)\right)\mathbf{Q}_i \\ &\quad = \rho T\left(\sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{Q}_j - (\sigma\mathbf{q}_i \odot_k \mathbf{q}_j)\mathbf{Q}_i]}{[\sigma - (\mathbf{q}_i \odot_k \mathbf{q}_j)^2]^{\frac{3}{2}}} - (\sigma\dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i)\mathbf{Q}_i\right) \end{aligned} \tag{2.14}$$

where I_2 is the 2×2 identity matrix.

For the last $k-2$ lines, we get

$$\ddot{Z} = \left(\sum_{j=1, j\neq i}^{n} \frac{m_j[1 - (\sigma\mathbf{q}_i \odot_k \mathbf{q}_j)]}{[\sigma - (\mathbf{q}_i \odot_k \mathbf{q}_j)^2]^{\frac{3}{2}}} - (\sigma\dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i)\right) Z. \tag{2.15}$$

Note that

$$\mathbf{q}_i \odot_k \mathbf{q}_j = \rho^2\langle \mathbf{Q}_i, \mathbf{Q}_j\rangle_2 + Z \odot_{k-2} Z. \tag{2.16}$$

As we have that $\langle \mathbf{Q}_i, \mathbf{Q}_i\rangle_2 = 1$ and as by (2.16),

$$\sigma^{-1} = \mathbf{q}_i \odot_k \mathbf{q}_i = \rho^2\langle \mathbf{Q}_i, \mathbf{Q}_i\rangle_2 + Z \odot_{k-2} Z,$$

we may rewrite (2.16) as

$$\mathbf{q}_i \odot_k \mathbf{q}_j = \sigma^{-1} + \rho^2\langle \mathbf{Q}_i, \mathbf{Q}_j\rangle_2 - \rho^2,$$

which can, in turn, be written as

$$\mathbf{q}_i \odot_k \mathbf{q}_j = \sigma^{-1} + \rho^2(\cos(\beta_i - \beta_j) - 1). \tag{2.17}$$

Furthermore,

$$\dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i = \langle \dot{\rho} T\mathbf{Q}_i + \rho\dot{T}\mathbf{Q}_i, \dot{\rho} T\mathbf{Q}_i + \rho\dot{T}\mathbf{Q}_i\rangle_2 + \dot{Z} \odot_{k-2} \dot{Z}. \tag{2.18}$$

As T is a rotation in $\mathbb{R}^2$, it is a unitary map, meaning that for $v, w \in \mathbb{R}^2$, $\langle Tv, Tw\rangle_2 = \langle v, w\rangle_2$, meaning that (2.18) can be written as

$$\dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i = \langle \dot{\rho}\mathbf{Q}_i + \rho T^{-1}\dot{T}\mathbf{Q}_i, \dot{\rho}\mathbf{Q}_i + \rho T^{-1}\dot{T}\mathbf{Q}_i\rangle_2 + \dot{Z} \odot_{k-2} \dot{Z}. \tag{2.19}$$

Using (2.12) with (2.19) gives

$$\begin{aligned}\dot{\mathbf{q}}_i \odot_k \dot{\mathbf{q}}_i &= \dot{\rho}^2 + 2\rho\dot{\rho}\dot{\theta}\left\langle \mathbf{Q}_i, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Q}_i \right\rangle_2 + \rho^2\dot{\theta}^2\|\mathbf{Q}_i\|^2 + \dot{Z} \odot_{k-2} \dot{Z} \\ &= \dot{\rho}^2 + 0 + \rho^2\dot{\theta}^2 + \dot{Z} \odot_{k-2} \dot{Z}.\end{aligned} \tag{2.20}$$

