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Multiple positive solutions for semi-linear elliptic systems with sign-changing weight

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ABSTRACT

In this paper, we study the multiplicity results of positive solutions for a semi-linear elliptic system involving both concave–convex and critical growth terms. With the help of the Nehari manifold and the Lusternik–Schnirelmann category, we investigate how the coefficient $h(x)$ of the critical nonlinearity affects the number of positive solutions of that problem and get a relationship between the number of positive solutions and the topology of the global maximum set of h.

1. Introduction and the main result

This paper is concerned with the multiplicity of positive solutions to the following elliptic system:

$$(E_{f,g})\quad \begin{cases} -\Delta u = f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^* = \frac{2N}{N-2}$ $(N \geq 3)$ and $1 < q < 2$. Moreover, we assume that f, g and h satisfy the following conditions.

(H_1) $f, g \in C(\overline{\Omega})$.
(H_2) There exist a non-empty closed set $M = \{x \in \overline{\Omega};\, h(x) = \max_{x\in\overline{\Omega}} h(x) = 1\}$ and a positive number $\rho > 2$ when $N \geq 6$, $\rho > \frac{N-2}{2}$ when $3 \leq N \leq 5$ such that $h(z) - h(x) = O(|x-z|^{\rho})$ as $x \to z$ and uniformly in $z \in M$.
(H_3) $f(x), g(x) > 0$ for $x \in M$.

Remark 1.1. Let $M_r = \{x \in \mathbb{R}^N;\, dist(x, M) < r\}$ for $r > 0$. Then by (H_1)–(H_3), there exist $C_0, r_0 > 0$ such that

$$f(x), g(x), h(x) > 0 \quad \text{for all } x \in M_{r_0} \subset \Omega$$

and

$$h(z) - h(x) \leq C_0|x-z|^{\rho} \quad \text{for all } x \in B_{r_0}(z)$$

uniformly in $z \in M$, where $B_{r_0}(z) = \{x \in \mathbb{R}^N;\, |x-z| < r_0\}$.

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For the systems of semi-linear elliptic equations with concave–convex nonlinearities, various studies concerning the solution structures have been presented (for example [10,1,15,4,3,8,5]). In particular, for $f \equiv \lambda, g \equiv \mu$, Hsu [10] proved that $(E_{f,g})$ permits at least two positive solutions when the pair of parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. Similar results were obtained by Adriouch and El Hamidi [1]. Further studies involving sign-changing weight functions were taken by Wu [15] and Chen and Wu [4] for example, where the two positive solutions were obtained for the subcritical case $2 < \alpha + \beta < 2^*$ in [15] while these for the critical case $\alpha + \beta = 2^*$ were obtained in [4]. The tool of them is the decomposition of the Nehari manifold.

For $2 < q < 2^*$, if $N > 4$, $0 \in \Omega$, f, g and h satisfy the following conditions.

(A_1) f, g and h are positive continuous functions in $\overline{\Omega}$.

(A_2) There exist k points $a^1, a^2, \dots, a^k$ in Ω such that

$$h(a^i) = \max_{x \in \overline{\Omega}} h(x) = 1 \quad \text{for } 1 \le i \le k,$$

and for some $\rho > N$, $h(x) - h(a^i) = O(|x - a^i|^\rho)$ as $x \to a^i$ and uniformly in i.

(A_3) Choose $\rho_0 > 0$ such that

$$\overline{B_{\rho_0}(a^i)} \bigcap \overline{B_{\rho_0}(a^j)} = \emptyset \quad \text{for } i \ne j \text{ and } 1 \le i, j \le k,$$

and $\bigcup_{i=1}^k \overline{B_{\rho_0}(a^i)} \subset \Omega$, where $\overline{B_{\rho_0}(a^i)} = \{x \in \mathbb{R}^N;\ |x - z| \le \rho_0\}$.

Lin [12] recently proved that $(E_{f,g})$ admits at least k positive solutions when f and g are small enough. A similar result was obtained in Li and Yang [11].

Motivated by [12,11], we aim to investigate how the coefficient $h(x)$ of the critical nonlinearity affects the number of positive solutions of $(E_{f,g})$ when $1 < q < 2$ in this work. We try to consider the relationship between the number of positive solutions and the topology of the global maximum set of h by the idea of category. Furthermore, by borrowing some techniques from [10,1,15,4,3,8,5], we will study $(E_{f,g})$ under the conditions (H_1)–(H_3), i.e., we do not need to assume f, g, h are positive solutions and $0 \in \Omega$ as [12,11]. The main result of this paper is as follows.

Theorem 1.1. *Assume* (H_1)–(H_3) *hold. Then for each* $\delta < r_0$, *there exists* $\Lambda_\delta > 0$ *such that if* $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$, $(E_{f,g})$ *has at least* $\operatorname{cat}_{M_\delta}(M) + 1$ *distinct positive solutions, where* $f_+ = \max\{f, 0\}$, $g_+ = \max\{g, 0\}$, $q^* = \frac{2^*}{2^* - q}$ *and cat means the Lusternik–Schnirelmann category (see* [13]*).*

Remark 1.2. Suppose (A_1)–(A_3) hold. By Theorem 1.1, we obtain that $(E_{f,g})$ has at least $k+1$ positive solutions when $\|f\|_{L^{q^*}}$ and $\|g\|_{L^{q^*}}$ are small enough.

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we discuss some concentration behavior. In Section 4, we prove Theorem 1.1.

2. Notations and preliminaries

We propose to study $(E_{f,g})$ in the framework of the Sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ using the standard norm

$$\|(u, v)\|_H = \left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx \right)^{\frac{1}{2}}.$$

Denote

$$S_{\alpha,\beta} := \inf_{(u,v) \in H \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx}{\left(\int_\Omega |u|^\alpha |v|^\beta dx \right)^{\frac{2}{\alpha+\beta}}}.$$

Working as in the proof of [2, Theorem 5], we deduce that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \right) S,$$

where S is the best Sobolev constant, that is

$$S := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx \right)^{\frac{2}{2^*}}}.$$

It is well known that S is independent of Ω, and for each $\varepsilon > 0$,

$$v_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \tag{2.1}$$

is a positive solution of critical problem

$$-\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N$$

with $\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} dx = \frac{1}{N} S^{N/2}$. Actually, S is never attained on a domain $\Omega \neq \mathbb{R}^N$.

Positive solutions to $(E_{f,g})$ will be obtained as critical points of the corresponding energy functional $I_{f,g} : H \to \mathbb{R}$ given by

$$I_{f,g}(u, v) = \frac{1}{2}\|(u, v)\|_H^2 - \frac{1}{q}\int_\Omega (fu_+^q + gv_+^q)dx - \frac{1}{\alpha+\beta}\int_\Omega hu_+^\alpha v_+^\beta dx,$$

where $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. From the assumption, it is easy to prove that $I_{f,g}$ is well defined in H and $I_{f,g} \in C^2(H, \mathbb{R})$.

As $I_{f,g}$ is not bounded below on H, we consider the behaviors of $I_{f,g}$ on the Nehari manifold

$$N_{f,g} = \{(u, v) \in H \setminus \{0\}; I'_{f,g}(u, v)(u, v) = 0\}.$$

Clearly, $(u, v) \in N_{f,g}$ if and only if

$$\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - \int_\Omega (fu_+^q + gv_+^q)dx - \int_\Omega hu_+^\alpha v_+^\beta dx = 0.$$

On the Nehari manifold $N_{f,g}$, from the Sobolev embedding theorem and the Young inequality,

$$\begin{aligned} I_{f,g}(u, v) &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega (fu_+^q + gv_+^q)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u, v)\|_H^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})C\|(u, v)\|_H^q \qquad (2.2) \\ &\geq -(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})^{2/(2-q)}C, \qquad (2.3) \end{aligned}$$

where C denotes positive constants (possibly different) independent of $(u, v) \in H$. Let

$$\begin{aligned} \psi_{f,g}(u, v) &:= I'_{f,g}(u, v)(u, v) \\ &= \int_\Omega |\nabla u|^2|\nabla v|^2 dx - \int_\Omega (fu_+^q + gv_+^q)dx - \int_\Omega hu_+^\alpha v_+^\beta dx. \end{aligned}$$

Then for $(u, v) \in N_{f,g}$,

$$\begin{aligned} \psi'_{f,g}(u, v)(u, v) &= (2-q)\|(u, v)\|_H^2 - (2^* - q)\int_\Omega hu_+^\alpha v_+^\beta dx \qquad (2.4) \\ &= (2-2^*)\|(u, v)\|_H^2 + (2^* - q)\int_\Omega (fu_+^q + gv_+^q)dx. \qquad (2.5) \end{aligned}$$

Similarly to the method used in [10], we split $N_{f,g}$ into three parts:

$$N_{f,g}^+ = \{(u, v) \in N_{f,g}; \psi'_{f,g}(u, v)(u, v) > 0\};$$
$$N_{f,g}^0 = \{(u, v) \in N_{f,g}; \psi'_{f,g}(u, v)(u, v) = 0\};$$
$$N_{f,g}^- = \{(u, v) \in N_{f,g}; \psi'_{f,g}(u, v)(u, v) < 0\}.$$

In the sequel, we shall use Λ_* to denote different small parameters. Then we have the following results.

Lemma 2.1. *Suppose that (u_0, v_0) is a local minimum for $I_{f,g}$ on $N_{f,g}$. Then, if $(u_0, v_0) \notin N_{f,g}^0$, (u_0, v_0) is a critical point of $I_{f,g}$.*

Lemma 2.2. *There exists $\Lambda_* > 0$ such that if $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} \in (0, \Lambda_*)$, $N_{f,g}^0 = \emptyset$.*

For the proofs of the two lemmas above, we refer the reader to [4, Lemmas 2.1, 2.2]. By Lemma 2.2, for $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} \in (0, \Lambda_*)$, we write $N_{f,g} = N_{f,g}^+ \bigcup N_{f,g}^-$ and define

$$\theta_{f,g}^+ = \inf_{(u,v)\in N_{f,g}^+} I_{f,g}(u, v); \qquad \theta_{f,g}^- = \inf_{(u,v)\in N_{f,g}^-} I_{f,g}(u, v).$$

For each $(u, v) \in H$ with $\int_\Omega hu_+^\alpha v_+^\beta dx > 0$, set

$$t_{\max} = \left(\frac{(2-q)\|(u, v)\|_H^2}{(2^* - q)\int_\Omega hu_+^\alpha v_+^\beta dx}\right)^{\frac{1}{\alpha+\beta-2}} > 0.$$

Then

Lemma 2.3. *For each $(u, v) \in H$ with $\int_\Omega h u_+^\alpha v_+^\beta dx > 0$, we have the following.*

(i) *If $\int_\Omega (f u_+^q + g v_+^q) dx \le 0$, there is a unique $t^- > t_{\max}$ such that $(t^- u, t^- v) \in N_{f,g}^-$ and*

$$I_{f,g}(t^- u, t^- v) = \sup_{t \ge 0} I_{f,g}(tu, tv).$$

(ii) *If $\int_\Omega (f u_+^q + g v_+^q) dx > 0$, there are unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+ u, t^+ v) \in N_{f,g}^+$, $(t^- u, t^- v) \in N_{f,g}^-$ and*

$$I_{f,g}(t^+ u, t^+ v) = \inf_{0 \le t \le t_{\max}} I_{f,g}(tu, tv); \; I_{f,g}(t^- u, t^- v) = \sup_{t \ge 0} I_{f,g}(tu, tv).$$

Lemma 2.4. *If $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} \in (0, \Lambda_*)$, then*

(i) $\theta_{f,g}^+ < 0$;

(ii) $\theta_{f,g}^- \ge \rho_0$ *for some* $\rho_0 > 0$.

For the proofs of Lemmas 2.3 and 2.4, the readers are referred to [4] for similar proofs.

Remark 2.1. From Lemmas 2.3 and 2.4, it is easy to know if $(u, v) \in N_{f,g}^-$,

$$\int_\Omega h u_+^\alpha v_+^\beta dx > 0.$$

Next we establish that $I_{f,g}$ satisfies the $(PS)_c$-condition for $c \in (-\infty, \theta_{f,g}^+ + \frac{1}{N} S_{\alpha,\beta}^{N/2})$, which was proved in [8] and we sketch the proof here for reader's convenience.

Lemma 2.5. *For $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} \in (0, \Lambda_*)$, $I_{f,g}$ satisfies the $(PS)_c$-condition for $c \in (-\infty, \theta_{f,g}^+ + \frac{1}{N} S_{\alpha,\beta}^{N/2})$.*

Proof. Let $\{(u_k, v_k)\} \subset H$ be a $(PS)_c$-sequence for $I_{f,g}$ and $c \in (-\infty, \theta_{f,g}^+ + \frac{1}{N} S_{\alpha,\beta}^{N/2})$. After a standard argument (see [14]), we know that $\{(u_k, v_k)\}$ is bounded in H. Thus, there exist a subsequence still denoted by $\{(u_k, v_k)\}$ and $(u, v) \in H$ such that $(u_k, v_k) \rightharpoonup (u, v)$ weakly in H. By the compactness of Sobolev embedding and [9, Lemma 2.1], we get

- $\int_\Omega (f (u_k)_+^q + g (v_k)_+^q) dx = \int_\Omega (f u_+^q + g v_+^q) dx + o(1)$;
- $\|(u_k - u, v_k - v)\|_H^2 = \|(u_k, v_k)\|_H^2 - \|(u, v)\|_H^2 + o(1)$;
- $\int_\Omega h (u_k - u)_+^\alpha (v_k - v)_+^\beta dx = \int_\Omega h (u_k)_+^\alpha (v_k)_+^\beta dx - \int_\Omega h u_+^\alpha v_+^\beta dx + o(1)$.

Moreover, we can obtain $I'_{f,g}(u, v) = 0$ in H^{-1} (the dual space of H). Since $I_{f,g}(u_k, v_k) = c + o(1)$ and $I'_{f,g}(u_k, v_k) = o(1)$ in H^{-1}, we deduce that

$$\frac{1}{2} \|(u_k - u, v_k - v)\|_H^2 - \frac{1}{2^*} \int_\Omega h (u_k - u)_+^\alpha (v_k - v)_+^\beta dx = c - I_{f,g}(u, v) + o(1) \tag{2.6}$$

and

$$\begin{aligned} o(1) &= I'_{f,g}(u_k, v_k)(u_k - u, v_k - v) = (I'_{f,g}(u_k, v_k) - I'_{f,g}(u, v))(u_k - u, v_k - v) \\ &= \|(u_k - u, v_k - v)\|_H^2 - \int_\Omega h (u_k - u)_+^\alpha (v_k - v)_+^\beta dx + o(1). \end{aligned}$$

Now we may assume that

$$\|(u_k - u, v_k - v)\|_H^2 \to l \quad \text{and} \quad \int_\Omega h (u_k - u)_+^\alpha (v_k - v)_+^\beta dx \to l \quad \text{as } k \to \infty,$$

for some $l \in [0, +\infty)$.

Suppose $l \ne 0$ and notice the fact $h \le 1$, using the Sobolev embedding theorem and passing to the limit as $k \to \infty$, we have

$$l \ge S_{\alpha,\beta} l^{\frac{2}{2^*}},$$

that is,

$$l \ge S_{\alpha,\beta}^{N/2}. \tag{2.7}$$

Then by (2.6), (2.7) and $(u, v) \in N_{f,g} \bigcup \{0\}$,

$$c = I_{f,g}(u, v) + \frac{1}{N} l \geq \theta^+_{f,g} + \frac{1}{N} S^{N/2}_{\alpha,\beta},$$

which contradicts the definition of c. Hence $l = 0$, i.e., $(u_k, v_k) \to (u, v)$ strongly in H. $\square$

Then we obtain the existence of a local minimizer for $I_{f,g}$ on $N^+_{f,g}$.

Lemma 2.6. *For $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} \in (0, \Lambda_*)$, the functional $I_{f,g}$ has a minimizer $(u^+_{f,g}, v^+_{f,g}) \in N^+_{f,g}$ and it satisfies:*

(i) $I_{f,g}(u^+_{f,g}, v^+_{f,g}) = \theta^+_{f,g}$;
(ii) $(u^+_{f,g}, v^+_{f,g})$ *is a positive solution of* $(E_{f,g})$;
(iii) $I_{f,g}(u^+_{f,g}, v^+_{f,g}) \to 0$ *as* $\|f_+\|_{L^{q^*}}, \|g_+\|_{L^{q^*}} \to 0$;
(iv) $\|(u^+_{f,g}, v^+_{f,g})\|_H \to 0$ *as* $\|f_+\|_{L^{q^*}}, \|g_+\|_{L^{q^*}} \to 0$.

Proof. (i)–(ii) are consequences of [15,4]. Moreover, by (2.3) and Lemma 2.4,

$$0 > I_{f,g}(u^+_{f,g}, v^+_{f,g}) \geq -(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})^{2/(2-q)} C.$$

We obtain $I_{f,g}(u^+_{f,g}, v^+_{f,g}) \to 0$ as $\|f_+\|_{L^{q^*}}, \|g_+\|_{L^{q^*}} \to 0$.

Now we show (iv). By $(u^+_{f,g}, v^+_{f,g}) \in N^+_{f,g}$ and (2.5),

$$\begin{aligned} \|(u^+_{f,g}, v^+_{f,g})\|^2_H &\leq \frac{2^* - q}{2^* - 2} \int_\Omega (f_+(u^+_{f,g})^q + g_+(u^+_{f,g})^q) dx \\ &\leq C(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}}) \|(u^+_{f,g}, v^+_{f,g})\|^q_H. \end{aligned} \tag{2.8}$$

Since $I_{f,g}$ is coercive and bounded below on $N_{f,g}$, $(u^+_{f,g}, v^+_{f,g})$ is bounded in H and so that by (2.8) we know

$$\|(u^+_{f,g}, v^+_{f,g})\|^{2-q}_H \leq C(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}}).$$

Then

$$\|(u^+_{f,g}, v^+_{f,g})\|_H \to 0 \quad \text{as } \|f_+\|_{L^{q^*}}, \|g_+\|_{L^{q^*}} \to 0. \quad \square$$

3. Concentration behavior

In this section, we will recall and prove some lemmas which are crucial in the proof of the main theorem.

For $b > 0$, we define

$$J^b_\infty(u, v) = \frac{1}{2} \|(u, v)\|^2_H - \frac{b}{2^*} \int_\Omega h u^\alpha_+ v^\beta_+ dx$$

and

$$N^b_\infty(u, v) = \{(u, v) \in H \setminus \{0\}; (J^b_\infty)'(u, v)(u, v) = 0\}.$$

Then we have the following.

Lemma 3.1. *For each $(u, v) \in N^-_{f,g}$, we have the following.*

(i) *There is a unique $t^b_{(u,v)}$ such that $(t^b_{(u,v)} u, t^b_{(u,v)} v) \in N^b_\infty$ and*

$$\max_{t \geq 0} J^b_\infty(tu, tv) = J^b_\infty(t^b_{(u,v)} u, t^b_{(u,v)} v) = \frac{1}{N} b^{\frac{2-N}{2}} \left(\frac{\|(u, v)\|^{2^*}_H}{\int_\Omega h u^\alpha_+ v^\beta_+ dx} \right)^{\frac{N-2}{2}}.$$

(ii) *For $\mu \in (0, 1)$, there is a unique $t^1_{(u,v)}$ such that $(t^1_{(u,v)} u, t^1_{(u,v)} v) \in N^1_\infty$. Moreover,*

$$J^1_\infty(t^1_{(u,v)} u, t^1_{(u,v)} v) \leq (1 - \mu)^{-\frac{N}{2}} \left(I_{f,g}(u, v) + \frac{2 - q}{2q} \mu^{\frac{q}{q-2}} C(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})^{\frac{2}{2-q}} \right).$$

Proof. (i) For each $u \in N^-_{f,g}$, let

$$\overline{h}(t) = J^b_\infty(tu, tv) = \frac{1}{2} t^2 \|(u, v)\|^2_H - \frac{b}{2^*} t^{2^*} \int_\Omega h u^\alpha_+ v^\beta_+ dx.$$

Then since Remark 2.1, we have $\overline{h}(t) \to -\infty$ as $t \to \infty$,

$$\overline{h}'(t) = t\|(u,v)\|_H^2 - bt^{2^*-1}\int_\Omega hu_+^\alpha v_+^\beta dx$$

and

$$\overline{h}''(t) = t\|(u,v)\|_H^2 - b(2^*-1)t^{2^*-2}\int_\Omega hu_+^\alpha v_+^\beta dx.$$

Set

$$t_{(u,v)}^b = \left(\frac{\|(u,v)\|_H^2}{\int_\Omega bhu_+^\alpha v_+^\beta dx}\right)^{\frac{1}{2^*-2}} > 0.$$

Then $h'(t_{(u,v)}^b) = 0$, $t_{(u,v)}^b u \in N_\infty^b$ and $h''(t_{(u,v)}^b) = (2-2^*)\|(u,v)\|_H^2 < 0$. Hence there is a unique $t_{(u,v)}^b$ such that $(t_{(u,v)}^b u, t_{(u,v)}^b v) \in N_\infty^b$ and

$$\max_{t\geq 0} J_\infty^b(tu,tv) = J_\infty^b(t_{(u,v)}^b u, t_{(u,v)}^b v) = \frac{1}{N}b^{\frac{2-N}{2}}\left(\frac{\|(u,v)\|_H^{2^*}}{\int_\Omega hu_+^\alpha v_+^\beta dx}\right)^{\frac{N-2}{2}}.$$

(ii) For $\mu \in (0,1)$, we have

$$\begin{aligned}
\int_\Omega f_+(t_{(u,v)}^b u)_+^q + g_+(t_{(u,v)}^b v)_+^q dx &\leq (\|f_+\|_{q^*} + \|g_+\|_{q^*})C\|(t_{(u,v)}^b u, t_{(u,v)}^b v)\|_H^q \\
&\leq \frac{2-q}{2}\left((\|f_+\|_{q^*} + \|g_+\|_{q^*})C\mu^{-\frac{q}{2}}\right)^{\frac{2}{2-q}} + \frac{q}{2}\left(\mu^{\frac{q}{2}}\|(t_{(u,v)}^b u, t_{(u,v)}^b v)\|_H^q\right)^{\frac{2}{q}} \\
&= \frac{2-q}{2}\mu^{\frac{q}{q-2}}C(\|f_+\|_{q^*} + \|g_+\|_{q^*})^{\frac{2}{2-q}} + \frac{q\mu}{2}\|(t_{(u,v)}^b u, t_{(u,v)}^b v)\|_H^2.
\end{aligned}$$

Then let $b = \frac{1}{1-\mu}$ and by part (i),

$$\begin{aligned}
I_{f,g}(u,v) &= \max_{t\geq 0} I_{f,g}(tu,tv) \geq I_{f,g}\left(t_{(u,v)}^{\frac{1}{1-\mu}}u, t_{(u,v)}^{\frac{1}{1-\mu}}v\right) \\
&\geq \frac{1-\mu}{2}\left\|\left(t_{(u,v)}^{\frac{1}{1-\mu}}u, t_{(u,v)}^{\frac{1}{1-\mu}}v\right)\right\|_H^2 - \frac{1}{2^*}\left(t_{(u,v)}^{\frac{1}{1-\mu}}\right)^{2^*}\int_\Omega hu_+^\alpha v_+^\beta dx - \frac{2-q}{2q}\mu^{\frac{q}{q-2}}C(\|f_+\|_{q^*} + \|g_+\|_{q^*})^{\frac{2}{2-q}} \\
&= (1-\mu)J_\infty^{\frac{1}{1-\mu}}\left(t_{(u,v)}^{\frac{1}{1-\mu}}u, t_{(u,v)}^{\frac{1}{1-\mu}}v\right) - \frac{2-q}{2q}\mu^{\frac{q}{q-2}}C(\|f_+\|_{q^*} + \|g_+\|_{q^*})^{\frac{2}{2-q}} \\
&= (1-\mu)^{\frac{N}{2}}\frac{1}{N}\left(\frac{\|(u,v)\|_H^{2^*}}{\int_\Omega hu_+^\alpha v_+^\beta dx}\right)^{\frac{N-2}{2}} - \frac{2-q}{2q}\mu^{\frac{q}{q-2}}C(\|f_+\|_{q^*} + \|g_+\|_{q^*})^{\frac{2}{2-q}} \\
&= (1-\mu)^{\frac{N}{2}}J_\infty^1(t_{(u,v)}^1 u, t_{(u,v)}^1 v) - \frac{2-q}{2q}\mu^{\frac{q}{q-2}}C(\|f_+\|_{q^*} + \|g_+\|_{q^*})^{\frac{2}{2-q}}.
\end{aligned}$$

This completes the proof. □

Following the same method as in [4] and Remark 1.1, let $\eta(x) \in C_0^\infty(\mathbb{R}^N)$ be a radially symmetric function with $0 \leq \eta \leq 1$, $|\nabla\eta| \leq C$, and

$$\eta(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{r_0}{2}, \\ 0, & \text{if } |x| \geq r_0. \end{cases}$$

For any $z \in M$, we define

$$\omega_{\varepsilon,z}(x) = \eta(x-z)v_\varepsilon(x-z)$$

where $v_\varepsilon(x)$ is given by (2.1). From the same arguments of [13] we know

$$\int_\Omega |\nabla\omega_{\varepsilon,z}|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) \quad \text{and} \quad \int_\Omega |\omega_{\varepsilon,z}|^{2^*} dx = S^{\frac{N}{2}} + O(\varepsilon^N). \tag{3.1}$$

Moreover, we have the following.

Lemma 3.2.

$$\int_\Omega h|\omega_{\varepsilon,z}|^{2^*}dx = \begin{cases} S^{N/2} + o(\varepsilon^2), & \text{if } N \geq 6, \\ S^{N/2} + o\left(\varepsilon^{\frac{N-2}{2}}\right), & \text{if } 3 \leq N \leq 5. \end{cases}$$

Proof. See [4, Lemma 3.2]. □

Then we have the following results.

Lemma 3.3. *There exist $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in (0, \varepsilon_0)$, we have $\sigma(\varepsilon_0) > 0$ and*

$$\sup_{t\geq 0} I_{f,g}(u^+_{f,g} + t\sqrt{\alpha}\omega_{\varepsilon,z}, v^+_{f,g} + t\sqrt{\beta}\omega_{\varepsilon,z}) < \theta^+_{f,g} + \frac{1}{N}S^{N/2}_{\alpha,\beta} - \sigma(\varepsilon_0) \quad \textit{uniformly in } z \in M.$$

Furthermore, there exists $t^-_z > 0$ such that

$$(u^+_{f,g} + t^-_z\sqrt{\alpha}\omega_{\varepsilon,z}, v^+_{f,g} + t^-_z\sqrt{\beta}\omega_{\varepsilon,z}) \in N^-_{f,g} \quad \textit{for all } z \in M.$$

Proof. Noting the conditions (H_1)–(H_3) and the compactness of M, the proof is almost identical to the proof of [4, Lemma 3.3] and is omitted here for brevity. □

Lemma 3.4. *We have*

$$\inf_{(u,v)\in N^1_\infty} J^1_\infty(u, v) = \inf_{(u,v)\in N^\infty} J^\infty(u, v) = \frac{1}{N}S^{N/2}_{\alpha,\beta},$$

where $J^\infty(u, v) = \frac{1}{2}\|(u, v)\|^2_H - \frac{1}{2^*}\int_\Omega u^\alpha_+ v^\beta_+ dx$ and $N^\infty = \{(u, v) \in H \setminus \{0\}; (J^\infty)'(u, v)(u, v) = 0\}$.

Proof. By [12, Lemma 4.4], we see

$$\inf_{(u,v)\in N^\infty} J^\infty(u, v) = \frac{1}{N}S^{N/2}_{\alpha,\beta}.$$

Thus it suffices to show that $\inf_{(u,v)\in N^1_\infty} J^1_\infty(u, v) = \frac{1}{N}S^{N/2}_{\alpha,\beta}$. Since

$$\max_{t\geq 0}\left(\frac{a}{2}t^2 - \frac{b}{2^*}t^{2^*}\right) = \frac{1}{N}\left(\frac{a}{b^{2/2^*}}\right)^{N/2} \quad \text{for any } a > 0 \text{ and } b > 0,$$

by (3.1) and Lemma 3.2 we deduce that

$$\begin{aligned} \sup_{t\geq 0} J^1_\infty(t\sqrt{\alpha}\omega_{\varepsilon,z}, t\sqrt{\beta}\omega_{\varepsilon,z}) &= \frac{1}{N}\left(\frac{(\alpha+\beta)\int_\Omega |\nabla\omega_{\varepsilon,z}|^2 dx}{\left(\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}\int_\Omega h|\omega_{\varepsilon,z}|^{2^*}dx\right)^{2/2^*}}\right)^{N/2} \\ &= \frac{1}{N}S^{N/2}_{\alpha,\beta} + O(\varepsilon^{N-2}). \end{aligned}$$

Then we obtain

$$\inf_{(u,v)\in N^1_\infty} J^1_\infty(u, v) \leq \frac{1}{N}S^{N/2}_{\alpha,\beta}, \quad \text{as } \varepsilon \to 0^+.$$

Since $h \leq 1$, for each $(u, v) \in H \setminus \{0\}$, we have

$$\sup_{t\geq 0} J^\infty(tu, tv) \leq \sup_{t\geq 0} J^1_\infty(tu, tv).$$

Hence

$$\begin{aligned} \frac{1}{N}S^{N/2}_{\alpha,\beta} &= \inf_{(u,v)\in N^\infty} J^\infty(u, v) = \inf_{(u,v)\in H\setminus\{0\}} \sup_{t\geq 0} J^\infty(tu, tv) \\ &\leq \inf_{(u,v)\in H\setminus\{0\}} \sup_{t\geq 0} J^1_\infty(tu, tv) = \inf_{(u,v)\in N^1_\infty} J^1_\infty(u, v) \leq \frac{1}{N}S^{N/2}_{\alpha,\beta}. \end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.1

In this section, we use the idea of category to get positive solutions of $E_{f,g}$ in H and give the proof of Theorem 1.1. Initially, we give the following two lemmas related to the category.

Proposition 4.1. *Let R be a $C^{1,1}$ complete Riemannian manifold (modeled on a Hilbert space) and assume $F \in C^1(R, \mathbb{R})$ bounded from below. Let $-\infty < \inf_R F < a < b < +\infty$. Suppose that h satisfies the (PS)-condition on the sublevel $\{u \in R; F(u) \le b\}$ and that a is not a critical level for F. Then*

$$\sharp\{u \in F^a; \nabla F(u) = 0\} \ge \mathrm{cat}_{F^a}(F^a),$$

where $h^a \equiv \{u \in H; h(u) \le a\}$.

Proof. See [6, Theorem 2.1]. □

Proposition 4.2. *Let Q, Ω^+ and Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\phi : Q \to \Omega^+$, $\varphi : \Omega^- \to Q$ be two continuous maps such that $\phi \circ \varphi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$. Then $\mathrm{cat}_Q(Q) \ge \mathrm{cat}_{\Omega^+}(\Omega^-)$.*

Proof. See [6, Lemma 2.2]. □

The proof of Theorem 1.1 is based on Propositions 4.1 and 4.2. To argue further, we need to introduce the following lemma.

Lemma 4.1. *Let $\{(u_k, v_k)\} \subset H$ be a nonnegative function sequence with $\int_\Omega (u_k)_+^\alpha (v_k)_+^\beta dx = 1$ and $\|(u_k, v_k)\|_H^2 \to S_{\alpha,\beta}$. Then there exists a sequence $\{(x_k, \varepsilon_k)\} \in \mathbb{R}^N \times \mathbb{R}^+$ such that*

$$\omega_k(x) = (\omega_k^1(x), \omega_k^2(x)) := \varepsilon_k^{\frac{N-2}{2}} (u_k(\varepsilon_k x + x_k), v_k(\varepsilon_k x + x_k))$$

contains a convergent subsequence denoted again by $\{\omega_k\}$ such that $\omega_k \to \omega = (\omega^1, \omega^2)$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $\omega^1(x) > 0$ and $\omega^2(x) > 0$ in $\mathbb{R}^N$. Moreover, we have $\varepsilon_k \to 0$ and $x_k \to x_0 \in \overline{\Omega}$ as $k \to \infty$.

Proof. See [7, Lemma 3.1]. □

Next we define the continuous map $\Phi : H \setminus G \to \mathbb{R}^N$ by

$$\Phi(u, v) := \frac{\int_\Omega x(u - u_{f,g}^+)_+^\alpha (v - v_{f,g}^+)_+^\beta dx}{\int_\Omega (u - u_{f,g}^+)_+^\alpha (v - v_{f,g}^+)_+^\beta dx},$$

where $G = \{(u, v) \in H; \int_\Omega (u - u_{f,g}^+)_+^\alpha (v - v_{f,g}^+)_+^\beta dx = 0\}$. Then we have the following.

Lemma 4.2. *For each $0 < \delta < r_0$, there exist $\Lambda_\delta, \delta_0 > 0$ such that if $(u, v) \in N_\infty^1, J_\infty^1(u, v) < \frac{1}{N} S_{\alpha,\beta}^{N/2} + \delta_0$ and $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$, then $\Phi(u, v) \in M_\delta$.*

Proof. Suppose the contrary. Then there exists a sequence $\{(u_k, v_k)\} \subset N_\infty^1$ such that $J_\infty^1(u_k, v_k) = \frac{1}{N} S_{\alpha,\beta}^{N/2} + o(1)$, $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} = o(1)$, and

$$\Phi(u_k, v_k) \notin M_\delta \quad \text{for all } k.$$

It is easy to show that $\{(u_k, v_k)\}$ is bounded in H and there is a sequence $\{t_k^\infty\} \subset \mathbb{R}^+$ such that $(t_k^\infty u_k, t_k^\infty v_k) \in N^\infty$ and

$$\frac{1}{N} S_{\alpha,\beta}^{N/2} \le J^\infty(t_k^\infty u_k, t_k^\infty v_k) \le J_\infty^1(t_k^\infty u_k, t_k^\infty v_k) \le J_\infty^1(u_k, v_k) = \frac{1}{N} S_{\alpha,\beta}^{N/2} + o(1).$$

We obtain $t_k^\infty = 1 + o(1)$ as $k \to \infty$ and

$$\begin{aligned} \lim_{k\to\infty} J^\infty(u_k, v_k) &= \lim_{k\to\infty} \frac{1}{N} \|(u_k, v_k)\|_H^2 = \lim_{k\to\infty} \frac{1}{N} \int_\Omega (u_k)_+^\alpha (v_k)_+^\beta dx \\ &= \lim_{k\to\infty} \frac{1}{N} \int_\Omega h(u_k)_+^\alpha (v_k)_+^\beta dx = \frac{1}{N} S_{\alpha,\beta}^{N/2} + o(1). \end{aligned} \tag{4.1}$$

Define

$$U_k = \left(\frac{(u_k)_+}{\left(\int_\Omega (u_k)_+^\alpha (v_k)_+^\beta dx\right)^{1/(\alpha+\beta)}}, \frac{(v_k)_+}{\left(\int_\Omega (u_k)_+^\alpha (v_k)_+^\beta dx\right)^{1/(\alpha+\beta)}} \right).$$

We see that $\int_\Omega (U_k^1)_+^\alpha (U_k^2)_+^\beta dx = 1$. It follows from (4.1) and the definition of $S_{\alpha,\beta}$ that

$$\lim_{k\to\infty} \|(U_k^1, U_k^2)\|_H^2 = S_{\alpha,\beta}.$$

By Lemma 4.3, there is a sequence $\{(x_k, \varepsilon_k)\} \in \mathbb{R}^N \times \mathbb{R}^+$ such that $\varepsilon_k \to 0$, $x_k \to x_0 \in \overline{\Omega}$ and $\omega_k(x) = \varepsilon_k^{\frac{N-2}{2}} (U_k^1(\varepsilon_k x + x_k), U_k^2(\varepsilon_k x + x_k)) \to (\omega_1, \omega_2)$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $\omega_1 > 0$ and $\omega_2 > 0$ in $\mathbb{R}^N$ as $k \to \infty$. Then by (4.1),

$$1 = o(1) + \int_\Omega h(U_k^1)_+^\alpha (U_k^2)_+^\beta dx = \varepsilon_k^{-N} \int_\Omega h \left(\omega_k^1 \left(\frac{x-x_k}{\varepsilon_k}\right)\right)_+^\alpha \left(\omega_k^2 \left(\frac{x-x_k}{\varepsilon_k}\right)\right)_+^\beta dx + o(1) = h(x_0),$$

as $k \to \infty$, which implies $x_0 \in M$. By the Lebesgue dominated convergence theorem again, we have

$$\begin{aligned}
\Phi(u_k, v_k) &= \frac{\int_\Omega x(u_k - u^+_{f_k,g_k})_+^\alpha (v_k - v^+_{f_k,g_k})_+^\beta dx}{\int_\Omega (u_k - u^+_{f_k,g_k})_+^\alpha (v_k - v^+_{f_k,g_k})_+^\beta dx} \\
&= \frac{\int_\Omega x(u_k)_+^\alpha (v_k)_+^\beta dx}{\int_\Omega (u_k)_+^\alpha (v_k)_+^\beta dx} + o(1), \quad \text{as } \|(f_k)_+\|_{L^{q^*}}, \|(g_k)_+\|_{L^{q^*}} \to 0 \\
&= \frac{\varepsilon_k^{-N} \int_\Omega x \left(\omega_k^1 \left(\frac{x-x_k}{\varepsilon_k}\right)\right)_+^\alpha \left(\omega_k^2 \left(\frac{x-x_k}{\varepsilon_k}\right)\right)_+^\beta dx}{\varepsilon_k^{-N} \int_\Omega \left(\omega_k^1 \left(\frac{x-x_k}{\varepsilon_k}\right)\right)_+^\alpha \left(\omega_k^2 \left(\frac{x-x_k}{\varepsilon_k}\right)\right)_+^\beta dx} + o(1), \\
&\to x_0 \in M \quad \text{as } k \to \infty,
\end{aligned}$$

which is a contradiction. $\square$

Lemma 4.3. *There exists $\Lambda_\delta > 0$ small enough such that if $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$ and $(u, v) \in N^-_{f,g}$ with $I_{f,g}(u, v) < \frac{1}{N} S_{\alpha,\beta}^{N/2} + \frac{\delta_0}{2}$ (δ_0 is given in Lemma 4.2), then $\Phi(u, v) \in M_\delta$.*

Proof. By Lemma 3.1, for $\mu \in (0, 1)$, there is a unique $t^1_{(u,v)}$ such that $(t^1_{(u,v)} u, t^1_{(u,v)} v) \in N^1_\infty$ and

$$J^1_\infty(t^1_{(u,v)} u, t^1_{(u,v)} v) \le (1-\mu)^{-\frac{N}{2}} \left(I_{f,g}(u, v) + \frac{2-q}{2q} \mu^{\frac{q}{q-2}} C(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})^{\frac{2}{2-q}} \right).$$

Thus there exists $\Lambda_\delta > 0$ small enough such that if $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$ and $I_{f,g}(u, v) < \frac{1}{N} S_{\alpha,\beta}^{N/2} + \frac{\delta_0}{2}$,

$$J^1_\infty(t^1_{(u,v)} u, t^1_{(u,v)} v) \le \frac{1}{N} S_{\alpha,\beta}^{N/2} + \delta_0.$$

By Lemma 4.2 and $\|(u^+_{f,g}, v^+_{f,g})\|_H \to 0$ as $\|(f_k)_+\|_{L^{q^*}}, \|(g_k)_+\|_{L^{q^*}} \to 0$, we complete the proof. $\square$

Now we denote $c_{f,g} := \theta^+_{f,g} + \frac{1}{N} S_{\alpha,\beta}^{N/2} - \sigma(\varepsilon_0)$ and consider the filtration of the manifold of $N^-_{f,g}$ as follows:

$$N^-_{f,g}(c_{f,g}) := \{(u, v) \in N^-_{f,g};\ I_{f,g} \le c_{f,g}\}.$$

Then $\mathrm{cat}_{M_\delta}(M)$ critical points of $I_{f,g}$ will be obtained from $N^-_{f,g}(c_{f,g})$ in the following.

Lemma 4.4. *Let $\delta, \Lambda_\delta > 0$ be as in Lemmas 4.2 and 4.3. Then for $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$, $I_{f,g}$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N^-_{f,g}(c_{f,g})$.*

Proof. For $z \in M$, by Lemma 3.3, we can define

$$F(z) = (u^+_{f,g} + t^-_z \sqrt{\alpha}\omega_{\varepsilon,z},\ v^+_{f,g} + t^-_z \sqrt{\beta}\omega_{\varepsilon,z}) \in N^-_{f,g}(c_{f,g}).$$

Furthermore, $I_{f,g}$ satisfies the (PS)-condition on $N^-_{f,g}(c_{f,g})$. Moreover, it follows from Lemma 4.3 that $\Phi(N^-_{f,g}(c_{f,g})) \subset M_\delta$ for $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$. Define $\xi : [0, 1] \times M \to M_\delta$ by

$$\xi(\theta, z) = \Phi\left(u^+_{f,g} + t^-_z \sqrt{\alpha}\omega_{(1-\theta)\varepsilon,z},\ v^+_{f,g} + t^-_z \sqrt{\beta}\omega_{(1-\theta)\varepsilon,z}\right) \in N^-_{f,g}(c_{f,g}).$$

Then straightforward calculations provide that $\xi(0, z) = \Phi \circ F(z)$ and $\lim_{\theta\to 1^-} \xi(\theta, z) = z$. Hence $\Phi \circ F$ is homotopic to the inclusion $j : M \to M_\delta$. By Propositions 4.1 and 4.2, $I_{f,g}$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N^-_{f,g}(c_{f,g})$. $\square$

Finally we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Note Lemmas 2.6 and 4.4, and applying $N^+_{f,g} \bigcap N^-_{f,g} = \emptyset$ and the strong maximum principle, we obtain the conclusion of Theorem 1.1. $\square$

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