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Oscillation and asymptotic behavior of higher-order delay differential equations with p-Laplacian like operators

Chenghui Zhang[a,*], Ravi P. Agarwal[b], Tongxing Li[a]

[a] Shandong University, School of Control Science and Engineering, Jinan, Shandong 250061, PR China

[b] Texas A&M University-Kingsville, Department of Mathematics, 700 University Blvd., Kingsville, TX 78363-8202, USA

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ABSTRACT

The p-Laplace equations have some applications in continuum mechanics. On the basis of this background detail, we study oscillation and asymptotic behavior of solutions to two classes of higher-order delay damped differential equations with p-Laplacian like operators. Some new criteria are presented that improve the related contributions to the subject. Several examples are provided to illustrate the relevance of new theorems.

1. Introduction

In the natural sciences, technology, and population dynamics, differential equations find many application fields; see [9]. For instance, the p-Laplace equations have some applications in continuum mechanics as seen from [4,5]. Recently, there has been an increasing interest in studying oscillation and nonoscillation of different classes of differential equations. We refer the reader to [1–4,6–8,10–27] and the references cited therein.

In this paper, we shall be concerned with the problem of oscillation and asymptotic behavior of a higher-order delay damped differential equation with p-Laplacian like operators

$$(a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t))' + r(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t) + q(t)|x(g(t))|^{p-2}x(g(t)) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, and we will assume that the following assumptions hold:

(H_1) p is a real number satisfying $p > 1$, $g \in C[t_0, \infty)$, $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$;
(H_2) $a \in C^1[t_0, \infty)$, $r, q \in C[t_0, \infty)$, $a(t) > 0$, $a'(t) + r(t) \geq 0$, $q(t) > 0$.

By a solution of (1.1) we mean a function $x \in C^{n-1}[T_x, \infty)$, $T_x \geq t_0$, which has the property $a|x^{(n-1)}|^{p-2}x^{(n-1)} \in C^1[T_x, \infty)$ and satisfies (1.1) on $[T_x, \infty)$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and tacitly assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is said to be nonoscillatory. Eq. (1.1) is termed oscillatory if all its solutions are oscillatory.

In what follows, we present some related results that serve and motivate the contents of this paper. Agarwal et al. [3] studied the higher-order differential equation

$$(|x^{(n-1)}(t)|^{\alpha-1}x^{(n-1)}(t))' + q(t)|x(g(t))|^{\alpha-1}x(g(t)) = 0, \tag{1.2}$$

* Corresponding author.
E-mail addresses: zchui@sdu.edu.cn (C. Zhang), agarwal@tamuk.edu (R.P. Agarwal), litongx2007@163.com (T. Li).

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where $n \geq 2$ is even, $\alpha > 0$ is a constant, $q, g \in C([t_0, \infty), \mathbb{R})$, $q(t) > 0$, and $\lim_{t\to\infty} g(t) = \infty$. They obtained the following criterion (note that some inaccuracies have been corrected).

Theorem 1.1 (*See [3, Theorem 2.1]*). *If there exist functions $\sigma, \ \rho \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\sigma(t) \leq \inf\{t, g(t)\}, \qquad \lim_{t\to\infty} \sigma(t) = \infty, \qquad \sigma'(t) > 0 \quad \text{for } t \geq t_0,$$

and

$$\int^{\infty} \left[\rho(t)q(t) - \theta \frac{(2(n-2)!)^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{(\rho'(t))^{\alpha+1}}{(\sigma^{n-2}(t)\sigma'(t)\rho(t))^{\alpha}}\right] dt = \infty$$

for all constants $\theta \in (1, \infty)$, then Eq. (1.2) is oscillatory.

Grace and Lalli [8], Karpuz et al. [10], Zafer [21], Zhang and Yan [26], and Zhang et al. [27] considered oscillation of a higher-order equation

$$x^{(n)}(t) + q(t)x(g(t)) = 0, \tag{1.3}$$

where $n \geq 2$ is even, $q, \ g \in C([t_0, \infty), \mathbb{R})$, $q(t) > 0$, and $\lim_{t\to\infty} g(t) = \infty$. They established the following results.

Theorem 1.2 (*See [8, Theorems 2 and 3]*). *If there exist functions $\sigma, \ \rho \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\sigma(t) \leq \min\{t, 2g(t)\}, \qquad \lim_{t\to\infty} \sigma(t) = \infty, \qquad \sigma'(t) > 0 \quad \text{for } t \geq t_0,$$

and

$$\int^{\infty} \left[\rho(t)q(t) - \frac{(n-1)!}{2^{3-2n}} \frac{(\rho'(t))^2}{\sigma^{n-2}(t)\sigma'(t)\rho(t)}\right] dt = \infty,$$

then Eq. (1.3) is oscillatory.

Theorem 1.3 (*See [21, Theorem 2]*). *Let $g(t) \leq t$. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} q(s)g^{n-1}(s) ds > \frac{(n-1)2^{(n-1)(n-2)}}{e},$$

then Eq. (1.3) is oscillatory.

Theorem 1.4 (*See [10, Corollary 1], [26, Corollary 1], and [27, Corollary 1]*). *Let $g(t) \leq t$. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} q(s)g^{n-1}(s) ds > \frac{(n-1)!}{e},$$

then Eq. (1.3) is oscillatory.

Theorem 1.5 (*See [26,27, Theorem 2]*). *Assume $g(t) \leq t$ and define $\delta(t) = \max_{t_0 \leq s \leq t} g(s)$, $\delta^{-1}(t) = \sup\{s \geq t_0 : \delta(s) = t\}$, $\delta^{-(k+1)}(t) = \delta^{-1}(\delta^{-k}(t)) = \sup\{s \geq \delta^{-k}(t_0) : \delta^{-k}(s) = t\}$. Set $Q(t) = q(t)g^{n-1}(t)/(n-1)!$. Further, define a sequence $\{Q_k(t)\}$ of functions as follows: $Q_1(t) = \int_{\delta(t)}^{t} Q(s) ds$, $t \geq \delta^{-1}(t_0)$, $Q_{k+1}(t) = \int_{\delta(t)}^{t} Q(s)Q_k(s) ds$, $t \geq \delta^{-(k+1)}(t_0)$, $k = 1, 2, \ldots$. Assume that there exists a positive integer K such that*

$$\liminf_{t\to\infty} Q_K(t) > \frac{1}{e^K}.$$

Then Eq. (1.3) is oscillatory.