Inserting (2.20) and (2.17) into (2.14) and multiplying both sides by T^{-1} provides us with

$$\begin{aligned}&\left((\ddot{\rho} - \rho\dot{\theta}^2)I_2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\mathbf{Q}_i \\ &= \rho \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{Q}_j - (1 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))\mathbf{Q}_i]}{[\rho^2(1 - \cos(\beta_i - \beta_j))(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}} \\ &\quad - (\sigma\rho\dot{\rho}^2 + \sigma\rho^3\dot{\theta}^2 + \sigma\rho\dot{Z} \odot_{k-2} \dot{Z})\mathbf{Q}_i.\end{aligned} \tag{2.21}$$

Taking the Euclidean inner product with $\mathbf{Q}_i$ on both sides of (2.21) and using that $\|\mathbf{Q}_i\|_2 = \|\mathbf{Q}_j\|_2 = 1$ provides us with

$$\ddot{\rho} - \rho\dot{\theta}^2 + \sigma\rho\dot{\rho}^2 + \sigma\rho^3\dot{\theta}^2 + \sigma\rho\dot{Z} \odot_{k-2} \dot{Z} = \left(\sigma - \frac{1}{\rho^2}\right) \sum_{j=1, j\neq i}^{n} \frac{m_j\left[(1 - \cos(\beta_i - \beta_j))^{-\frac{1}{2}}\right]}{[(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}}. \tag{2.22}$$

Taking the Euclidean inner product of (2.21) with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{Q}_i$ and using that $\|\mathbf{Q}_i\|_2 = \|\mathbf{Q}_j\|_2 = 1$ gives us that

$$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = \sum_{j=1, j\neq i}^{n} \frac{m_j \sin(\beta_i - \beta_j)}{[(1 - \cos(\beta_i - \beta_j))(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}}. \tag{2.23}$$

Let

$$\begin{aligned}b_i &:= \sum_{j=1, j\neq i}^{n} \frac{m_j\left[(1 - \cos(\beta_i - \beta_j))^{-\frac{1}{2}}\right]}{[(2 - \sigma\rho^2(1 - \cos\alpha_{ij}))]^{\frac{3}{2}}} \quad \text{and} \\ c_i &:= \sum_{j=1, j\neq i}^{n} \frac{-m_j \sin(\beta_i - \beta_j)}{[(1 - \cos(\beta_i - \beta_j))(2 - \sigma\rho^2(1 - \cos(\beta_i - \beta_j)))]^{\frac{3}{2}}}.\end{aligned}$$

Inserting (2.20) and (2.17) into (2.15), combined with (2.22) and (2.23), gives the following system of differential equations:

$$\begin{cases} \ddot{\rho} = \rho\dot{\theta}^2 - \sigma\rho\dot{\rho}^2 - \sigma\rho^3\dot{\theta}^2 - \sigma\rho\dot{Z} \odot_{k-2} \dot{Z} + \left(\sigma - \dfrac{1}{\rho^2}\right) b_i \\ \ddot{\theta} = \dfrac{c_i}{\rho} - 2\dfrac{\dot{\rho}}{\rho}\dot{\theta} \\ \ddot{Z} = \left(b_i - \sigma\dot{\rho}^2 - \sigma\rho^2\dot{\theta}^2 - \sigma\dot{Z} \odot_{k-2} \dot{Z}\right) Z. \end{cases} \tag{2.24}$$

For (2.24) to make sense, we need that

$$b_1 = \cdots = b_n \quad \text{and} \quad c_1 = \cdots = c_n \tag{2.25}$$

which shows the necessity of (2.25).

Furthermore, that (2.24) has a global solution holds by the same argument as the argument used in the proof of Criterion 1 in [4] to prove global existence of a solution of (15) and (17). By the uniqueness of solutions to ordinary differential equations given suitable initial conditions, the solution to (2.24) must be a rotopulsating orbit, as every step from (2.14) and (2.15) to (2.24) is invertible.

Thus (2.25) is both necessary and sufficient. Finally, as by Lemma 2.1 $\rho^2\dot{\theta} = \rho^2(0)\dot{\theta}(0)$, we have that $\frac{d}{dt}(\rho^2\dot{\theta}) = 0$, which means that the left hand side of (2.23) equals zero, which means that $c_i = 0$. This completes the proof. $\square$

3. Proof of Theorem 1.1

Criterion 1 in [4] tells us that for $n \geq 3$ bodies of masses $m_1, m_2, \ldots, m_n > 0$ moving on the surface $\mathbf{M}^2_\kappa$, necessary and sufficient conditions for a rotopulsating orbit as described by (2.3) to be a solution of Eq. (2.2) are given by the equations

$$\delta_1 = \delta_2 = \cdots = \delta_n \quad \text{and} \quad \gamma_1 = \gamma_2 = \cdots = \gamma_n,$$

where

$$\delta_i = \sum_{j=1, j\neq i}^{n} m_j \mu_{ji}, \qquad \gamma_i = \sum_{j=1, j\neq i}^{n} m_j \nu_{ji}, \quad i = \overline{1, n},$$

$$\mu_{ji} = \frac{1}{c_{ji}^{\frac{1}{2}}(2 - c_{ji}\kappa r^2)^{\frac{3}{2}}}, \qquad \nu_{ji} = \frac{s_{ji}}{c_{ji}^{\frac{3}{2}}(2 - c_{ji}\kappa r^2)^{\frac{3}{2}}}$$

$$s_{ji} = \sin(\alpha_j - \alpha_i), \qquad c_{ji} = 1 - \cos(\alpha_j - \alpha_i), \quad i, j = \overline{1, n},\ i \neq j$$

where $\alpha_i = \beta_i$, $r = \rho$, $\delta_i := b_i$ and $\gamma_i := c_i$. Thus the conditions of Criterion 1 are exactly the conditions of Criterion 1 in [4] with the added bonus that $\gamma_i = 0$. The proof of Theorem 1.1 in [22] is therefore a proof for Theorem 1.1 as well.

Remark 3.1. It should be noted that in [4,22] rotopulsators as described by (2.3) are called 'polygonal homographic orbits'. However, as in [7] it was argued that the term 'homographic' should be replaced by 'rotopulsating', as we did so here.

4. Proof of Theorem 1.2

Let again $r := \rho$, $\alpha_i := \beta_i$, $\delta_i := b_i$ and $\gamma_i := c_i$ in Criterion 1. Then, as before in the proof of Theorem 1.1, the conditions of Criterion 1 become exactly the conditions of Criterion 1 in [4] with the added bonus that $\gamma_i = 0$. In [4], Diacu proved that

Theorem 1. *Consider the curved n-body problem, $n \geq 3$, given by (2.2). If n bodies of equal masses, $m := m_1 = m_2 = \cdots = m_n$, lie initially at the vertices of a regular n-gon parallel with the (x, y)-plane, then there is a class of initial velocities for which the corresponding solutions are rotopulsators. These orbits also satisfy the equalities $\gamma_1 = \gamma_2 = \cdots = \gamma_n = 0$.*

and

Theorem 2. *If the masses $m_1, \ldots, m_n$, $n \geq 3$, form a rotopulsating solution of the curved n-body problem given by Eq. (2.2), such that the polygon is regular, then $m_1 = m_2 = \cdots = m_n$.*

The proofs in [4] of Theorems 1 and 2 use Criterion 1 of [4] alone, thus as Criterion 1 of [4] and Criterion 1 coincide, Theorem 1.2 now follows directly from Theorems 1 and 2 in [4].

Remark 4.1. It should be noted that the expressions 'rotopulsators' and 'rotopulsating solution' as described by (2.3) are not used in Theorems 1 and 2 in [4]. In [4], the expressions used are 'homographic' and 'polygonal homographic solution' respectively. However, as in [7] it was argued that the term 'homographic' should be replaced by 'rotopulsating', as we did so here.

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