



The Borel summable solutions of heat operators on a real analytic manifold



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ARTICLE INFO

Article history:

Received 11 January 2013

Available online 16 August 2013

Submitted by A. Cianchi

Keywords:

Heat operators

Commuting vector fields

Borel summability

Integral means

Pizzetti's formulas

ABSTRACT

We study heat type equations $\partial_t u = \tilde{\Delta} u$, where the operator $\tilde{\Delta}$ is given by a sum of squares of commuting, real analytic vector fields acting on a real analytic manifold. We give necessary and sufficient conditions for convergence and Borel summability of formal power series solutions in terms of generalized integral means of the initial data. The results are also valid for affine perturbations of $\tilde{\Delta}$.

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1. Introduction

During the last few years there is a growing interest in the study of summability of formal power series solutions of non-Kowalevskian equations. The first paper in this direction was done by Lutz, Miyake and Schäfer [12], where they provided necessary and sufficient conditions for the Borel summability of formal solutions to the one-dimensional heat equation $\partial_t u = \partial_x^2 u$ in terms of the initial data, see also [2]. The result was extended to quasi-homogeneous equations by Ichinobe [10], who also gave explicit integral representations of the Borel sums of solutions. General linear partial differential equations with constant coefficients in two variables were studied by Balser [4]. He determined initial conditions for which the (unique) formal solution of a normalized Cauchy problem is multisummable. Another proof of Balser's result in a more general framework of fractional equations was given by Michalik [16].

In the case of the multidimensional heat equation $\partial_t u = \Delta u$, where Δ is n -dimensional Laplace operator, Balser and Malek [7] obtained conditions for k -summability in terms of some auxiliary function expressed in terms of formal solutions; Michalik proved in [14] that the formal solution is Borel summable if and only if some auxiliary function $\Phi(t, x)$, expressed by some integrals of the initial data, is holomorphic at the origin in x variables and can be analytically continued with respect to t in two opposite sectors in \mathbb{C} , and this continuation is of exponential order at most 2, while in [17], using results of Łysik [13], he observed that the function $\Phi(t, x)$ can be replaced by the integral means of the initial data over the closed ball $B(x, t)$ or the sphere $S(x, t)$. Borel summability of formal power series solutions to the inhomogeneous heat equation was studied by Balser [5] and Michalik [15].

Recently the general one space variable heat equation $\partial_t u - a(x)\partial_x^2 u = f(t, x)$ with a variable coefficient $a(x)$ and inhomogeneity $f(t, x)$ was studied by Balser and Loday-Richaud [6]. They delivered necessary and sufficient conditions for the Borel summability of a formal solution \hat{u} in terms of the data and the first two terms of the solution \hat{u} itself. In the case

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$a(x) = x$ the conditions are given only in terms of the initial data. Borel summability of the equation $\partial_t u = a(x)\partial_x^2 u$, where $a(x)$ is a quartic polynomial, was studied by Costin, Park and Takei [8]. They also gave the detailed resurgence structure of the singular manifold in the case $a(x) = x$.

On the other hand there are multiple results on the well-posedness in the analytic-Gevrey spaces of the Cauchy problem for linear evolution partial differential equations. Let us only mention here the paper of Guordin and Gramchev [9] (see also references therein) where they established the well-posedness of the Cauchy problem in scales of analytic-Gevrey spaces for quite general linear non-Kowalevskian equations. In particular, in case of heat equations with polynomial coefficients they obtained convergence results for the initial data being entire functions of exponential type 2.

Here we study heat type equations $\partial_t u = \tilde{\Delta} u$, where the operator $\tilde{\Delta}$ is given by a sum of squares of commuting, real analytic vector fields acting on a real analytic manifold. We give necessary and sufficient conditions for the convergence and Borel summability of formal power series solutions of the Cauchy problem. The conditions are given in terms of generalized integral means of the initial data. In fact, the results are also valid for affine perturbations of $\tilde{\Delta}$.

2. Convergent solutions

2.1. Integral means

For a continuous function u defined on an open subset Ω of \mathbb{R}^n , any $x \in \Omega$ and $0 < R < \text{dist}(x, \partial\Omega)$ denote by $M(u; x, R)$ and $N(u; x, R)$ the integral means of u over the closed ball $B(x, R) \subset \Omega$ and the sphere $S(x, R) \subset \Omega$ respectively, i.e.,

$$M(u; x, R) = \frac{1}{\sigma(n)R^n} \int_{B(x, R)} u(\xi) d\xi,$$

$$N(u; x, R) = \frac{1}{n\sigma(n)R^{n-1}} \int_{S(x, R)} u(\xi) dS(\xi),$$

where $\sigma(n) = \pi^{n/2}/\Gamma(n/2 + 1)$ (with Γ the Euler Γ -function) is the volume of the unit ball in \mathbb{R}^n and dS denotes the surface measure on $S(x, R)$. If u is real analytic on Ω , then $M(u; x, R)$ and $N(u; x, R)$ as functions of R are holomorphic at the origin and for $|R|$ small enough can be expressed by the Pizzetti formulas (see [13, Theorem 3.1])

$$M(u; x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k (\frac{n}{2} + 1)_k k!} R^{2k},$$

$$N(u; x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k (\frac{n}{2})_k k!} R^{2k},$$

where for $a \in \mathbb{R}$, $(a)_0 = 1$ and $(a)_k = a(a+1) \cdots (a+k-1)$ for $k \in \mathbb{N}$.

2.2. The heat equation on \mathbb{R}^n

Let us consider the initial value problem for the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (1)$$

where $u_0 \in \mathcal{A}(\Omega)$, i.e., u_0 is a real analytic function on a domain $\Omega \subset \mathbb{R}^n$. Then its unique formal power series solution is given by

$$\hat{u}(t, x) = \sum_{k=0}^{\infty} \frac{\Delta^k u_0(x)}{k!} t^k. \quad (2)$$

Before stating a theorem on convergent solutions to (1), recall

Definition 1. Let $\varrho > 0$ and $\tau \geq 0$. We say that a function F defined on $\Omega \times \mathbb{C}$, $\Omega \subset \mathbb{R}^n$, is *entire of exponential growth* (ϱ, τ) *locally uniformly in Ω* if for any compact set $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C_{K, \varepsilon}$ such that

$$\sup_{x \in K} \sup_{|z| \leq R} |F(x, z)| \leq C_{K, \varepsilon} \exp\{(\tau + \varepsilon)R^\varrho\} \quad \text{for any } R < \infty.$$

In [13] we have proved the following

Theorem 1. (See [13, Theorem 5.1].) Let $0 < T \leq \infty$. If the formal power series solution (2) of the initial value problem (1) is convergent for $|t| < T$ locally uniformly in Ω , then $M(u_0; x, R)$ and $N(u_0; x, R)$ as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω . Conversely, if $M(u_0; x, R)$ or $N(u_0; x, R)$ can be holomorphically extended to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω , then the solution (2) of (1) is convergent for $|t| < T$ locally uniformly in Ω .

2.3. A perturbed heat equation

Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Denote by $\Delta^{a,b}$ the Laplace operator perturbed by an affine transformation, i.e., $\Delta^{a,b} = \Delta + \langle a, \nabla \rangle + b$, where $\langle \cdot, \cdot \rangle$ is the Euclidian scalar product and ∇ the gradient operator. Then the unique formal power series solution $\hat{w}(t, x)$ to the initial value problem

$$\begin{cases} \partial_t w - \Delta^{a,b} w = 0, \\ w|_{t=0} = w_0, \end{cases} \quad (3)$$

with $w_0 \in \mathcal{A}(\Omega)$, is given by

$$\hat{w}(t, x) = \sum_{k=0}^{\infty} \frac{(\Delta^{a,b})^k w_0(x)}{k!} t^k. \quad (4)$$

On the other hand $w(t, x)$ satisfies (3) if and only if $u(t, x) = \exp\{\frac{1}{2}\langle a, x \rangle - ct\} w(t, x)$ with $c = \frac{1}{4}a^2 - \frac{1}{2}\sum_{i=1}^n a_i + b$ being a solution of (1). Denote by $M^a(w_0; x, R)$ and $N^a(w_0; x, R)$ the integral means of w_0 with respect to the measure $\exp\{\frac{1}{2}\langle a, \xi \rangle\} d\xi$ over the closed ball $B(x, R)$ and the sphere $S(x, R)$, respectively. Since the multiplication by an exponential function has no influence on convergence/divergence properties by Theorem 1 we get

Corollary 1. