J. Math. Anal. Appl. 407 (2013) 398–412

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Growth and distortion theorems for linearly invariant families on homogeneous unit balls in $\mathbb{C}^n$

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ARTICLE INFO

Article history:
Received 16 February 2013
Available online 23 May 2013
Submitted by A.V. Isaev

This work is dedicated to Professor Ian Graham on the occasion of his 65th birthday.

Keywords:
Affine invariance
Close-to-convex mapping
Convex mapping
JB*-triple
Linear invariance
Pluriharmonic mapping
Two-point distortion

ABSTRACT

Let B be a homogeneous unit ball in $X = \mathbb{C}^n$. In this paper, we obtain growth and distortion theorems for linearly invariant families $\mathcal{F}$ of locally biholomorphic mappings on the unit ball B with finite norm-order $\|\mathrm{ord}\|_{e,1}\mathcal{F}$. We use the Euclidean norm for the target space instead of the norm of X, because we are able to obtain lower bounds in the two-point distortion theorems for linearly invariant families on any homogeneous unit ball in $\mathbb{C}^n$. We also obtain similar results for affine and linearly invariant families (A.L.I.F.s) of pluriharmonic mappings of the unit ball B into $\mathbb{C}^n$. Again, in most of these results, we use the Euclidean norm for the target space, to obtain lower bounds in the two-point distortion theorems for A.L.I.F.s on B. These results are generalizations to homogeneous unit balls of recent results due to Graham, Kohr and Pfaltzgraff, the authors of this paper, and Duren, Hamada and Kohr. In the last section, we consider two-point distortion theorems for L.I.F.s and A.L.I.F.s on the unit polydisc U^n in $\mathbb{C}^n$.

1. Introduction

The notion of a linearly invariant family (L.I.F.) was introduced by Pommerenke [24]. He obtained various properties of linearly invariant families on the unit disc, including growth, distortion and coefficient bounds of L.I.F.s, which are generalizations of related results in the theory of univalent functions. Generalizations of this notion to higher dimensions were obtained by Barnard, FitzGerald and Gong [1], Pfaltzgraff [20], Pfaltzgraff and Suffridge [21–23], Gong (see [10] and the references therein), Godula, Liczberski and Starkov [9], Hamada and Kohr [15,16], and the authors (see [13,14]). Pfaltzgraff and Suffridge [23] proved a number of interesting results concerning the norm-order of L.I.F.s on the Euclidean unit ball in $\mathbb{C}^n$ and connections with univalence (starlikeness, convexity).

There are important differences between the theory of linearly invariant families of locally univalent functions on the unit disc U and that of locally biholomorphic mappings on the unit ball in $\mathbb{C}^n$. Among them, we mention the following:

- In the case of one complex variable, a well known result due to Pommerenke [24] yields that the family $\mathcal{K}$ of normalized convex (univalent) functions on U is a L.I.F. with minimum order 1. In contrast to the one variable case, Pfaltzgraff and Suffridge [22] proved that the family $\mathcal{K}(B^n)$ $(n \geq 2)$ of normalized convex mappings of the Euclidean unit ball B^n is a

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0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.05.040

L.I.F. that does not have the minimum trace-order $(n+1)/2$ for L.I.F.s on B^n, i.e. ord $\mathcal{K}(B^n) > (n+1)/2$, and there exist L.I.F.s $\mathcal{F}$ on B^n of minimum trace-order, which are not contained in $\mathcal{K}(B^n)$ for $n \geq 2$.

- In dimension $n \geq 2$, ord $\mathcal{K}(B^n)$ is still unknown (see [22]). However, in the case of the unit polydisc U^n of $\mathbb{C}^n$, the family $\mathcal{K}(U^n)$ is a L.I.F. of minimum trace-order n (see [22]). Also, there exist L.I.F.s on U^n of trace-order n, which are not subsets of $\mathcal{K}(U^n)$ (see [22]; cf. [16]).
- The Cayley transform does not provide sharp bounds for the growth of the Jacobian determinant of the L.I.F. $\mathcal{K}(B^n)$ for $n \geq 2$ (see [22]). On the other hand, in dimension $n \geq 2$, the sharp lower bound in the following distortion result for the family $\mathcal{K}(B^n)$:

$$\frac{1}{(1+\|z\|_e)^2} \leq \|Df(z)\|_e \leq \frac{1}{(1-\|z\|_e)^2}, \quad z \in B^n,$$

where $\|\cdot\|_e$ denotes the Euclidean norm on $\mathbb{C}^n$, is unknown (see [23]; see also [11], and the references therein).

- In contrast with the case $n = 1$ (ord $S(B^1) = 2$), the L.I.F. $S(B^n)$ (resp. $S(U^n)$) consisting of all normalized biholomorphic mappings on B^n (resp. U^n) has infinite order (both trace-order and norm-order are ∞) for $n \geq 2$ (see [23]). Hence, in higher dimensions it is of interest to study properties of L.I.F.s $\mathcal{F}$ such that $\mathcal{F} \subsetneq S(B^n)$ (resp. $\mathcal{F} \subsetneq S(U^n)$).
- In dimension $n \geq 2$, the exponents in the bounds for the Jacobian determinant for L.I.F.s on the unit balls B^n and U^n are different (see [20,22]). The authors [13] gave a clear explanation for this phenomenon, based on the fact that these exponents depend on the Bergman metric at the origin. We obtained a unified approach to the above results and proved a general distortion result for L.I.F.s of finite trace-order on the unit ball of an n-dimensional JB*-triple.

Recently, Duren, Hamada and Kohr [8] extended the notion of linear invariance on the Euclidean unit ball B^n in $\mathbb{C}^n$ to the case of affine and linearly invariant families of pluriharmonic mappings of B^n into $\mathbb{C}^n$. To this end, they obtained various results concerning two-point distortion theorems for affine and linearly invariant families of harmonic functions on the unit disc U and of pluriharmonic mappings of B^n into $\mathbb{C}^n$. We mention that affine and linearly invariant families of harmonic functions on the unit disc U were introduced by Sheil-Small [25]. Other results about linearly invariant families in $\mathbb{C}^n$ may be found in [10,11] and the references therein. Also, recent results related to two-point distortion results for harmonic mappings of the unit disc and necessary and sufficient conditions for univalence of pluriharmonic mappings of the Euclidean unit ball B^n in $\mathbb{C}^n$ may be found in [4,5].

In this paper, we continue the above work on L.I.F.s and we obtain growth and distortion theorems for linearly invariant families $\mathcal{F}$ of locally biholomorphic mappings on the unit ball B of an n-dimensional JB*-triple X with finite norm-order $\|\mathrm{ord}\|_{e,1}\mathcal{F}$, where

$$\|\mathrm{ord}\|_{e,1}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{\frac{1}{2}\|D^2f(0)(w,\cdot)\|_{X,e}\right\}$$

and

$$\|A\|_{X,e} = \sup\{\|Az\|_e : \|z\|_X = 1\}, \quad A \in L(\mathbb{C}^n).$$

Note that the reason for which we use the Euclidean norm $\|\cdot\|_e$ for the target space instead of the norm on X is that we are able to obtain lower bounds in the two-point distortion theorems for linearly invariant families on any homogeneous unit ball in $\mathbb{C}^n$. Next, we obtain similar results for affine and linearly invariant families (A.L.I.F.s) of pluriharmonic mappings of the unit ball B into $\mathbb{C}^n$. Various particular cases are also obtained. Again, in most of these results, we use the Euclidean norm for the target space, to obtain lower bounds in the two-point distortion theorems for A.L.I.F.s on B. We remark that in the case of the upper bounds in the two-point distortion theorems, we may also obtain similar results to those in Theorems 5.1 and 5.4, by replacing the Euclidean norm $\|\cdot\|_e$ by the norm $\|\cdot\|_X$ on X. In the last section of this paper, we obtain two-point distortion theorems for L.I.F.s and A.L.I.F.s on the unit polydisc U^n in $\mathbb{C}^n$.

The main results in this paper are generalizations to homogeneous unit balls of recent results obtained in [8,12,13].

2. Preliminaries

Let X, Y be complex Banach spaces. We denote by $L(X, Y)$ the space of continuous linear operators from X into Y with the standard operator norm. Let I be the identity operator in $L(X)$, where $L(X) = L(X, X)$.

The set of holomorphic mappings from a domain $\Omega \subset X$ into Y is denoted by $\mathcal{H}(\Omega, Y)$. The set $\mathcal{H}(\Omega, X)$ is denoted by $\mathcal{H}(\Omega)$. A mapping $f \in \mathcal{H}(\Omega, Y)$ is said to be biholomorphic if $f(\Omega)$ is a domain, the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. When Ω contains the origin, we say that a mapping $f \in \mathcal{H}(\Omega)$ is normalized if $f(0) = 0$ and $Df(0) = I$.

The family of normalized biholomorphic mappings in $\mathcal{H}(\Omega)$ will be denoted by $S(\Omega)$. In the case of one complex variable, $S(U)$ is the usual family S of normalized univalent functions on the unit disc U. Let $\mathcal{L}S(\Omega)$ be the family of normalized locally biholomorphic mappings of Ω into X. Also, let $K(B)$ be the subfamily of $S(B)$ consisting of convex mappings.

For each $z \in X \setminus \{0\}$, let

$$T(z) = \{l_z \in L(X, \mathbb{C}) : \; l_z(z) = \|z\|_X, \; \|l_z\|_X = 1\},$$

where $\|\cdot\|_X$ is the norm on X. This set is nonempty by the Hahn–Banach theorem.

Let $\mathrm{Aut}(\Omega)$ denote the set of biholomorphic automorphisms of Ω. A domain Ω is called homogeneous if for any $x, y \in \Omega$, there exists some mapping $f \in \mathrm{Aut}(\Omega)$ such that $f(x) = y$.

Definition 2.1. A complex Banach space X is called a JB*-*triple* if there exists a triple product $\{\cdot,\cdot,\cdot\} : X^3 \to X$ which is conjugate linear in the middle variable, but linear and symmetric in the other variables, and satisfies

(i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$;
(ii) the map $a\Box a : x \in X \mapsto \{a, a, x\} \in X$ is Hermitian with nonnegative spectrum;
(iii) $\|\{a, a, a\}\|_X = \|a\|_X^3$;

for $a, b, x, y, z \in X$.

Remark 2.2. Every bounded symmetric domain in a complex Banach space is homogeneous. Conversely, the open unit ball B of a Banach space admits a symmetry $s(z) = -z$ at 0 and if B is homogeneous, then B is a symmetric domain. Banach spaces with a homogeneous open unit ball are precisely the JB*-triples (see [18]). We refer to [3,26,27] for relevant details of JB*-triples and references.

For every $a \in X$, let $Q_a : X \to X$ be the conjugate linear operator defined by $Q_a(z) = \{a, z, a\}$. This operator is called the quadratic representation and it satisfies the fundamental formula

$$Q_{Q_a(b)} = Q_a Q_b Q_a$$

for all $a, b \in X$. For every $z, w \in X$, the Bergman operator $B(z, w) \in L(X, X)$ is defined by

$$B(z, w) = I - 2z\Box w + Q_z Q_w,$$

where $z\Box w(x) = \{z, w, x\}$. Let B be the unit ball of a JB*-triple X. Then, for each $a \in B$, the Möbius transformation g_a defined by

$$g_a(z) = a + B(a, a)^{1/2}(I + z\Box a)^{-1}z, \tag{2.1}$$

is a biholomorphic mapping of B onto itself with $g_a(0) = a$, $g_a(-a) = 0$, $g_{-a} = g_a^{-1}$ and $Dg_a(0) = B(a, a)^{1/2}$.

Let $\|\cdot\|_X$ be a norm on X and $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{C}^n$. For $A \in L(X, \mathbb{C}^n)$, let

$$\|A\|_{X,e} = \sup\{\|Az\|_e : \|z\|_X = 1\}$$

and if $X = \mathbb{C}^n$, let

$$\|A\|_X = \sup\{\|Az\|_X : \|z\|_X = 1\}$$

and

$$\|A\|_e = \sup\{\|Az\|_e : \|z\|_e = 1\}.$$

3. Linearly invariant families of holomorphic mappings

We begin this section with the notion of linearly invariant families on the unit ball B of a complex Banach space X. Then we give the notion of norm-order and obtain distortion and growth results for L.I.F.'s on the unit ball of finite dimensional JB*-triples (cf. [13,23]).

Definition 3.1. Let B be the unit ball of a complex Banach space X. Then a family $\mathcal{F}$ is called a linearly invariant family (L.I.F.) if the following conditions hold:

(i) $\mathcal{F} \subset \mathcal{LS}(B)$;
and
(ii) $\Lambda_\phi(f) \in \mathcal{F}$, for all $f \in \mathcal{F}$ and $\phi \in \mathrm{Aut}(B)$.

Here $\Lambda_\phi(f)$ is the Koebe transform of f given by

$$\Lambda_\phi(f)(z) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}(f(\phi(z)) - f(\phi(0))), \quad z \in B.$$

Note that the Koebe transform has the group property $\Lambda_\psi \circ \Lambda_\phi = \Lambda_{\phi\circ\psi}$.

If $X = \mathbb{C}^n$ and $\mathcal{F}$ is a linearly invariant family, we define two types of *norm-order* of $\mathcal{F}$ (cf. [13,23]), given by

$$\|\mathrm{ord}\|_{e,1}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{\frac{1}{2}\|D^2 f(0)(w,\cdot)\|_{X,e}\right\}$$

and

$$\|\mathrm{ord}\|_{e,2}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{\frac{1}{2}\|D^2 f(0)(w,w)\|_{e}\right\}.$$

It is clear that $\|\mathrm{ord}\|_{e,1}\mathcal{F} \geq \|\mathrm{ord}\|_{e,2}\mathcal{F}$. On the other hand, since

$$D^2f(0)(z,w) = \frac{1}{2}\Big\{D^2f(0)(z+w,z+w) - D^2f(0)(z,z) - D^2f(0)(w,w)\Big\},$$

we obtain $\|\mathrm{ord}\|_{e,1}\mathcal{F} \leq 3\|\mathrm{ord}\|_{e,2}\mathcal{F}$. Moreover, if X is a finite dimensional complex Hilbert space, then $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \|\mathrm{ord}\|_{e,2}\mathcal{F}$ by [17, Theorem 4].

We also define the trace-order of $\mathcal{F}$ (cf. [14,20]) given by

$$\mathrm{ord}\,\mathcal{F} = \sup_{f\in\mathcal{F}}\sup_{\|w\|_X=1}\left\{\frac{1}{2}\left|\mathrm{tr}\left[D^2f(0)(w,\cdot)\right]\right|\right\}.$$

We now give some examples of linearly invariant families on the unit ball B of a complex Banach space X (cf. [13,14,22]).

Example 3.2. (i) $K(B)$, the set of convex mappings in $\mathcal{LS}(B)$. If X is a finite dimensional complex Hilbert space, then $\|\mathrm{ord}\|_{e,1}K(B) = 1$ (see [23,15]). On the other hand, it is known that in the case of an n-dimensional complex Hilbert space with $n \geq 2$, $\mathrm{ord}\,K(B) > (n+1)/2$ and $\mathrm{ord}\,K(B)$ is unknown (see [22]).

(ii) $S(B)$, the set of all biholomorphic mappings in $\mathcal{LS}(B)$. If X is a complex Hilbert space of dimension n, where $n > 1$, the linearly invariant family $S(B)$ does not have finite trace-order (see [1]; cf. [20]).

(iii) $\mathcal{U}_\alpha(B)$, the union of all linearly invariant families contained in $\mathcal{LS}(B)$ with trace-order not greater than α. This is a generalization of the universal linearly invariant families $\mathcal{U}_\alpha = \mathcal{U}_\alpha(\Delta)$ considered in [24].

(iv) If $\mathcal{G}$ is a nonempty subset of $\mathcal{LS}(B)$, then the linearly invariant family generated by $\mathcal{G}$ is the family

$$\Lambda[\mathcal{G}] = \{\Lambda_\phi(g) : g \in \mathcal{G},\ \phi \in \mathrm{Aut}(B)\}.$$

The linear invariance is a consequence of the group property of the Koebe transform. Obviously, $\Lambda[\mathcal{G}] = \mathcal{G}$ if and only if $\mathcal{G}$ is a linearly invariant family. In the cases of the unit Euclidean ball and the unit polydisc of $\mathbb{C}^n$, this example provided a useful technique for generating many interesting mappings (see [20–22]). For example, we can use a single mapping f from $\mathcal{LS}(B)$ to generate the linearly invariant family $\Lambda[\{f\}]$. The family $\Lambda[\{i\}]$, generated by the identity mapping $i(z) = z$, consists of all the Koebe transforms of $i(z)$.

In the rest of this paper, unless otherwise stated, let B be the homogeneous unit ball of $X = \mathbb{C}^n$, that is B is the unit ball of a finite dimensional JB*-triple X. We also assume that

$$\inf\{\|z\|_e : z \in \partial B\} = 1. \tag{3.1}$$

This assumption is not so strong, because for any unit ball B of a finite dimensional JB*-triple X, there exists a constant $c > 0$ such that cB satisfies the equality (3.1). Also, let

$$C_1 = \sup\{\|z\|_e : z \in \partial B\}. \tag{3.2}$$

Taking into account the relations (3.1) and (3.2), we deduce that

$$\|z\|_X \leq \|z\|_e \leq C_1\|z\|_X, \quad z \in X.$$

Also, since $|\mathrm{tr}(A)| \leq n\|A\|_e$ for all $A \in L(X,\mathbb{C}^n)$ by (3.1), we have

$$\mathrm{ord}\,\mathcal{F} \leq n\|\mathrm{ord}\|_{e,1}\mathcal{F}.$$

Theorem 3.3. *Let B be the unit ball of a finite dimensional JB*-triple X which satisfies the condition (3.1). Let $\mathcal{F}$ be a linearly invariant family on B. Then $\|\mathrm{ord}\|_{e,1}\mathcal{F} \geq 1$ holds.*

Proof. Let

$$\|\mathrm{ord}\|_{X,2}\mathcal{F} = \sup_{f\in\mathcal{F}}\sup_{\|z\|_X=1}\left\{\frac{1}{2}\|D^2f(0)(z,z)\|_X\right\}.$$

Then $\|\mathrm{ord}\|_{X,2}\mathcal{F} \geq 1$ by [13, Theorem 3.9]. Since $\|\mathrm{ord}\|_{e,1}\mathcal{F} \geq \|\mathrm{ord}\|_{X,2}\mathcal{F}$ by (3.1), we obtain the theorem. □

Let h_0 be the Bergman metric on B at 0 and let

$$c(B) = \frac{1}{2}\sup_{z,w\in B}|h_0(z,w)|.$$

By [19, Theorem 6.5] (see also [14, Proposition 2.3]), we deduce that $c(B) = h_0(e,e)/2$, where e is an arbitrary maximal tripotent in X.

The following result was obtained in [14, Theorem 4.1] (compare [20]).

Theorem 3.4. *Let $\mathcal{F}$ be a linearly invariant family on the unit ball B of a finite dimensional* JB**-triple X. If* $\operatorname{ord}\mathcal{F} = \alpha_t < \infty$, *then*

$$\frac{(1-\|z\|_X)^{\alpha_t - c(B)}}{(1+\|z\|_X)^{\alpha_t + c(B)}} \le |\det Df(z)| \le \frac{(1+\|z\|_X)^{\alpha_t - c(B)}}{(1-\|z\|_X)^{\alpha_t + c(B)}}, \quad z \in B \tag{3.3}$$

for all $f \in \mathcal{F}$. If B is the Euclidean unit ball or the unit polydisc of $\mathbb{C}^n$, then the above estimates are sharp.

In view of Theorem 3.4, we may prove the lower bound for $\|Df(z)\|_{X,e}$, when f belongs to a L.I.F. on the unit ball of a finite dimensional JB*-triple X.

Theorem 3.5. *Let B be the unit ball of an n-dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F}$ be a linearly invariant family on B. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_X)^{\alpha - c(B)/n}}{(1+\|z\|_X)^{\alpha + c(B)/n}} \le \|Df(z)\|_{X,e} \le C_1 \frac{(1+\|z\|_X)^{\alpha - 1}}{(1-\|z\|_X)^{\alpha + 1}}, \quad z \in B, \tag{3.4}$$

for all $f \in \mathcal{F}$, where C_1 is a constant defined by (3.2).

Proof. Let $\operatorname{ord}\mathcal{F} = \alpha_t$. Since $|\det Df(z)| \le \|Df(z)\|_{X,e}^n$ and $\alpha_t \le n\alpha$ by the condition (3.1), the lower bound in (3.4) follows from the relation (3.3).

Next, let $\alpha_X = \|\operatorname{ord}\|_{X,1}\mathcal{F}$, where

$$\|\operatorname{ord}\|_{X,1}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{ \frac{1}{2} \left\| D^2 f(0)(w,\cdot) \right\|_X \right\}.$$

Then $\alpha_X \le \alpha$ in view of the relation (3.1), and

$$\|Df(z)\|_{X,e} \le C_1 \|Df(z)\|_X \le C_1 \frac{(1+\|z\|_X)^{\alpha_X - 1}}{(1-\|z\|_X)^{\alpha_X + 1}} \le C_1 \frac{(1+\|z\|_X)^{\alpha - 1}}{(1-\|z\|_X)^{\alpha + 1}},$$

by [13, Theorem 4.2]. □

Let B^n be the Euclidean unit ball in $\mathbb{C}^n$. Then Theorem 3.5 yields the following particular case (compare [13,23]). In view of [23, Theorem 4.1], the upper estimate in (3.5) is sharp and the lower estimate in (3.5) is not sharp.

Corollary 3.6. *Let $\mathcal{F}$ be a linearly invariant family on B^n. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_e)^{\alpha - \frac{n+1}{2n}}}{(1+\|z\|_e)^{\alpha + \frac{n+1}{2n}}} \le \|Df(z)\|_e \le \frac{(1+\|z\|_e)^{\alpha - 1}}{(1-\|z\|_e)^{\alpha + 1}}, \quad z \in B^n, \tag{3.5}$$

for all $f \in \mathcal{F}$.

If U^n is the unit polydisc in $\mathbb{C}^n$, then we obtain the following corollary, in view of Theorem 3.5 (compare [16] and [22]).

Corollary 3.7. *Let $\mathcal{F}$ be a linearly invariant family on U^n. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_\infty)^{\alpha - 1}}{(1+\|z\|_\infty)^{\alpha + 1}} \le \|Df(z)\|_{X,e} \le \sqrt{n} \frac{(1+\|z\|_\infty)^{\alpha - 1}}{(1-\|z\|_\infty)^{\alpha + 1}}, \quad z \in U^n, \tag{3.6}$$

for all $f \in \mathcal{F}$, where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{C}^n$.

Question 3.8. *Are the estimates in the inequalities (3.6) sharp?*

As in the proof of [23, Theorem 4.2], we may use Theorem 3.5 to deduce the following growth result for L.I.F.'s on the unit ball of finite dimensional JB*-triples.

Theorem 3.9. *Let B be the unit ball of a finite dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{LS}(B)$ be a linearly invariant family of norm order $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and let $f \in \mathcal{F}$. Then*

$$\|f(z)\|_e \le \frac{C_1}{2\alpha} \left\{ \left(\frac{1+\|z\|_X}{1-\|z\|_X} \right)^{\alpha} - 1 \right\}, \quad z \in B, \tag{3.7}$$

where C_1 is a constant defined by (3.2).

Theorem 3.10. *Let B be the unit ball of an n-dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F}$ be a linearly invariant family on B. If $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_X)^{(2n-1)\alpha+n-1-c(B)}}{(1+\|z\|_X)^{(2n-1)\alpha-n+1+c(B)}}\|w\|_X \le C_1^{n-1}\,\|Df(z)w\|_e\,,\quad z\in B,\ w\in X,$$

for all $f \in \mathcal{F}$, where C_1 is a constant defined by (3.2).

Proof. We use an argument similar to that in [23, Theorem 4.1]. If $A \in L(\mathbb{C}^n)$, then $\|A\|_{X,e} \ge \sqrt{\lambda_n}$, where $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the eigenvalues of A^*A and

$$\sqrt{\lambda_1} \le \inf\{\|Aw\|_e : \|w\|_X = 1\}.$$

Also, $|\det A| = \sqrt{\lambda_1\cdots\lambda_n} \le \sqrt{\lambda_1}\lambda_n^{(n-1)/2}$. Since $\alpha_t \le n\alpha$, we obtain from Theorems 3.4 and 3.5 that

$$\begin{aligned}\frac{(1-\|z\|_X)^{n\alpha-c(B)}}{(1+\|z\|_X)^{n\alpha+c(B)}} &\le \frac{(1-\|z\|_X)^{\alpha_t-c(B)}}{(1+\|z\|_X)^{\alpha_t+c(B)}}\\ &\le |\det Df(z)| = \sqrt{\lambda_1\cdots\lambda_n}\\ &\le \sqrt{\lambda_1}\lambda_n^{(n-1)/2}\\ &\le \sqrt{\lambda_1}\left(C_1\frac{(1+\|z\|_X)^{\alpha-1}}{(1-\|z\|_X)^{\alpha+1}}\right)^{n-1}\end{aligned}$$

for all $z \in B$. Therefore, we have

$$\frac{(1-\|z\|_X)^{(2n-1)\alpha+n-1-c(B)}}{(1+\|z\|_X)^{(2n-1)\alpha-n+1+c(B)}} \le C_1^{n-1}\sqrt{\lambda_1} \le C_1^{n-1}\|Df(z)w\|_e$$

for all $z \in B$ and $w \in X$ with $\|w\|_X = 1$. This completes the proof. □

In view of Theorem 3.10, we may obtain the following result, which is a generalization of [12, Theorem 5] to the case of finite dimensional JB*-triples.

Theorem 3.11. *Let B be the unit ball of an n-dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F}$ be a linearly invariant family on B. If $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and $f \in \mathcal{F}$ is biholomorphic on B, then*

$$\|f(z)\|_e \ge \Psi_{n,\alpha}(\operatorname{artanh}\|z\|_X),\quad z\in B, \tag{3.8}$$

where

$$\Psi_{n,\alpha}(v) = C_1^{1-n}\int_0^v \frac{e^{-2(2n-1)\alpha u}}{(\cosh u)^{2n-2c(B)}}\,du,\quad 0\le v<\infty. \tag{3.9}$$

Proof. Let $\delta = \|\mathrm{ord}\|_{e,1}\Lambda[\{f\}]$. Since $\Lambda[\{f\}] \subseteq \mathcal{F}$ by the fact that $\mathcal{F}$ is a L.I.F., it is clear that $\delta \le \alpha$. Now, fix $r \in (0, 1)$ and let $\rho(r) = \min\{\|f(z)\|_e : \|z\|_X = r\}$. Then, there exists a $z_0 \in \partial B_r$ such that $\|f(z_0)\|_e = \rho(r)$. Let $\Gamma = \{tf(z_0) : 0 \le t \le 1\}$. Then, as in the proof of [12, Theorem 5], we deduce that

$$\rho(r) \ge \int_\gamma \left\|Df(\zeta)\frac{d\zeta}{\|d\zeta\|_X}\right\|_e d\|\zeta\|_X,$$

where $\gamma = f^{-1}(\Gamma)$. In view of Theorem 3.10, we obtain that

$$\|f(z)\|_e \ge \rho(r) \ge C_1^{1-n}\int_0^r \frac{(1-t)^{(2n-1)\delta+n-1-c(B)}}{(1+t)^{(2n-1)\delta-n+1+c(B)}}dt = \Psi_{n,\delta}(\operatorname{artanh} r),$$

for $\|z\|_X = r$. Since $\Psi_{n,\delta}(v) \ge \Psi_{n,\alpha}(v)$ for $v \in [0, \infty)$, the result follows, as desired. □

In view of Theorem 3.11, we obtain the following two-point distortion result, which is a generalization of [12, Theorem 7] to the case of finite dimensional JB*-triples.

Theorem 3.12. *Let B be the unit ball of an n-dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{L}S(B)$ be a linearly invariant family of norm-order $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and let $f \in \mathcal{F}$ be biholomorphic. Then*

$$\|f(a)-f(b)\|_e \ge \Psi_{n,\alpha}(C_B(a,b))\max\{T_f(a), T_f(b)\},\quad a,b\in B, \tag{3.10}$$

where $\Psi_{n,\alpha}$ is defined by (3.9), $C_B(a,b)$ denotes the Carathéodory metric in B, and

$$T_f(z) = \|B(z,z)^{-1/2}[Df(z)]^{-1}\|_e^{-1}, \quad z \in B. \tag{3.11}$$

Conversely, if a locally biholomorphic mapping f on B satisfies the inequality (3.10), for all $a, b \in B$ and for some $\alpha > 0$, then f is biholomorphic on B.

Proof. We use an argument similar to that in the proof of [12, Theorem 7]. Fix $a, b \in B$ and let $g_b \in \mathrm{Aut}(B)$ be given by (2.1). Also, let

$$F(z) = [Dg_b(0)]^{-1}[Df(g_b(0))]^{-1}(f(g_b(z)) - f(b)), \quad z \in B. \tag{3.12}$$

Since $\mathcal{F}$ is a L.I.F., it follows that $F \in \mathcal{F}$. In view of (3.8), we deduce that

$$\|F(z)\|_e \geq \Psi_{n,\alpha}(\operatorname{artanh}\|z\|_X), \quad z \in B.$$

Letting $z = g_b^{-1}(a)$ in the above, we deduce that

$$\|f(a) - f(b)\|_e \cdot \|[Dg_b(0)]^{-1}[Df(b)]^{-1}\|_e \geq \|F(z)\|_e \geq \Psi_{n,\alpha}(C_B(z,0)).$$

Finally, since $Dg_b(0) = B(b,b)^{1/2}$ and $C_B(z,0) = C_B(a,b)$, by the fact that the Carathéodory metric is invariant under the biholomorphic automorphisms of B, we obtain the relation (3.10) by interchanging the roles of a and b, as desired.

For the converse part, it suffices to see that if f is a locally biholomorphic mapping on B, which satisfies (3.10), and if $f(a) = f(b)$ for $a, b \in B$, then we must have $a = b$, in view of the fact that the C_B is a metric. □

We close this section with the following upper bound for the distortion $\|f(a) - f(b)\|_e$, when f belongs to a L.I.F. on the unit ball of a finite dimensional JB*-triple X (cf. [13, Theorem 4.7] and [2, Lemma 2.7]).

Theorem 3.13. *Let B be the unit ball of a finite dimensional JB*-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{LS}(B)$ be a linearly invariant family of norm-order $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and let $f \in \mathcal{F}$. Then*

$$\|f(a) - f(b)\|_e \leq \frac{C_1}{2\alpha}[\exp(2\alpha C_B(a,b)) - 1]\min\{\tilde{T}_f(a), \tilde{T}_f(b)\}, \quad a, b \in B,$$

where

$$\tilde{T}_f(z) = \|Df(z)B(z,z)^{1/2}\|_e, \quad z \in B, \tag{3.13}$$

and C_1 is a constant given by (3.2).

Proof. We use an argument similar to that in the proof of [13, Theorem 4.7]. Fix $a, b \in B$ and let $g_b \in \mathrm{Aut}(B)$ be given by (2.1). In view of the linear invariance of the family $\mathcal{F}$, we deduce that $F \in \mathcal{F}$, where F is given by (3.12). Taking into account the relation (3.7), we deduce that

$$\|F(z)\|_e \leq \frac{C_1}{2\alpha}\left\{\left(\frac{1+\|z\|_X}{1-\|z\|_X}\right)^\alpha - 1\right\}, \quad z \in B.$$

Now, if $z = g_b^{-1}(a)$ in the above, we deduce that

$$\begin{aligned}\|f(a) - f(b)\|_e &= \|Df(b)Dg_b(0)DF(z)\|_e \\ &\leq \|Df(b)Dg_b(0)\|_e \frac{C_1}{2\alpha}[\exp(2\alpha C_B(z,0)) - 1].\end{aligned}$$

Finally, since $Dg_b(0) = B(b,b)^{1/2}$, $C_B(z,0) = C_B(a,b)$ and interchanging the roles of a and b in the above relation, the result follows, as desired. □

4. Families of pluriharmonic mappings

In this section, we define the notions of affine and linear invariance for families of locally univalent pluriharmonic mappings on the unit ball of a finite dimensional complex Banach space, and we present some basic results related to these notions. The results will be applied in Section 5 to prove two-point distortion theorems for pluriharmonic mappings.

Let B be the unit ball of a finite dimensional complex Banach space $X = \mathbb{C}^n$. A complex-valued function f of class C^2 on B is said to be pluriharmonic if its restriction to every complex line is harmonic. This happens if and only if $\frac{\partial^2}{\partial z_j \partial \overline{z}_k} f(z) \equiv 0$ in B for all $j, k = 1, 2 \dots, n$. A real-valued function f of class C^2 on B is pluriharmonic if and only if it is the real part of some holomorphic function on B. Every real-valued harmonic function on the unit disc U is the real part of a holomorphic function on U, but this is no longer true for harmonic functions on B. Thus, in dimension $n \geq 2$, the family of all pluriharmonic functions is a proper subclass of the family of all harmonic functions on B. Every pluriharmonic mapping $f : B \to X$ can be written as $f = h + \overline{g}$, where g and h are holomorphic mappings of B into X, $\overline{g}$ is the usual complex conjugate of g in $X = \mathbb{C}^n$, and this representation is unique if $g(0) = 0$.

Let $S_H(B)$ be the family of all univalent pluriharmonic mappings $f = h + \overline{g}$ on B, where $h, g \in \mathcal{H}(B)$, $g(0) = 0$, and $h \in \mathcal{L}S(B)$. We remark that a mapping $f \in S_H(B)$ is not necessarily sense-preserving; that is, its real Jacobian (when f is regarded as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$) need not be positive. Let $S_H^0(B)$ be the subfamily of mappings $f = h+\overline{g} \in S_H(B)$ such that $Dg(0) = 0$. Also, let $\mathcal{L}S_H(B)$ be the family of all pluriharmonic mappings $f = h + \overline{g}$ on B with $h \in \mathcal{L}S(B)$ and $g(0) = 0$. In the case of one complex variable, the family $S_H(U)$ is a normal family, while $S_H^0(U)$ is compact (see [6,7]). However, if B is the n-dimensional Euclidean unit ball with $n \geq 2$, the family $S_H(B)$ is not normal (see [8]).

We recall that in the case $n = 1$, a harmonic mapping $f = h + \overline{g}$ is sense-preserving on the unit disc U if and only if $|g'(z)| < |h'(z)|, z \in U$. This condition is equivalent to the statement that h is locally univalent on U and $|\omega_f(z)| < 1$ for $z \in U$, where $\omega_f = g'/h'$. Hence, the analytic function $h + ag$ is locally univalent on U for $|a| \leq 1$ (see e.g. [7]).

The following theorems are generalizations of [8, Theorems 5 and 6] to the unit ball of a finite dimensional complex Banach space. Since the proofs are similar to those in [8, Theorems 5 and 6], we omit them.

Theorem 4.1. *Let B be the unit ball of an n-dimensional complex Banach space $X = \mathbb{C}^n$ and let $f = h + \overline{g} : B \to \mathbb{C}^n$ be a pluriharmonic mapping such that h is locally biholomorphic on B. If*

$$\|Dg(z)[Dh(z)]^{-1}\|_e < 1, \quad z \in B, \tag{4.1}$$

then $h + Ag$ is locally biholomorphic in B for each $A \in L(\mathbb{C}^n)$ with $\|A\|_e \leq 1$. Furthermore, f is a sense-preserving mapping on B.

The following result provides concrete examples of univalent pluriharmonic mappings on B (see [8, Theorem 6] and [5]).

Theorem 4.2. *Let $h : B \to \mathbb{C}^n$ be a convex (biholomorphic) mapping and let $g \in \mathcal{H}(B)$ be such that the condition (4.1) holds. Then $f = h + \overline{g}$ is a sense-preserving univalent mapping on B.*

Remark 4.3. Let $h : B \to \mathbb{C}^n$ be a locally biholomorphic (resp. biholomorphic) mapping and let $A \in L(\mathbb{C}^n)$ be such that $\|A\|_e < 1$. Then it is not difficult to deduce that $f = h + A\overline{h}$ is a pluriharmonic sense-preserving locally univalent (resp. univalent) mapping on B, in view of Theorem 4.1. Moreover, if h is also convex (biholomorphic) on B, then f is a convex pluriharmonic mapping on B.

For families $\mathcal{F} \subset \mathcal{L}S_H(B)$ of pluriharmonic mappings, we introduce the following notions of linear and affine invariance (cf. [8]; see [25] for $n = 1$).

Linear invariance. If $f = h + \overline{g} \in \mathcal{F}$ then for each $\varphi \in \mathrm{Aut}(B)$ the mapping

$$F(z) = [D\varphi(0)]^{-1}[Dh(\varphi(0))]^{-1}\{f(\varphi(z)) - f(\varphi(0))\}, \quad z \in B,$$

also belongs to $\mathcal{F}$.

Affine invariance. If $f = h + \overline{g} \in \mathcal{F}$ and $A \in L(\mathbb{C}^n)$ has norm $\|A\|_e < 1$, and if $h + Ag$ is locally biholomorphic on B, then the mapping

$$\widetilde{F}(z) = [I + ADg(0)]^{-1}[f(z) + A\overline{f(z)}], \quad z \in B,$$

also belongs to $\mathcal{F}$.

Remark 4.4. The family $\mathcal{F}$ of sense-preserving mappings in $\mathcal{L}S_H(B)$ has the "affine invariance" property. Indeed, if $\|A\|_e < 1$, then the mapping $w \mapsto w + A\overline{w}$ is a sense-preserving mapping. Therefore, if $f = h + \overline{g}$ is sense-preserving and $A \in L(\mathbb{C}^n)$ has norm $\|A\|_e < 1$, and if $h + Ag$ is locally biholomorphic on B, then the mapping

$$\widetilde{F}(z) = [I + ADg(0)]^{-1}[f(z) + A\overline{f(z)}], \quad z \in B,$$

is sense-preserving, because it is a composition of sense-preserving mappings.

We now define the order of a linearly invariant family $\mathcal{F} \subset \mathcal{L}S_H(B)$ of pluriharmonic mappings by (cf. [8])

$$\alpha = \alpha(\mathcal{F}) = \sup\left\{\frac{1}{2}\|D^2h(0)(w, \cdot)\|_{X,e} : f = h + \overline{g} \in \mathcal{F},\ \|w\|_X = 1\right\}.$$

Note that if $\mathcal{F}$ is a L.I.F. of locally biholomorphic mappings, then $\alpha(\mathcal{F})$ is the norm-order as introduced in Section 3 (cf. [23]).

Remark 4.5. (i) It is not difficult to deduce that if $\mathcal{F} \subset \mathcal{L}S_H(B)$ is a linearly invariant family of order $\alpha(\mathcal{F})$, then the family $\mathcal{F}^\star = \{h : h + \overline{g} \in \mathcal{F}\} \subset \mathcal{L}S(B)$ is also a linearly invariant family with norm-order $\alpha(\mathcal{F}^\star) = \alpha(\mathcal{F})$. Since Theorem 3.3 (cf. [13, Theorem 3.9], [23, Theorem 3.1]) yields that $\alpha(\mathcal{F}^\star) \geq 1$, we deduce that $\alpha(\mathcal{F}) \geq 1$ for every linearly invariant family $\mathcal{F} \subset \mathcal{L}S_H(B)$ of pluriharmonic mappings.

(ii) The family $K_H(B) \subset S_H(B)$ of pluriharmonic mappings with convex image is affine and linearly invariant. It would be interesting to estimate $\alpha(K_H(B))$ in dimension $n > 1$. For $n = 1$, the family K_H of convex mappings in S_H has order 2.

5. Distortion of pluriharmonic mappings

In this section, we generalize the results in Section 3 to the case of pluriharmonic mappings on B. The first result is a generalization of Theorem 3.9 to pluriharmonic mappings (see [8, Theorem 7] in the case of the Euclidean space $X = \mathbb{C}^n$).

Theorem 5.1. *Let B be the unit ball of an n-dimensional* JB*-*triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{L}S_H(B)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$. Let $f = h + \overline{g} \in \mathcal{F}$, and suppose that $h + Ag$ is locally biholomorphic on B for each $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$. Then*

$$\|f(z)\|_e \le \frac{C_1}{2\alpha}(1 + \|Dg(0)\|_e)\left[\left(\frac{1+\|z\|_X}{1-\|z\|_X}\right)^{\alpha} - 1\right], \quad z \in B,$$

where C_1 is a constant given by (3.2).

Proof. We shall use arguments similar to those in the proof of [8, Theorem 7]. In view of the hypothesis that $h+Ag$ is locally biholomorphic on B for $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$, the affine invariance of the family $\mathcal{F}$ shows that

$$[I + ADg(0)]^{-1}(f + A\overline{f}) = [I + ADg(0)]^{-1}[(h + Ag) + (A\overline{h} + \overline{g})]$$

also belongs to the family $\mathcal{F}$. Hence, in view of Theorem 3.5 and Remark 4.5, we obtain that

$$\|[I + ADg(0)]^{-1}[Dh(z) + ADg(z)]\|_{X,e} \le C_1\frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}, \quad r = \|z\|_X < 1,$$

and thus

$$\|Dh(z) + ADg(z)\|_{X,e} \le \mu(g)C_1\frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}},$$

where $\mu(g) = 1 + \|Dg(0)\|_e$. It follows from this that

$$\|Dh(z)(w) + ADg(z)(w)\|_e \le \mu(g)C_1\frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}\|w\|_X,$$

for all $w \in X$ and $A \in L(\mathbb{C}^n)$ with $\|A\|_e \le 1$. For fixed $z \in B$ and $w \in X \setminus \{0\}$, there exists a unitary matrix A such that $ADg(z)(w) = cDh(z)(w)$ for some $c \ge 0$. This implies that

$$\|Dh(z)(w)\|_e + \|Dg(z)(w)\|_e \le \mu(g)C_1\frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}\|w\|_X \tag{5.1}$$

for all $z \in B$ and $w \in X$, where $r = \|z\|_X < 1$.

Now fix an arbitrary point $z \in B$. If $f(z) = 0$, then Theorem 5.1 holds. So we may assume that $f(z) \ne 0$. Let $v : [0, 1] \to \mathbb{R}$ be given by

$$v(t) = \Re\phi(f(tz)), \quad 0 \le t \le 1,$$

for $\phi \in T(f(z))$. Then $v(0) = 0$ and

$$\begin{aligned}\|f(z)\|_e &= v(1) = \int_0^1 v'(t)dt = \int_0^1 \Re\phi\left(Dh(tz)(z) + \overline{Dg(tz)(z)}\right)dt \\ &\le \int_0^1 (\|Dh(tz)(z)\|_e + \|Dg(tz)(z)\|_e)\,dt.\end{aligned}$$

Hence, it follows from (5.1) that

$$\|f(z)\|_e \le \mu(g)C_1\int_0^1 \frac{(1+rt)^{\alpha-1}}{(1-rt)^{\alpha+1}}rdt = \mu(g)\frac{C_1}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right],$$

for $\|z\|_X = r < 1$. This completes the proof. □

The following result is a generalization of Theorem 3.11 to pluriharmonic mappings (see [8, Theorem 8] in the case of the Euclidean space $X = \mathbb{C}^n$).

Theorem 5.2. *Let B be the unit ball of an n-dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset S_H(B)$ be an affine and linearly invariant family of pluriharmonic mappings of order $\alpha = \alpha(\mathcal{F}) < \infty$. Let $f = h + \overline{g} \in \mathcal{F}$ and suppose that $\|Dg(0)\|_e < 1$ and that $h + Ag$ is locally biholomorphic on B for each $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$. Then*

$$\|f(z)\|_e \geq (1 - \|Dg(0)\|_e)\, \Psi_{n,\alpha}(\operatorname{artanh} \|z\|_X), \quad z \in B,$$

where $\Psi_{n,\alpha}$ is defined by (3.9).

Proof. We shall use arguments similar to those in the proof of [8, Theorem 8]. In view of the affine invariance of $\mathcal{F}$, Theorem 3.10 and Remark 4.5, we obtain that

$$\|[I + ADg(0)]^{-1}[Dh(z) + ADg(z)](w)\|_e \geq C_1^{1-n} \frac{(1-r)^{(2n-1)\alpha+n-1-c(B)}}{(1+r)^{(2n-1)\alpha-n+1+c(B)}} \|w\|_X,$$

for all $r = \|z\|_X < 1$, $w \in X$ and $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$. It follows that

$$\|Dh(z)(w) + ADg(z)(w)\|_e \geq \lambda(g) C_1^{1-n} \frac{(1-r)^{(2n-1)\alpha+n-1-c(B)}}{(1+r)^{(2n-1)\alpha-n+1+c(B)}} \|w\|_X,$$

for all $r = \|z\|_X < 1$, $w \in X$ and $A \in L(\mathbb{C}^n)$ with $\|A\|_e \leq 1$, where

$$\lambda(g) = \inf_{\|A\|_e \leq 1} \|[I + ADg(0)]^{-1}\|_e^{-1}.$$

This implies that

$$|\, \|Dh(z)(w)\|_e - \|Dg(z)(w)\|_e| \geq \lambda(g) C_1^{1-n} \frac{(1-r)^{(2n-1)\alpha+n-1-c(B)}}{(1+r)^{(2n-1)\alpha-n+1+c(B)}} \|w\|_X, \tag{5.2}$$

for $r = \|z\|_X < 1$ and $w \in X$.

Now, fix $r \in (0, 1)$ and let $\rho(r) = \min\{\|f(z)\|_e : \|z\|_X = r\}$. Then, there exists a $z_0 \in \partial B_r$ such that $\|f(z_0)\|_e = \rho(r)$. Let $\Gamma = \{tf(z_0) : 0 \leq t \leq 1\}$. Then, we deduce that

$$\rho(r) \geq \int_\gamma \left| \left\| Dh(\zeta) \frac{d\zeta}{\|d\zeta\|_X} \right\|_e - \left\| Dg(\zeta) \frac{d\zeta}{\|d\zeta\|_X} \right\|_e \right| d\|\zeta\|_X,$$

where $\gamma = f^{-1}(\Gamma)$. In view of (5.2), we obtain

$$\begin{aligned} \|f(z)\|_e &\geq \lambda(g) C_1^{1-n} \int_0^r \frac{(1-t)^{(2n-1)\delta+n-1-c(B)}}{(1+t)^{(2n-1)\delta-n+1+c(B)}} dt \\ &= \lambda(g) \Psi_{n,\alpha}(\operatorname{artanh} r), \end{aligned}$$

for $\|z\|_X = r$. Since $\lambda(g) = 1 - \|Dg(0)\|_e$ (see the proof of [8, Theorem 8]), we obtain the theorem. □

We denote by (cf. [8])

$$S(z) = \|B(z, z)^{-1/2}[Dh(z)]^{-1}\|_e^{-1}(1 - \|Dg(z)[Dh(z)]^{-1}\|_e), \quad z \in B,$$

and

$$\nu(g) = \inf_{\|A\|_e \leq 1} \left\{ \frac{1}{\|[I_n + ADg(0)]^{-1}\|_e \cdot \|I_n + ADg(0)\|_e} \right\}, \tag{5.3}$$

where $g \in \mathcal{H}(B)$ with $\|Dg(0)\|_e < 1$ and h is locally biholomorphic on B. Note that $\nu(g) \geq (1 - \|Dg(0)\|_e)/(1 + \|Dg(0)\|_e)$. The following result is a generalization of Theorem 3.12 to pluriharmonic mappings (see [8, Theorem 9] in the case of the Euclidean space $X = \mathbb{C}^n$).

Theorem 5.3. *Let B be the unit ball of an n-dimensional* JB**-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{L}S_H(B)$ be an affine and linearly invariant family of pluriharmonic mappings of order $\alpha = \alpha(\mathcal{F}) < \infty$, and let $f = h + \overline{g} \in \mathcal{F}$. Assume that $h + Ag$ is biholomorphic on B for each $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$, and the condition (4.1) holds. Then*

$$\|f(a) - f(b)\|_e \geq \Phi_{n,\alpha}(a, b), \quad a, b \in B, \tag{5.4}$$

where

$$\Phi_{n,\alpha}(a, b) = \nu(g)\, \Psi_{n,\alpha}(C_B(a, b))\, \max\{S(a), S(b)\}$$

and $\Psi_{n,\alpha}$ is defined by (3.9). Conversely, let $f = h + \overline{g}$ be a pluriharmonic mapping on B such that h is locally biholomorphic on B and $\|Dg(0)\|_e < 1$. If the relation (4.1) holds and f satisfies the relation (5.4) for some $\alpha > 0$ and for all $a, b \in B$, then f is a sense-preserving univalent mapping on B.

Proof. We shall use arguments similar to those in [8, Theorem 9]. Since $f = h + \overline{g} \in \mathcal{F}$ and $h + Ag$ is biholomorphic on B, it follows that $F = [I + ADg(0)]^{-1}(f + A\overline{f}) \in \mathcal{F}$ for $\|A\|_e < 1$ by the affine invariance property for $\mathcal{F}$. Let $F = H + \overline{G}$. Then

$$H = [I + ADg(0)]^{-1}(h + Ag) \in \mathcal{L}S(B)$$

and in view of Theorem 3.12 and Remark 4.5, we obtain that

$$\|H(a) - H(b)\|_e \geq \Psi_{n,\alpha}(C_B(a, b)) \max\{T_H(a), T_H(b)\}, \tag{5.5}$$

for all $a, b \in B$, where $T_f(z)$ is given by (3.11). On the other hand, we see that

$$T_H(z) \geq \frac{S(z)}{\|I + ADg(0)\|_e}, \quad z \in B.$$

Combining (5.5) and the above relation, we obtain that

$$\|h(a) - h(b) + A(g(a) - g(b))\|_e \geq \Phi_{n,\alpha}(a, b), \quad a, b \in B, \tag{5.6}$$

for all $A \in L(\mathbb{C}^n)$ with $\|A\|_e \leq 1$.

Finally, fix $a, b \in B$. Then we may choose a unitary matrix A and some $c \geq 0$ such that $A(g(a) - g(b)) = -c(h(a) - h(b))$. Then, in view of (5.6) and this equality, we obtain that

$$\begin{aligned}\|f(a) - f(b)\|_e &\geq \big|\, \|h(a) - h(b)\|_e - \|g(a) - g(b)\|_e \big| \\ &= \|h(a) - h(b) + A(g(a) - g(b))\|_e \\ &\geq \Phi_{n,\alpha}(a, b).\end{aligned}$$

This completes the proof. □

The next result provides an upper bound for $\|f(a) - f(b)\|_e$, where f belongs to an affine and linearly invariant family of pluriharmonic mappings on B. This result is a generalization of [8, Theorems 1 and 10] to the unit ball of a finite dimensional JB*-triple.

If $g, h \in \mathcal{H}(B)$, let

$$\tilde{S}(a) = \|B(a, a)^{1/2}\|_e(\|Dh(a)\|_e + \|Dg(a)\|_e), \quad a \in B.$$

For every $g \in \mathcal{H}(B)$ with $\|Dg(0)\|_e < 1$, let (cf. [8])

$$\chi(g) = \sup_{\|A\|_e \leq 1} \left\{ \|[I + ADg(0)]^{-1}\|_e \cdot \|I + ADg(0)\|_e \right\}. \tag{5.7}$$

Theorem 5.4. *Let B be the unit ball of an n-dimensional JB*-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{L}S_H(B)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$ and let $f = h + \overline{g} \in \mathcal{F}$ be such that the condition (4.1) holds. Then*

$$\|f(a) - f(b)\|_e \leq \chi(g) \frac{C_1}{2\alpha} [\exp(2\alpha C_B(a, b)) - 1] \min\{\tilde{S}(a), \tilde{S}(b)\}, \quad a, b \in B,$$

where C_1 is a constant given by (3.2).

Proof. We shall use arguments similar to those in [8, Theorem 10]. Since $f = h + \overline{g} \in \mathcal{F}$ and $h + Ag$ is locally biholomorphic on B by Theorem 4.1, it follows that $F = [I + ADg(0)]^{-1}(f + A\overline{f}) \in \mathcal{F}$ for $\|A\|_e < 1$ by the affine invariance property for $\mathcal{F}$. Let $F = H + \overline{G}$. Then

$$H = [I + ADg(0)]^{-1}(h + Ag) \in \mathcal{L}S(B)$$

and in view of Theorem 3.13 and Remark 4.5, we obtain that

$$\|H(a) - H(b)\|_e \leq \frac{C_1}{2\alpha} [\exp(2\alpha C_B(a, b)) - 1] \min\{\tilde{T}_H(a), \tilde{T}_H(b)\}, \tag{5.8}$$

for all $a, b \in B$, where $\tilde{T}_f(z)$ is given by (3.13). On the other hand, it is not difficult to deduce that

$$\tilde{T}_H(z) \leq \tilde{S}(z)\|[I + ADg(0)]^{-1}\|_e, \quad z \in B.$$

Combining (5.8) and the above relation, we obtain that

$$\|h(a) - h(b) + A(g(a) - g(b))\|_e \leq \chi(g) \frac{C_1}{2\alpha} [\exp(2\alpha C_B(a, b)) - 1] \min\{\tilde{S}(a), \tilde{S}(b)\}, \quad a, b \in B, \tag{5.9}$$

for all $A \in L(\mathbb{C}^n)$ with $\|A\|_e \leq 1$.

Finally, fix $a, b \in B$. Then we may choose a unitary matrix A and some $c \geq 0$ such that $A(g(a) - g(b)) = c(h(a) - h(b))$. Then, in view of (5.9) and this equality, we obtain that

$$\begin{aligned}\|f(a) - f(b)\|_e &\leq \|h(a) - h(b)\|_e + \|g(a) - g(b)\|_e \\ &= \|h(a) - h(b) + A(g(a) - g(b))\|_e \\ &\leq \chi(g)\frac{C_1}{2\alpha}[\exp(2\alpha C_B(a, b)) - 1]\min\{\tilde{S}(a), \tilde{S}(b)\}.\end{aligned}$$

This completes the proof. □

Let $\mathcal{F} \subset \mathcal{LS}_H(B)$ and let $\mathcal{F}^0$ be the subset of $\mathcal{F}$ consisting of all mappings $f = h + \overline{g} \in \mathcal{F}$ such that $Dg(0) = 0$. In view of Theorems 5.3, 5.4 and 4.1, we obtain the following consequence, which is a generalization of [8, Corollary, p. 6216] to finite dimensional JB*-triples.

Corollary 5.5. *Let B be the unit ball of an n-dimensional JB*-triple X which satisfies the condition (3.1). Let $\mathcal{F} \subset \mathcal{LS}_H(B)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$ and let $f = h + \overline{g} \in \mathcal{F}^0$ be such that the condition (4.1) holds. Then*

$$\|f(a) - f(b)\|_e \leq \frac{C_1}{2\alpha}[\exp(2\alpha C_B(a, b)) - 1]\min\{\tilde{S}(a), \tilde{S}(b)\}, \quad a, b \in B,$$

where C_1 is a constant given by (3.2). In addition, if $h + Ag$ is biholomorphic on B for $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$, then f is a sense-preserving univalent mapping on B and

$$\|f(a) - f(b)\|_e \geq \Phi^0_{n,\alpha}(a, b), \quad a, b \in B, \tag{5.10}$$

where $\Phi^0_{n,\alpha}(a, b) = \Psi_{n,\alpha}(C_B(a, b))\max\{S(a), S(b)\}$ and $\Psi_{n,\alpha}$ is given by the relation (3.9).

Conversely, let $f = h + \overline{g}$ be a pluriharmonic mapping on B such that h is locally biholomorphic on B and g is holomorphic on B. If the relation (4.1) holds and f satisfies the relation (5.10) for some $\alpha > 0$ and for all $a, b \in B$, then f is a sense-preserving univalent mapping on B.

Remark 5.6. If we define the order of a linearly invariant family $\mathcal{F} \subset \mathcal{LS}_H(B)$ of pluriharmonic mappings by

$$\alpha_X = \alpha_X(\mathcal{F}) = \sup\left\{\frac{1}{2}\|D^2h(0)(w, \cdot)\|_X : f = h + \overline{g} \in \mathcal{F},\ \|w\|_X = 1\right\},$$

and replace the condition $\|A\|_e < 1$ by $\|A\|_X < 1$ in the definition of "affine invariance", then we can obtain similar results to those in Theorems 5.1 and 5.4, by replacing $\|\cdot\|_e$ by $\|\cdot\|_X$ and the constant C_1 by 1.

6. L.I.F.s and A.L.I.F.s on the unit polydisc in $\mathbb{C}^n$

In this section, we consider L.I.F.s and A.L.I.F.s on the unit polydisc U^n in $\mathbb{C}^n$. Note that U^n is the unit ball of the JB*-triple with the triple product

$$\{x, y, z\} = (x_i\overline{y_i}z_i)_{1\leq i\leq n}, \quad x = (x_i),\ y = (y_i),\ z = (z_i) \in \mathbb{C}^n.$$

Then $Q_a(z) = (a_i\overline{z_i}a_i)_{1\leq i\leq n}$ and

$$B(a, a)z = (z_i - 2|a_i|^2z_i + |a_i|^4z_i)_{1\leq i\leq n}.$$

Therefore, we have

$$B(a, a)^{1/2}z = ((1 - |a_i|^2)z_i)_{1\leq i\leq n} \quad \text{and} \quad B(a, a)^{-1/2}z = ((1 - |a_i|^2)^{-1}z_i)_{1\leq i\leq n}.$$

Also, since $c(U^n) = n$ and $C_1 = \sqrt{n}$, we obtain the following results. Theorem 6.1 is a direct consequence of Theorem 3.10, and Theorem 6.2 may be obtained directly from Theorem 3.11.

Theorem 6.1. *Let $\mathcal{F}$ be a linearly invariant family on U^n. If $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1 - \|z\|_\infty)^{(2n-1)\alpha-1}}{(1 + \|z\|_\infty)^{(2n-1)\alpha+1}}\|w\|_\infty \leq \sqrt{n}^{n-1}\|Df(z)w\|_e, \quad z \in U^n,\ w \in \mathbb{C}^n,$$

for all $f \in \mathcal{F}$.

Theorem 6.2. *Let $\mathcal{F}$ be a linearly invariant family on U^n. If $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and $f \in \mathcal{F}$ is biholomorphic on U^n, then*

$$\|f(z)\|_e \geq \frac{\sqrt{n}^{1-n}}{2(2n-1)\alpha}\left[1 - \left(\frac{1 - \|z\|_\infty}{1 + \|z\|_\infty}\right)^{(2n-1)\alpha}\right], \quad z \in U^n. \tag{6.1}$$

Since $T_f(z) \geq (1 - \|z\|^2_\infty)\|[Df(z)]^{-1}\|_e^{-1}, z \in U^n$, we obtain the following two-point distortion theorem from Theorem 3.12.

Theorem 6.3. *Let $\mathcal{F} \subset \mathcal{L}S(U^n)$ be a linearly invariant family of norm-order $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and let $f \in \mathcal{F}$ be biholomorphic. Then*

$$\|f(a) - f(b)\|_e \geq \frac{\sqrt{n}^{1-n}}{2(2n-1)\alpha}\left[1 - \exp(-2(2n-1)\alpha C_{U^n}(a,b))\right]\max\{T_f^{\infty}(a), T_f^{\infty}(b)\}, \tag{6.2}$$

for $a, b \in U^n$, where $C_{U^n}(a,b)$ denotes the Carathéodory metric in U^n, and

$$T_f^{\infty}(z) = (1 - \|z\|_{\infty}^2)\|[Df(z)]^{-1}\|_e^{-1}.$$

Conversely, if a locally biholomorphic mapping f on U^n satisfies the inequality (6.2), for all $a, b \in U^n$ and for some $\alpha > 0$, then f is biholomorphic on U^n.

Since $\tilde{T}_f(z) \leq (1 - \min_{1\leq i\leq n}|z_i|^2)\|Df(z)\|_e$, $z \in U^n$, we obtain the following result from Theorem 3.13.

Theorem 6.4. *Let $\mathcal{F} \subset \mathcal{L}S(U^n)$ be a linearly invariant family of norm-order $\|\mathrm{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$ and let $f \in \mathcal{F}$. Then*

$$\|f(a) - f(b)\|_e \leq \frac{\sqrt{n}}{2\alpha}\left[\exp(2\alpha C_{U^n}(a,b)) - 1\right]\min\{\tilde{T}_f^{\infty}(a), \tilde{T}_f^{\infty}(b)\}, \quad a, b \in U^n,$$

where

$$\tilde{T}_f^{\infty}(z) = \left(1 - \min_{1\leq i\leq n}|z_i|^2\right)\|Df(z)\|_e, \quad z \in U^n. \tag{6.3}$$

Next, we consider A.L.I.F.s on the unit polydisc in $\mathbb{C}^n$. The following results are particular cases of Theorems 5.2, 5.3, 5.4 and Corollary 5.5.

Theorem 6.5. *Let $\mathcal{F} \subset S_H(U^n)$ be an affine and linearly invariant family of pluriharmonic mappings of order $\alpha = \alpha(\mathcal{F}) < \infty$. Let $f = h + \overline{g} \in \mathcal{F}$ and suppose that $\|Dg(0)\|_e < 1$ and that $h + Ag$ is locally biholomorphic on U^n for each $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$. Then*

$$\|f(z)\|_e \geq (1 - \|Dg(0)\|_e)\frac{\sqrt{n}^{1-n}}{2(2n-1)\alpha}\left[1 - \left(\frac{1 - \|z\|_{\infty}}{1 + \|z\|_{\infty}}\right)^{(2n-1)\alpha}\right], \quad z \in U^n.$$

Since $S(z) \geq (1 - \|z\|_{\infty}^2)\|[Dh(z)]^{-1}\|_e^{-1}(1 - \|Dg(z)[Dh(z)]^{-1}\|_e)$, we obtain the following theorem.

Theorem 6.6. *Let $\mathcal{F} \subset \mathcal{L}S_H(U^n)$ be an affine and linearly invariant family of pluriharmonic mappings of order $\alpha = \alpha(\mathcal{F}) < \infty$, and let $f = h + \overline{g} \in \mathcal{F}$. Assume that $h + Ag$ is biholomorphic on U^n for each $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$. If*

$$\|Dg(z)[Dh(z)]^{-1}\|_e < 1, \quad z \in U^n, \tag{6.4}$$

then

$$\|f(a) - f(b)\|_e \geq \Phi_{n,\alpha}^{\infty}(a,b), \quad a, b \in U^n, \tag{6.5}$$

where

$$\Phi_{n,\alpha}^{\infty}(a,b) = \frac{\nu(g)\sqrt{n}^{1-n}}{2(2n-1)\alpha}\left[1 - \exp(-2(2n-1)\alpha C_{U^n}(a,b))\right]\max\{S^{\infty}(a), S^{\infty}(b)\},$$

$\nu(g)$ is given by (5.3), and

$$S^{\infty}(z) = (1 - \|z\|_{\infty}^2)\|[Dh(z)]^{-1}\|_e^{-1}(1 - \|Dg(z)[Dh(z)]^{-1}\|_e), \quad z \in U^n.$$

Conversely, let $f = h + \overline{g}$ be a pluriharmonic mapping on U^n such that h is locally biholomorphic on U^n and $\|Dg(0)\|_e < 1$. If the relation (6.4) holds and f satisfies the relation (6.5) for some $\alpha > 0$ and for all $a, b \in U^n$, then f is a sense-preserving univalent mapping on U^n.

Since $\tilde{S}(z) \leq (1 - \min_{1\leq i\leq n}|z_i|^2)(\|Dh(z)\|_e + \|Dg(z)\|_e)$, we obtain the following theorem.

Theorem 6.7. *Let $\mathcal{F} \subset \mathcal{L}S_H(U^n)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$ and let $f = h + \overline{g} \in \mathcal{F}$ be such that the condition (6.4) holds. Then*

$$\|f(a) - f(b)\|_e \leq \chi(g)\frac{\sqrt{n}}{2\alpha}\left[\exp(2\alpha C_{U^n}(a,b)) - 1\right]\min\{\tilde{S}^{\infty}(a), \tilde{S}^{\infty}(b)\},$$

for $a, b \in U^n$, where $\chi(g)$ is given by (5.7), and

$$\tilde{S}^{\infty}(z) = \left(1 - \min_{1 \le i \le n} |z_i|^2\right) (\|Dh(z)\|_e + \|Dg(z)\|_e), \quad z \in U^n.$$

Corollary 6.8. *Let $\mathcal{F} \subset \mathcal{L}S_H(U^n)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$ and let $f = h + \overline{g} \in \mathcal{F}^0$ be such that the condition (6.4) holds. Then*

$$\|f(a) - f(b)\|_e \le \frac{\sqrt{n}}{2\alpha} [\exp(2\alpha C_{U^n}(a, b)) - 1] \min\{\tilde{S}^{\infty}(a), \tilde{S}^{\infty}(b)\}, \quad a, b \in U^n.$$

In addition, if $h + Ag$ is biholomorphic on U^n for $A \in L(\mathbb{C}^n)$ with $\|A\|_e < 1$, then f is a sense-preserving univalent mapping on U^n and

$$\|f(a) - f(b)\|_e \ge \Phi^{0,\infty}_{n,\alpha}(a, b), \quad a, b \in U^n, \tag{6.6}$$

where

$$\Phi^{0,\infty}_{n,\alpha}(a, b) = \frac{\sqrt{n}^{1-n}}{2(2n-1)\alpha} [1 - \exp(-2(2n-1)\alpha C_{U^n}(a, b))] \max\{S^{\infty}(a), S^{\infty}(b)\}.$$

Conversely, let $f = h + \overline{g}$ be a pluriharmonic mapping on U^n such that h is locally biholomorphic on U^n and g is holomorphic on U^n. If the relation (6.4) holds and f satisfies the relation (6.6) for some $\alpha > 0$ and for all $a, b \in U^n$, then f is a sense-preserving univalent mapping on U^n.

For $A \in L(\mathbb{C}^n)$, let

$$\|A\|_{\infty} = \sup\{\|Az\|_{\infty} : \|z\|_{\infty} = 1\}.$$

If we define the order of a linearly invariant family $\mathcal{F} \subset \mathcal{L}S_H(U^n)$ of pluriharmonic mappings by

$$\alpha_{\infty} = \alpha_{\infty}(\mathcal{F}) = \sup\left\{\frac{1}{2}\|D^2h(0)(w, \cdot)\|_{\infty} : f = h + \overline{g} \in \mathcal{F}, \ \|w\|_{\infty} = 1\right\},$$

and replace the condition $\|A\|_e < 1$ by $\|A\|_{\infty} < 1$ in the definition of "affine invariance", then we can obtain similar results to those in Theorems 5.1 and 6.7, by replacing $\|\cdot\|_e$ by $\|\cdot\|_{\infty}$ and the constant C_1 by 1.

Theorem 6.9. *Let $\mathcal{F} \subset \mathcal{L}S_H(U^n)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$. Let $f = h + \overline{g} \in \mathcal{F}$, and suppose that $h + Ag$ is locally biholomorphic on U^n for each $A \in L(\mathbb{C}^n)$ with $\|A\|_{\infty} < 1$. Then*

$$\|f(z)\|_{\infty} \le \frac{1}{2\alpha}(1 + \|Dg(0)\|_{\infty}) \left[\left(\frac{1 + \|z\|_{\infty}}{1 - \|z\|_{\infty}}\right)^{\alpha} - 1\right], \quad z \in U^n.$$

Theorem 6.10. *Let $\mathcal{F} \subset \mathcal{L}S_H(U^n)$ be an affine and linearly invariant family of order $\alpha = \alpha(\mathcal{F}) < \infty$ and let $f = h + \overline{g} \in \mathcal{F}$ be such that the condition*

$$\|Dg(z)[Dh(z)]^{-1}\|_{\infty} < 1, \quad z \in U^n,$$

holds. Then

$$\|f(a) - f(b)\|_{\infty} \le \chi_{\infty}(g) \frac{1}{2\alpha} [\exp(2\alpha C_{U^n}(a, b)) - 1] \min\{\tilde{S}^{\infty}_2(a), \tilde{S}^{\infty}_2(b)\},$$

for $a, b \in U^n$, where

$$\chi_{\infty}(g) = \sup_{\|A\|_{\infty} \le 1} \left\{\|[I + ADg(0)]^{-1}\|_{\infty} \cdot \|I + ADg(0)\|_{\infty}\right\},$$

and

$$\tilde{S}^{\infty}_2(z) = \left(1 - \min_{1 \le i \le n} |z_i|^2\right) (\|Dh(z)\|_{\infty} + \|Dg(z)\|_{\infty}), \quad z \in U^n.$$

Acknowledgments

Hidetaka Hamada was partially supported by JSPS KAKENHI Grant Number 25400151. Gabriela Kohr was supported by a grant from the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0899. The authors would like to thank the referee for valuable suggestions that improved the paper.

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