Zhang et al. [23] studied the even-order equation

$$(a(t)(x^{(n-1)}(t))^{\alpha})' + q(t)x^{\alpha}(g(t)) = 0, \tag{1.4}$$

where α is a quotient of odd positive integers, $a \in C^1[t_0, \infty)$, $q, \ g \in C[t_0, \infty)$, $a(t) > 0$, $a'(t) \geq 0$, $q(t) \geq 0$, $g(t) < t$, and $\lim_{t\to\infty} g(t) = \infty$. Assuming that $\int_{t_0}^{\infty} a^{-1/\alpha}(t) dt < \infty$ and x is an eventually positive solution of Eq. (1.4), the authors considered three possible cases in the proof of [23, Theorem 2.1]. As a matter of fact, they dealt only with the following case (as in [3,8,10,21,26,27])

$$x > 0, \qquad x' > 0, \qquad x^{(n-1)} > 0, \quad \text{and} \quad x^{(n)} \leq 0 \tag{1.5}$$

under the assumption that $\int_{t_0}^{\infty} a^{-1/\alpha}(t) dt = \infty$. Therefore, an application of [23, Corollary 2.1] yields the following result (note that it is also valid in the case where $n = 2$).

Theorem 1.6. *Let $n \geq 2$ be even. If $\int_{t_0}^{\infty} a^{-1/\alpha}(t)\mathrm{d}t = \infty$ and*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{q(s)}{a(g(s))}(g^{n-1}(s))^{\alpha}\mathrm{d}s > \frac{((n-1)!)^{\alpha}}{\mathrm{e}},$$

then Eq. (1.4) is oscillatory.

Very recently, Liu et al. [14] and Zhang et al. [25] proved several oscillation criteria for Eq. (1.1), some of which we present below for the convenience of the reader.

Theorem 1.7 (*See [14, Theorem 1]*). *Assume* (H_1), (H_2), $a'(t) \geq 0, r(t) \geq 0$, *and let* $n \geq 2$ *be even,* $g \in \mathrm{C}^1[t_0, \infty), g'(t) > 0$ *for* $t \geq t_0$, *and*

$$\int_{t_0}^{\infty} \left[\frac{1}{a(s)} \exp\left(-\int_{t_0}^{s} \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)\right]^{\frac{1}{p-1}} \mathrm{d}s = \infty. \tag{1.6}$$

Suppose that there exists a continuous function

$$H : \mathbb{D} \equiv \{(t, s)|t \geq s \geq t_0\} \to \mathbb{R}$$

such that

$$H(t, t) = 0, \quad t \geq t_0; \qquad H(t, s) > 0, \quad t > s \geq t_0,$$

and H has a nonpositive continuous partial derivative with respect to the second variable in $\mathbb{D}_0 \equiv \{(t, s)|t > s \geq t_0\}$. *Assume further that there exist functions* $h \in \mathrm{C}(\mathbb{D}_0, \mathbb{R}), K, \rho \in \mathrm{C}^1([t_0, \infty), (0, \infty))$ *such that*

$$-\frac{\partial}{\partial s}(H(t, s)K(s)) - H(t, s)K(s)\left(\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right) = h(t, s), \quad \forall (t, s) \in \mathbb{D}_0. \tag{1.7}$$

If for some constant $\theta \in (0, 1)$ *and for all constants* $M > 0$,

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[\rho(s)q(s)H(t, s)K(s) - \left(\frac{|h(t, s)|}{p}\right)^{p} \frac{\rho(s)a(s)}{(H(t, s)G(s)K(s))^{p-1}}\right]\mathrm{d}s = \infty,$$

where $G(s) := \theta M g^{n-2}(s)g'(s)$, *then Eq. (1.1) is oscillatory.*

As a special case, when $H(t, s) = (t-s)^{\lambda}$, $\lambda > p-1$, and $K(t) = \rho(t) = 1$, Theorem 1.7 reduces to the following result.

Theorem 1.8 (*See [25, Theorem 1]*). *Assume* (H_1), (H_2), (1.6), $a'(t) \geq 0, r(t) \geq 0$, *and let* $n \geq 2$ *be even,* $g \in \mathrm{C}^1[t_0, \infty)$, $g'(t) > 0$ *for* $t \geq t_0$. *Suppose also that there exists a constant* $\lambda > p - 1$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} (t-s)^{\lambda-p}\left[(t-s)^{p}q(s) - \left(\frac{\lambda}{p}\right)^{p} G^{1-p}(s)\left(1 + \frac{(t-s)r(s)}{\lambda a(s)}\right)^{p} a(s)\right]\mathrm{d}s = \infty$$

for some constant $\theta \in (0, 1)$ *and for all constants* $M > 0$, *where G is as in Theorem 1.7. Then Eq. (1.1) is oscillatory.*

If $H(t, s) = 1$ and $K(t) = 1$, then we have the following criterion due to Theorem 1.7.

Theorem 1.9 (*See [25, Theorem 2]*). *Assume* (H_1), (H_2), (1.6), $a'(t) \geq 0, r(t) \geq 0$, *and let* $n \geq 2$ *be even,* $g \in \mathrm{C}^1[t_0, \infty), g'(t) > 0$ *for* $t \geq t_0$. *If there exists a function* $\rho \in \mathrm{C}^1([t_0, \infty), (0, \infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)\left[q(s) - \frac{a(s)}{p^{p}}(\theta M g^{n-2}(s)g'(s))^{1-p}\left|\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right|^{p}\right]\mathrm{d}s = \infty$$

for some constant $\theta \in (0, 1)$ *and for all constants* $M > 0$, *then Eq. (1.1) is oscillatory.*

Theorems 1.7–1.9 gave some interesting ideas for the study of oscillatory properties of trinomial differential equations. Two unsolved problems for research can be formulated as follows.

(P1) Is it possible to establish oscillation criteria for (1.1) without requiring $a' \geq 0, r \geq 0, g' > 0$, and the randomicity of M?

(P2) Suggest a different method to investigate (1.1) in the case where n is odd.

Our aim in this paper is to give an affirmative answer to these questions. This paper is organized as follows. In Section 2, we give some answers to problem (P1) by refining the standard integral averaging technique. In Section 3, we study oscillation and asymptotic behavior of (1.1) relating these properties of this equation to the existence of positive solutions to associated first-order delay differential inequalities. The results obtained allow applications to (1.1) with even-order and odd-order. In Section 4, two selected examples show that the results in Sections 2 and 3 are of independent interest. In Section 5, we extend results obtained in Section 3 to a more general differential equation. In Section 6, we present some conclusions to summarize the contents of this paper.

In the sequel, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Oscillation results via the integral averaging technique

Before stating the main results, we begin with the following lemma.

Lemma 2.1 *(See [15]). Let $f \in \mathrm{C}^n([t_0,\infty),\mathbb{R}^+)$. If $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist a $t_x \ge t_0$ and an integer l, $0 \le l \le n$ with $n+l$ even for $f^{(n)}(t) \ge 0$, or $n+l$ odd for $f^{(n)}(t) \le 0$ such that*

$$l > 0 \text{ yields } f^{(k)}(t) > 0 \quad \text{for } t \ge t_x,\ k = 0, 1, \ldots, l-1, \quad \text{and}$$
$$l \le n-1 \text{ yields } (-1)^{l+k} f^{(k)}(t) > 0 \quad \text{for } t \ge t_x,\ k = l, l+1, \ldots, n-1.$$

Lemma 2.2 *(See [2, Lemma 2.2.3]). Assume that f is as in Lemma 2.1, $f^{(n)}(t)f^{(n-1)}(t) \le 0$ for $t \ge t_x$, and $\lim_{t\to\infty} f(t) \ne 0$. Then for every constant $\lambda \in (0,1)$, there exists $t_\lambda \in [t_x,\infty)$ such that*

$$f(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}(t)|$$

holds on $[t_\lambda,\infty)$.

Lemma 2.3. *Assume* (H_1), (H_2), *$n \ge 4$ is even, and let x be an eventually positive solution of (1.1). If (1.6) holds, then there exist two possible cases for $t \ge t_1$ large enough:*

(1) $x(t) > 0, x'(t) > 0, x''(t) > 0, x^{(n-1)}(t) > 0, x^{(n)}(t) < 0$;
(2) $x(t) > 0, x^{(j)}(t) > 0, x^{(j+1)}(t) < 0$ *for every odd integer* $j \in \{1, 2, \ldots, n-3\}$, $x^{(n-1)}(t) > 0, x^{(n)}(t) < 0$.

Proof. Since x is an eventually positive solution of (1.1), there exists a $t_1 \ge t_0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for $t \ge t_1$. By virtue of (1.1), we have

$$(a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t))' + r(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t) < 0,$$

which yields

$$\left(a(t)\exp\left(\int_{t_0}^{t}\frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t)\right)' < 0. \tag{2.1}$$

From the proof of [14, Lemma 4], we have $x^{(n-1)} > 0$ eventually. Then we can write (2.1) in the form

$$\left(a(t)\exp\left(\int_{t_0}^{t}\frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)(x^{(n-1)}(t))^{p-1}\right)' < 0,$$

which implies that

$$\exp\left(\int_{t_0}^{t}\frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)\left[a'(t)+r(t)\right](x^{(n-1)}(t))^{p-1} + (p-1)a(t)\exp\left(\int_{t_0}^{t}\frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)(x^{(n-1)}(t))^{p-2}x^{(n)}(t) < 0.$$

Thus, $x^{(n)} < 0$ eventually. Then by Lemma 2.1, we have two possible cases (1) and (2). This completes the proof. □

Theorem 2.4. *Assume* (H_1), (H_2), (1.6), *$n \ge 4$ is even, $r(t) \ge 0$, and let $\mathbb{D}, \mathbb{D}_0, H$ be as in Theorem 1.7. Assume further that there exist functions $h \in \mathrm{C}(\mathbb{D}_0,\mathbb{R})$, $K, \rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ such that (1.7) holds and for some constant $\lambda_0 \in (0,1)$,*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]\mathrm{d}s = \infty. \tag{2.2}$$

Suppose also that there exists a continuous function $H_* : \mathbb{D} \to \mathbb{R}$ such that

$$H_*(t,t) = 0, \quad t \geq t_0; \qquad H_*(t,s) > 0, \quad t > s \geq t_0, \tag{2.3}$$

and H_* has a nonpositive continuous partial derivative with respect to the second variable in $\mathbb{D}_0$. If there exist functions $h_* \in \mathrm{C}(\mathbb{D}_0, \mathbb{R})$, $K_*, \delta \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ such that

$$-\frac{\partial}{\partial s}(H_*(t,s)K_*(s)) - H_*(t,s)K_*(s)\frac{\delta'(s)}{\delta(s)} = h_*(t,s), \quad \forall (t,s) \in \mathbb{D}_0 \tag{2.4}$$

and

$$\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)} \int_{t_0}^{t} \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)} \right] \mathrm{d}s = \infty, \tag{2.5}$$

where

$$Q(t) := \frac{\int_t^\infty (\eta - t)^{n-4} \left[\frac{\int_\eta^\infty q(s)\left(\frac{g(s)}{s}\right)^{p-1} \mathrm{d}s}{a(\eta)} \right]^{1/(p-1)} \mathrm{d}\eta}{(n-4)!},$$

then Eq. (1.1) is oscillatory.

Proof. Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. From Lemma 2.3, we have two possible cases (1) and (2). We consider each of two cases separately.

Assume that (1) holds. We see that $\lim_{t\to\infty} x'(t) \neq 0$. By virtue of Lemma 2.2, for every constant $\lambda \in (0,1)$ and for all large t, we have

$$x'(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} x^{(n-1)}(t), \quad \text{by setting } f(t) := x'(t). \tag{2.6}$$

Now we introduce a Riccati substitution

$$u(t) := \rho(t) \frac{a(t)(x^{(n-1)}(t))^{p-1}}{(x(t))^{p-1}}, \quad t \geq t_1. \tag{2.7}$$

Then $u(t) > 0$ on $[t_1,\infty)$, and we have by (2.6) that

$$\begin{aligned} u'(t) &= \rho'(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}}{(x(t))^{p-1}} + \rho(t)\frac{(a(t)(x^{(n-1)}(t))^{p-1})'}{(x(t))^{p-1}} - \rho(t)\frac{(p-1)a(t)(x^{(n-1)}(t))^{p-1}x'(t)}{(x(t))^p} \\ &\leq -\rho(t)\frac{q(t)x^{p-1}(g(t))}{(x(t))^{p-1}} - \rho(t)\frac{r(t)(x^{(n-1)}(t))^{p-1}}{(x(t))^{p-1}} + \rho'(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}}{(x(t))^{p-1}} \\ &\quad - \frac{\lambda(p-1)}{(n-2)!} t^{n-2} \rho(t) \frac{a(t)(x^{(n-1)}(t))^p}{(x(t))^p}. \end{aligned} \tag{2.8}$$

By the Kiguradze Lemma [11], which shows that if a function y satisfies $y^{(i)} > 0$, $i = 0,1,2,\ldots,k$ and $y^{(k+1)} \leq 0$, then $y(t)/y'(t) \geq t/k$, we have

$$\frac{x(t)}{x'(t)} \geq \frac{t}{n-1}.$$

Thus, we obtain that x/t^{n-1} is nonincreasing, and so

$$\frac{x(g(t))}{x(t)} \geq \frac{g^{n-1}(t)}{t^{n-1}}. \tag{2.9}$$

It follows from (2.7)–(2.9) that

$$\rho(t)q(t)\left(\frac{g^{n-1}(t)}{t^{n-1}}\right)^{p-1} \leq -u'(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right)u(t) - \frac{\lambda(p-1)t^{n-2}}{(n-2)!(\rho(t)a(t))^{1/(p-1)}} u^{p/(p-1)}(t).$$

Replacing t by s, multiplying two sides by $H(t,s)K(s)$, and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}\mathrm{d}s \\
&\quad\le -\int_{t_1}^{t} H(t,s)K(s)u'(s)\mathrm{d}s + \int_{t_1}^{t} H(t,s)K(s)\left(\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right)u(s)\mathrm{d}s \\
&\qquad - \int_{t_1}^{t} H(t,s)K(s)\frac{\lambda(p-1)s^{n-2}}{(n-2)!(\rho(s)a(s))^{1/(p-1)}}u^{p/(p-1)}(s)\mathrm{d}s \\
&\quad= H(t,t_1)K(t_1)u(t_1) - \int_{t_1}^{t}\left[-\frac{\partial}{\partial s}(H(t,s)K(s)) - H(t,s)K(s)\left(\frac{\rho'(s)}{\rho(s)} - \frac{r(s)}{a(s)}\right)\right]u(s)\mathrm{d}s \\
&\qquad - \int_{t_1}^{t} H(t,s)K(s)\frac{\lambda(p-1)s^{n-2}}{(n-2)!(\rho(s)a(s))^{1/(p-1)}}u^{p/(p-1)}(s)\mathrm{d}s \\
&\quad\le H(t,t_1)K(t_1)u(t_1) + \int_{t_1}^{t}|h(t,s)|u(s)\mathrm{d}s - \int_{t_1}^{t} H(t,s)K(s)\frac{\lambda(p-1)s^{n-2}}{(n-2)!(\rho(s)a(s))^{1/(p-1)}}u^{p/(p-1)}(s)\mathrm{d}s.
\end{aligned}\tag{2.10}$$

Set $\gamma := p/(p-1)$,

$$X := \left[(p-1)H(t,s)K(s)\frac{\lambda s^{n-2}}{(n-2)!}\right]^{\frac{p-1}{p}}\frac{u(s)}{(\rho(s)a(s))^{1/p}},$$

and

$$Y := \left(\frac{p-1}{p}\right)^{p-1}|h(t,s)|^{p-1}\left(\frac{\rho(s)a(s)}{\left[(p-1)H(t,s)K(s)\frac{\lambda s^{n-2}}{(n-2)!}\right]^{p-1}}\right)^{\frac{p-1}{p}}.$$

Using the inequality

$$\gamma XY^{\gamma-1} - X^{\gamma} \le (\gamma-1)Y^{\gamma}, \quad \gamma > 1,\ X \ge 0,\ Y \ge 0,$$

we have

$$|h(t,s)|u(s) - H(t,s)K(s)\frac{\lambda(p-1)s^{n-2}}{(n-2)!(\rho(s)a(s))^{1/(p-1)}}u^{p/(p-1)}(s) \le \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}.$$

Putting the resulting inequality into (2.10), we obtain

$$\begin{aligned}
&\int_{t_1}^{t}\left[H(t,s)K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]\mathrm{d}s \\
&\quad\le H(t,t_1)K(t_1)u(t_1) \le H(t,t_0)K(t_1)u(t_1).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]\mathrm{d}s \\
&\quad\le K(t_1)u(t_1) + \int_{t_0}^{t_1} K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1}\mathrm{d}s < \infty,
\end{aligned}$$

which contradicts (2.2).

Assume that (2) holds. By virtue of $x' > 0$ and $x'' < 0$, we get $x(t) \geq tx'(t)$ due to the Kiguradze Lemma [11], and so we have that x/t is nonincreasing. Hence by (1.1), we obtain

$$-a(t)(x^{(n-1)})^{p-1}(t) + \int_t^{\infty} q(s)x^{p-1}(s)\left(\frac{g(s)}{s}\right)^{p-1} \mathrm{d}s \leq 0.$$

It follows from $x' > 0$ that

$$-x^{(n-1)}(t) + \frac{x(t)}{a^{1/(p-1)}(t)}\left[\int_t^{\infty} q(s)\left(\frac{g(s)}{s}\right)^{p-1} \mathrm{d}s\right]^{1/(p-1)} \leq 0.$$

Integrating the above inequality from t to ∞ for a total of $(n-3)$ times, we have

$$x''(t) + \frac{\int_t^{\infty}(\eta-t)^{n-4}\left[\frac{\int_\eta^\infty q(s)\left(\frac{g(s)}{s}\right)^{p-1}\mathrm{d}s}{a(\eta)}\right]^{1/(p-1)}\mathrm{d}\eta}{(n-4)!}x(t) \leq 0. \tag{2.11}$$

Now, we define a Riccati substitution

$$w(t) := \delta(t)\frac{x'(t)}{x(t)}, \quad t \geq t_1. \tag{2.12}$$

Then $w(t) > 0$ for $t \geq t_1$, and

$$w'(t) = \delta'(t)\frac{x'(t)}{x(t)} + \delta(t)\frac{x''(t)x(t) - (x'(t))^2}{x^2(t)}.$$

It follows from (2.11) and (2.12) that

$$\delta(t)Q(t) \leq -w'(t) + \frac{\delta'(t)}{\delta(t)}w(t) - \frac{1}{\delta(t)}w^2(t).$$

Replacing t by s, multiplying two sides by $H_*(t,s)K_*(s)$, and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned}
\int_{t_1}^t H_*(t,s)K_*(s)\delta(s)Q(s)\mathrm{d}s &\leq -\int_{t_1}^t H_*(t,s)K_*(s)w'(s)\mathrm{d}s \\
&\quad + \int_{t_1}^t H_*(t,s)K_*(s)\frac{\delta'(s)}{\delta(s)}w(s)\mathrm{d}s - \int_{t_1}^t \frac{H_*(t,s)K_*(s)}{\delta(s)}w^2(s)\mathrm{d}s \\
&= H_*(t,t_1)K_*(t_1)w(t_1) - \int_{t_1}^t \frac{H_*(t,s)K_*(s)}{\delta(s)}w^2(s)\mathrm{d}s \\
&\quad - \int_{t_1}^t\left[-\frac{\partial}{\partial s}(H_*(t,s)K_*(s)) - H_*(t,s)K_*(s)\frac{\delta'(s)}{\delta(s)}\right]w(s)\mathrm{d}s \\
&\leq H_*(t,t_1)K_*(t_1)w(t_1) + \int_{t_1}^t |h_*(t,s)|w(s)\mathrm{d}s - \int_{t_1}^t \frac{H_*(t,s)K_*(s)}{\delta(s)}w^2(s)\mathrm{d}s.
\end{aligned}$$

Hence we have

$$\int_{t_1}^t \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)}\right]\mathrm{d}s \leq H_*(t,t_1)K_*(t_1)w(t_1) \leq H_*(t,t_0)K_*(t_1)w(t_1).$$

Then

$$\frac{1}{H_*(t,t_0)}\int_{t_0}^t \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)}\right]\mathrm{d}s \leq K_*(t_1)w(t_1) + \int_{t_0}^{t_1} K_*(s)\delta(s)Q(s)\mathrm{d}s < \infty,$$

which contradicts (2.5). Therefore, every solution of (1.1) is oscillatory. □

Example 2.5. For $t \geq 1$, consider a fourth-order delay damped differential equation

$$x^{(4)}(t) + \frac{1}{t^2}x^{(3)}(t) + \frac{q_0}{t^4}x\left(\frac{t}{\sqrt[3]{4}}\right) = 0, \tag{2.13}$$

where $q_0 > 0$ is a constant. Let $n = 4$, $p = 2$, $a(t) = 1$, $r(t) = 1/t^2$, $q(t) = q_0/t^4$, and $g(t) = t/\sqrt[3]{4}$. It is easy to see that condition (1.6) is satisfied. Set $H(t,s) = (t-s)^2$, $K(t) = 1$, and $\rho(t) = t^3$. Then $h(t,s) = (t-s)[5 - s^{-1} + t(s^{-2} - 3s^{-1})]$, and so

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)K(s)\rho(s)q(s) \left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right]^{p-1}} \left(\frac{|h(t,s)|}{p}\right)^{p} \right] \mathrm{d}s$$
$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[\frac{q_0}{4} t^2 s^{-1} + \frac{q_0}{4} s - \frac{q_0}{2} t - \frac{s}{2\lambda_0} (25 + s^{-2} - 10s^{-1} \right.$$
$$\left. + t^2 s^{-4} + 9t^2 s^{-2} - 6t^2 s^{-3} + 16ts^{-2} - 2ts^{-3} - 30ts^{-1}) \right] \mathrm{d}s = \infty,$$

if $q_0 > 18/\lambda_0$ for some constant $\lambda_0 \in (0,1)$. In particular, we can take $q_0 \geq 19$ (by letting $\lambda_0 \in (18/19, 1)$). On the other hand, we see that $Q(t) \geq q_0/(12t^2)$. Let $\delta(t) = t$, $K_*(t) = 1$, and $H_*(t,s) = (t-s)^2$. Then $h_*(t,s) = (t-s)(3 - ts^{-1})$, and hence

$$\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)} \int_{t_0}^{t} \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)} \right] \mathrm{d}s$$
$$\geq \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[\frac{q_0}{12} t^2 s^{-1} + \frac{q_0}{12} s - \frac{q_0}{6} t - \frac{s}{4} (9 - 6ts^{-1} + t^2 s^{-2}) \right] \mathrm{d}s = \infty,$$

if $q_0 > 3$. In conclusion, Eq. (2.13) is oscillatory if $q_0 \geq 19$ when using Theorem 2.4. However, Theorems 1.7–1.9 cannot be used to get this conclusion due to the arbitrariness of M.

As a special case, when $r(t) = 0$, Eq. (1.1) reduces to

$$(a(t)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t))' + q(t)|x(g(t))|^{p-2}x(g(t)) = 0. \tag{2.14}$$

Using Theorem 2.4 in Eq. (2.14), we have the following result.

Corollary 2.6. *Let all assumptions of Theorem 2.4 be satisfied except that condition $r(t) \geq 0$ be replaced with $r(t) = 0$. Then Eq. (2.14) is oscillatory.*

Consider the binomial differential equation

$$\left(a(t)\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)|x^{(n-1)}(t)|^{p-2}x^{(n-1)}(t)\right)' + q(t)\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)|x(g(t))|^{p-2}x(g(t)) = 0. \tag{2.15}$$

Application of Corollary 2.6 yields the following criterion for Eq. (2.15).

Corollary 2.7. *Assume (H_1), (H_2), (1.6), $n \geq 4$ is even, and let $\mathbb{D}$, $\mathbb{D}_0$, H be as in Theorem 1.7. Assume further that there exist functions $h^* \in C(\mathbb{D}_0, \mathbb{R})$, $K, \rho \in C^1([t_0,\infty), (0,\infty))$ such that*

$$-\frac{\partial}{\partial s}(H(t,s)K(s)) - H(t,s)K(s)\frac{\rho'(s)}{\rho(s)} = h^*(t,s), \quad \forall (t,s) \in \mathbb{D}_0, \tag{2.16}$$

and, for some constant $\lambda_0 \in (0,1)$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \exp\left(\int_{t_0}^{s} \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)$$
$$\times \left[H(t,s)K(s)\rho(s)q(s) \left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right]^{p-1}} \left(\frac{|h^*(t,s)|}{p}\right)^{p} \right] \mathrm{d}s = \infty. \tag{2.17}$$

Suppose also that there exists a continuous function $H_* : \mathbb{D} \to \mathbb{R}$ such that (2.3) holds and H_* has a nonpositive continuous partial derivative with respect to the second variable in $\mathbb{D}_0$. If there exist functions $h_* \in C(\mathbb{D}_0, \mathbb{R})$, $K_*, \delta \in C^1([t_0,\infty), (0,\infty))$ such that (2.4) holds and

$$\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)} \int_{t_0}^{t} \left[H_*(t,s)K_*(s)\delta(s)\bar{Q}(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)} \right] \mathrm{d}s = \infty, \tag{2.18}$$

where

$$\bar{Q}(t) := \frac{\int_t^\infty (\eta - t)^{n-4}\left[\frac{\int_\eta^\infty q(s)\exp\left(\int_{t_0}^s \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)\left(\frac{g(s)}{s}\right)^{p-1}\mathrm{d}s}{a(\eta)\exp\left(\int_{t_0}^\eta \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)}\right]^{1/(p-1)}\mathrm{d}\eta}{(n-4)!},$$

then Eq. (2.15) is oscillatory.

Since Eqs. (1.1) and (2.15) are equivalent, we have the following result.

Corollary 2.8. *Let all assumptions of Corollary 2.7 be satisfied. Then Eq. (1.1) is oscillatory.*

Example 2.9. For $t \ge 1$, consider a fourth-order delay differential equation

$$x^{(4)}(t) + \frac{q_0}{t^4}x\left(\frac{9t}{10}\right) = 0, \tag{2.19}$$

where $q_0 > 0$ is a constant. Let $n = 4$, $p = 2$, $a(t) = 1$, $r(t) = 0$, $q(t) = q_0/t^4$, and $g(t) = 9t/10$. It is easy to see that condition (1.6) is satisfied. Set $H(t,s) = (t-s)^2$, $K(t) = 1$, and $\rho(t) = t^3$. Then $h(t,s) = h^*(t,s) = (t-s)(5-3ts^{-1})$, and so

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \left[H(t,s)K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^p\right]\mathrm{d}s \\
&\quad = \limsup_{t\to\infty} \frac{1}{(t-1)^2}\int_1^t \left[\frac{729q_0}{1000}t^2s^{-1} + \frac{729q_0}{1000}s - \frac{729q_0}{500}t - \frac{s}{2\lambda_0}(25 + 9t^2s^{-2} - 30ts^{-1})\right]\mathrm{d}s = \infty,
\end{aligned}$$

if $q_0 > 500/(81\lambda_0)$ for some constant $\lambda_0 \in (0,1)$. In particular, we can take $q_0 \ge 41000/9^4$ (by letting $\lambda_0 \in (81/82, 1)$). On the other hand, we have that $Q(t) = 3q_0/(20t^2)$. Let $\delta(t) = t$, $K_*(t) = 1$, and $H_*(t,s) = (t-s)^2$. Then $h_*(t,s) = (t-s)(3-ts^{-1})$, and hence

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)}\int_{t_0}^t \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)}\right]\mathrm{d}s \\
&\quad = \limsup_{t\to\infty} \frac{1}{(t-1)^2}\int_1^t \left[\frac{3q_0}{20}t^2s^{-1} + \frac{3q_0}{20}s - \frac{3q_0}{10}t - \frac{s}{4}(9 - 6ts^{-1} + t^2s^{-2})\right]\mathrm{d}s = \infty,
\end{aligned}$$

if $q_0 > 5/3$. In conclusion, Eq. (2.19) is oscillatory if $q_0 \ge 6.25$ when using Theorem 2.4 or Corollary 2.8.

Remark 2.10. Applying Theorem 2.4 or Corollary 2.8, the oscillation result ($q_0 \ge 6.25$) obtained for equation (2.19) is new; see the following details.

(a) Theorem 1.1 cannot be applied to Eq. (2.19) due to the arbitrariness in the choice of θ;
(b) Let $\sigma(t) = t$ and $\rho(t) = t^3$. Using Theorem 1.2, we see that Eq. (2.19) is oscillatory if $q_0 > 1728$;
(c) Using Theorem 1.3, we find that Eq. (2.19) is oscillatory if $q_0 > 64000/\left(243\mathrm{e}\ln\frac{10}{9}\right) > 919.6$;
(d) An application of Theorem 1.4 implies that Eq. (2.19) is oscillatory when $q_0 > 2000/\left(243\mathrm{e}\ln\frac{10}{9}\right) > 28.73$;
(e) Using Theorem 1.5, $\delta(t) = 9t/10$, $Q_K(t) = \left(243\ln\frac{10}{9}q_0/2000\right)^K$, and so Eq. (2.19) is oscillatory if $q_0 > 2000/\left(243\mathrm{e}\ln\frac{10}{9}\right) > 28.73$;
(f) Let $\alpha = 1$ and $a(t) = 1$. Using Theorem 1.6, we obtain that Eq. (2.19) is oscillatory if $q_0 > 2000/\left(243\mathrm{e}\ln\frac{10}{9}\right) > 28.73$;
(g) Theorems 1.7–1.9 cannot be applied to Eq. (2.19) due to the arbitrary choice of M.

Remark 2.11. In this section, assuming $a'(t) + r(t) \ge 0$ and using a modified technique, we establish two new oscillation criteria (Theorem 2.4 and Corollary 2.8) that provide answers to problem (P1) formulated in Section 1. We stress that, contrary to Theorem 2.4, we do not need in Corollary 2.8 the restrictive condition $r(t) \ge 0$ (in the second part of the proof of Theorem 2.4) which, in some sense, is a significant improvement. We also improve some of results reported there, cf. Examples 2.5 and 2.9. In particular, Example 2.9 shows that Theorem 2.4 and Corollary 2.8 are new in the case where $r(t) = 0$.

Remark 2.12. Let $\rho(t)$ be replaced with $\rho(t)\exp\left(-\int_{t_0}^t \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)$. Then assumptions (2.16) and (2.17) reduce to conditions (1.7) and (2.2), respectively. It is not difficult to find that, contrary to assumption (2.18), condition (2.5) is easier to be verified in the case $r(t) \ge 0$.

Remark 2.13. Under the assumption that $n \geq 4$ is even, we stress that the study of oscillatory behavior of Eq. (1.1) is more difficult in comparison with second-order differential equations. Since the sign of the derivative x'' is not known, our criteria for oscillation of (1.1) involve double assumptions, as for instance, (2.2) and (2.5). On the other hand, we point out that (2.6) cannot be given by using (1.5) and Lemma 2.2, since we need condition $\lim_{t\to\infty} x'(t) \neq 0$.

3. Asymptotic results via the comparison technique

Theorem 3.1. *Assume* (H_1), (H_2), (1.6), *and let* $n \geq 2$ *be even. If there exists a constant* $\lambda_0 \in (0, 1)$ *such that the first-order differential inequality*

$$y(t)\left[y'(t) + \frac{r(t)}{a(t)}y(t) + \frac{q(t)}{a(g(t))}\left(\frac{\lambda_0 g^{n-1}(t)}{(n-1)!}\right)^{p-1} y(g(t))\right] \leq 0 \tag{3.1}$$

has no positive solutions, then Eq. (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, x can be assumed to be eventually positive. Similar as in the proof of Lemma 2.3, we have

$$x(t) > 0, \qquad x'(t) > 0, \qquad x^{(n-1)}(t) > 0, \quad \text{and} \quad x^{(n)}(t) < 0 \tag{3.2}$$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large. It follows from Lemma 2.2 that

$$x(t) \geq \frac{\lambda}{(n-1)!}\frac{t^{n-1}}{a^{1/(p-1)}(t)}a^{1/(p-1)}(t)x^{(n-1)}(t) \tag{3.3}$$

for every $\lambda \in (0, 1)$ and for all sufficiently large t. Set $y := a(x^{(n-1)})^{p-1}$. Using (3.3) in (1.1), we obtain inequality

$$y'(t) + \frac{r(t)}{a(t)}y(t) + \frac{q(t)}{a(g(t))}\left(\frac{\lambda g^{n-1}(t)}{(n-1)!}\right)^{p-1} y(g(t)) \leq 0. \tag{3.4}$$

That is, y is a positive solution of inequality (3.1), which is a contradiction. The proof is complete. □

On the basis of Theorem 3.1 and [20, Corollary 1], we obtain the following result.

Corollary 3.2. *Assume* (H_1), (H_2), (1.6), *and let* $n \geq 2$ *be even. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{q(s)}{a(g(s))}(g^{n-1}(s))^{p-1} \exp\left(\int_{g(s)}^{s} \frac{r(v)}{a(v)}\mathrm{d}v\right)\mathrm{d}s > \frac{((n-1)!)^{p-1}}{\mathrm{e}}, \tag{3.5}$$

then Eq. (1.1) *is oscillatory.*

Example 3.3. For $t \geq 1$, consider a fourth-order delay damped differential equation

$$x^{(4)}(t) + \frac{1}{t}x^{(3)}(t) + \frac{q_0}{t^4}x\left(\frac{t}{3}\right) = 0, \tag{3.6}$$

where $q_0 > 0$ is a constant. It is not difficult to check that (1.6) holds. Furthermore,

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{q(s)}{a(g(s))}g^{n-1}(s) \exp\left(\int_{g(s)}^{s} \frac{r(v)}{a(v)}\mathrm{d}v\right)\mathrm{d}s = \frac{q_0 \ln 3}{9} > \frac{6}{\mathrm{e}},$$

if $q_0 > 54/(\mathrm{e}\ln 3) \approx 18.08$. Hence by Corollary 3.2, Eq. (3.6) is oscillatory when $q_0 > 18.1$.

Theorem 3.4. *Assume* (H_1), (H_2), (1.6), *and let* $n \geq 3$ *be odd. If there exists a constant* $\lambda_0 \in (0, 1)$ *such that the first-order differential inequality* (3.1) *has no positive solutions, then every solution of Eq.* (1.1) *is oscillatory or converges to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1), which does not tend to zero asymptotically. Without loss of generality, we can assume that x is eventually positive. On one hand, similar as in the proof of [14, Lemma 4] and Lemma 2.3, we obtain

$$x(t) > 0, \qquad x^{(n-1)}(t) > 0, \quad \text{and} \quad x^{(n)}(t) < 0$$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large. On the other hand, using Lemma 2.2, we have that (3.3) holds for every $\lambda \in (0, 1)$ and for all sufficiently large t. Set $y := a(x^{(n-1)})^{p-1}$. By virtue of (1.1) and (3.3), we get (3.4). That is, y is a positive solution of inequality (3.1), which is a contradiction. This completes the proof. □

Combining Theorem 3.4 and [20, Corollary 1], we have the following result.

Corollary 3.5. *Assume* (H_1), (H_2), (1.6), *and let* $n \geq 3$ *be odd. If* (3.5) *holds, then every solution of Eq.* (1.1) *is oscillatory or converges to zero as* $t \to \infty$.

Example 3.6. For $t \geq 1$, consider a third-order differential equation

$$x'''(t) + t^{-1}x''(t) + \frac{t-1}{\mathrm{e}t}x(t-1) = 0. \tag{3.7}$$

It is not difficult to verify that (1.6) is satisfied. Furthermore,

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{q(s)}{a(g(s))} g^{n-1}(s) \exp\left(\int_{g(s)}^{s} \frac{r(v)}{a(v)} \mathrm{d}v\right) \mathrm{d}s = \frac{1}{\mathrm{e}} \liminf_{t\to\infty} \int_{t-1}^{t} (s-1)^2 \mathrm{d}s = \infty.$$

Hence by Corollary 3.5, every solution of (3.7) is oscillatory or tends to zero as $t \to \infty$. As a matter of fact, one such solution is $x(t) = \mathrm{e}^{-t}$.

Remark 3.7. Assuming x is an eventually positive solution of (2.15) and using Lemma 2.2 and [13, Theorem 2.1.1] in Eq. (2.15), we also have Corollaries 3.2 and 3.5. The details are left to the reader.

Remark 3.8. In this section, assuming $a'(t) + r(t) \geq 0$ and using a comparison method, we establish several new criteria that include answers to problem (P2) formulated in Section 1. On one hand, we point out that, contrary to Theorem 2.4, we do not need in Theorem 3.1 and Corollary 3.2 the restrictive condition $r(t) \geq 0$ which, in certain sense, is a significant improvement. On the other hand, the results obtained in this section cannot be applied to (1.1) in the case where $g(t) = t$. However, Theorem 2.4 and Corollary 2.8 are valid in this case. As yet, we cannot extend the method used in Section 2 to (1.1) in the case where n is odd.

4. Comments on results in Sections 2 and 3

The following two examples illustrate applications of theoretical results in the previous sections.

Example 4.1. For $t \geq 1$, consider a fourth-order delay damped differential equation

$$x^{(4)}(t) + \frac{1}{t}x^{(3)}(t) + \frac{q_0}{t^4}x\left(\frac{99t}{100}\right) = 0, \tag{4.1}$$

where $q_0 > 0$ is a constant. It is not difficult to check that (1.6) holds. Furthermore,

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{q(s)}{a(g(s))} g^{n-1}(s) \exp\left(\int_{g(s)}^{s} \frac{r(v)}{a(v)} \mathrm{d}v\right) \mathrm{d}s = q_0 \left(\frac{99}{100}\right)^2 \ln\frac{100}{99} > \frac{6}{\mathrm{e}},$$

if $q_0 > 6\,(100/99)^2 \big/ \left(\mathrm{e}\ln\frac{100}{99}\right) \approx 224.08$. Hence by Corollary 3.2, Eq. (4.1) is oscillatory when $q_0 > 224.1$.

Let $n = 4, p = 2, a(t) = 1, r(t) = 1/t, q(t) = q_0/t^4$, and $g(t) = 99t/100$. It is easy to see that condition (1.6) is satisfied. Set $H(t,s) = (t-s)^2$, $K(t) = 1$, and $\rho(t) = t^3$. Then $h(t,s) = (t-s)(4-2ts^{-1})$, and hence

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)K(s)\rho(s)q(s)\left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right]^{p-1}}\left(\frac{|h(t,s)|}{p}\right)^{p}\right]\mathrm{d}s \\
&= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[\left(\frac{99}{100}\right)^3 q_0 t^2 s^{-1} + \left(\frac{99}{100}\right)^3 q_0 s - 2\left(\frac{99}{100}\right)^3 q_0 t - \frac{s}{\lambda_0}(8 + 2t^2 s^{-2} - 8ts^{-1})\right]\mathrm{d}s = \infty,
\end{aligned}$$

if $q_0 > 2(100/99)^3/\lambda_0$ for some constant $\lambda_0 \in (0,1)$. In particular, we can take $q_0 \geq 2(100/99)^4$ (by letting $\lambda_0 \in (99/100, 1)$). On the other hand, we have that $Q(t) = 33q_0/(200t^2)$. Let $\delta(t) = t$, $K_*(t) = 1$, and $H_*(t,s) = (t-s)^2$. Then $h_*(t,s) = (t-s)(3-ts^{-1})$, and so

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)} \int_{t_0}^{t} \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)}\right]\mathrm{d}s \\
&= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[\frac{33q_0}{200}t^2 s^{-1} + \frac{33q_0}{200}s - \frac{33q_0}{100}t - \frac{s}{4}(9 - 6ts^{-1} + t^2 s^{-2})\right]\mathrm{d}s = \infty,
\end{aligned}$$

if $q_0 > 50/33$. In conclusion, Eq. (4.1) is oscillatory if $q_0 \geq 2.09$ when using Theorem 2.4. Therefore, Theorem 2.4 improves Corollary 3.2 for Eq. (4.1).

Example 4.2. For $t \geq 1$, consider Eq. (3.6). As in Example 3.3, Eq. (3.6) is oscillatory if $q_0 > 18.1$. Let now $n = 4, p = 2, a(t) = 1, r(t) = 1/t, q(t) = q_0/t^4$, and $g(t) = t/3$. It is easy to see that condition (1.6) holds. Set $H(t,s) = (t-s)^2, K(t) = 1$, and $\rho(t) = t^3$. Then $h(t,s) = (t-s)(4-2ts^{-1})$, and hence

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)K(s)\rho(s)q(s) \left(\frac{g^{n-1}(s)}{s^{n-1}}\right)^{p-1} - \frac{\rho(s)a(s)}{\left[H(t,s)K(s)\frac{\lambda_0 s^{n-2}}{(n-2)!}\right]^{p-1}} \left(\frac{|h(t,s)|}{p}\right)^{p} \right] \mathrm{d}s$$
$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[\frac{q_0}{27} t^2 s^{-1} + \frac{q_0}{27} s - \frac{2q_0}{27} t - \frac{s}{\lambda_0}(8 + 2t^2 s^{-2} - 8ts^{-1}) \right] \mathrm{d}s = \infty,$$

if $q_0 > 54/\lambda_0$ for some constant $\lambda_0 \in (0,1)$. In particular, we can take $q_0 \geq 55$ (by letting $\lambda_0 \in (54/55, 1)$). On the other hand, we have that $Q(t) = q_0/(18t^2)$. Let $\delta(t) = t, K_*(t) = 1$, and $H_*(t,s) = (t-s)^2$. Then $h_*(t,s) = (t-s)(3-ts^{-1})$, and so

$$\limsup_{t\to\infty} \frac{1}{H_*(t,t_0)} \int_{t_0}^{t} \left[H_*(t,s)K_*(s)\delta(s)Q(s) - \frac{\delta(s)|h_*(t,s)|^2}{4H_*(t,s)K_*(s)} \right] \mathrm{d}s$$
$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} \left[\frac{q_0}{18} t^2 s^{-1} + \frac{q_0}{18} s - \frac{q_0}{9} t - \frac{s}{4}(9 - 6ts^{-1} + t^2 s^{-2}) \right] \mathrm{d}s = \infty,$$

if $q_0 > 9/2$. In conclusion, Eq. (3.6) is oscillatory if $q_0 \geq 55$ when using Theorem 2.4. Therefore, Corollary 3.2 improves Theorem 2.4 for Eq. (3.6).

In conclusion, the results obtained in Sections 2 and 3 are of independent interest. Furthermore, it follows from Examples 2.5, 2.9, 4.1, and 4.2 that condition (2.5) plays an auxiliary role in certain sense. That is, condition (2.2) may have more important role in Theorem 2.4.

5. Extension of the results in Section 3

In what follows, consider a higher-order differential equation

$$(a(t)(x^{(n-1)}(t))^{\gamma})' + r(t)(x^{(n-1)}(t))^{\gamma} + q(t)\prod_{i=1}^{m} x^{\gamma_i}(g_i(t)) = 0, \tag{5.1}$$

where we assume that (H_2) is satisfied and

(H_3) $i = 1, \dots, m$, γ and γ_i are the ratios of odd natural numbers, $\sum_{i=1}^{m} \gamma_i = \gamma$, $g_i \in \mathrm{C}[t_0, \infty)$, $g_i(t) \leq t$, $\lim_{t\to\infty} g_i(t) = \infty$.

Eq. (5.1) may be also viewed as a special case of higher-order damped differential equations with a p-Laplacian. Now we establish the following results for (5.1) using the similar method given in Section 3.

Theorem 5.1. *Assume* (H_2), (H_3), *and let $n \geq 2$ be even. Suppose further that*

$$\int_{t_0}^{\infty} \left[\frac{1}{a(s)} \exp\left(-\int_{t_0}^{s} \frac{r(\tau)}{a(\tau)} \mathrm{d}\tau \right) \right]^{\frac{1}{\gamma}} \mathrm{d}s = \infty. \tag{5.2}$$

If there exists a constant $\lambda_0 \in (0,1)$ such that the first-order differential inequality

$$y(t)\left[y'(t) + \frac{r(t)}{a(t)} y(t) + \left(\frac{\lambda_0}{(n-1)!}\right)^{\gamma} q(t) \prod_{i=1}^{m} \frac{(g_i^{n-1}(t))^{\gamma_i}}{a^{\gamma_i/\gamma}(g_i(t))} \prod_{i=1}^{m} y^{\gamma_i/\gamma}(g_i(t)) \right] \leq 0 \tag{5.3}$$

has no positive solutions, then Eq. (5.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (5.1). Without loss of generality, x can be assumed to be eventually positive. Similar as in the proof of Lemma 2.3, we have (3.2) for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large. It follows from Lemma 2.2 that

$$x(t) \geq \frac{\lambda}{(n-1)!} \frac{t^{n-1}}{a^{1/\gamma}(t)} a^{1/\gamma}(t) x^{(n-1)}(t) \tag{5.4}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Set $y := a(x^{(n-1)})^{\gamma}$. Using (5.4) in (5.1), we obtain inequality

$$y'(t) + \frac{r(t)}{a(t)} y(t) + \left(\frac{\lambda}{(n-1)!}\right)^{\gamma} q(t) \prod_{i=1}^{m} \frac{(g_i^{n-1}(t))^{\gamma_i}}{a^{\gamma_i/\gamma}(g_i(t))} \prod_{i=1}^{m} y^{\gamma_i/\gamma}(g_i(t)) \leq 0.$$

That is, y is a positive solution of inequality (5.3), which is a contradiction. The proof is complete. □

On the basis of Theorem 5.1 and [20, Corollary 1], we obtain the following result.

Corollary 5.2. *Assume* (H_2), (H_3), (5.2), *and let* $n \geq 2$ *be even. If*

$$\sum_{k=1}^{m} \liminf_{t\to\infty} \frac{\gamma_k}{\gamma} \int_{g_k(t)}^{t} q(s) \prod_{i=1}^{m} \frac{(g_i^{n-1}(s))^{\gamma_i}}{a^{\gamma_i/\gamma}(g_i(s))} \exp\left(\sum_{j=1}^{m} \frac{\gamma_j}{\gamma} \int_{g_j(s)}^{s} \frac{r(v)}{a(v)} \mathrm{d}v\right) \mathrm{d}s > \frac{((n-1)!)^{\gamma}}{\mathrm{e}}, \tag{5.5}$$

then Eq. (5.1) *is oscillatory.*

Similarly, we have the following result.

Theorem 5.3. *Assume* (H_2), (H_3), (5.2), *and let* $n \geq 3$ *be odd. Suppose further that either*

(i) (5.5) *holds or*
(ii) *there exists a constant* $\lambda_0 \in (0, 1)$ *such that the first-order differential inequality* (5.3) *has no positive solutions.*

Then every solution of Eq. (5.1) *is oscillatory or tends to zero as* $t \to \infty$.

Remark 5.4. One can easily obtain that Eqs. (5.1) and

$$\left(a(t)\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)(x^{(n-1)}(t))^{\gamma}\right)' + q(t)\exp\left(\int_{t_0}^{t} \frac{r(\tau)}{a(\tau)}\mathrm{d}\tau\right)\prod_{i=1}^{m} x^{\gamma_i}(g_i(t)) = 0 \tag{5.6}$$

are equivalent. Assuming x is an eventually positive solution of (5.6) and using Lemma 2.2 and [20, Corollary 1] in Eq. (5.6), we also get Corollary 5.2. The details are left to the reader.

Remark 5.5. The results obtained in this section include those reported in Section 3 under the assumptions that $m = 1$ and $p - 1 = \gamma$.

6. Summary

Most oscillation results in the literature for the damped differential equation (1.1) and its particular cases have been obtained with the help of the original results due to Philos [15]. In this paper, using the integral averaging technique and a comparison method, we establish new criteria for oscillation and asymptotic behavior of Eqs. (1.1) and (5.1). We point out that, contrary to [3,8,10,14,21,23,25–27], we do not need in our results restrictive condition that M is an arbitrary constant and other similar assumptions which, in certain sense, is a significant improvement compared to the results in the cited papers.

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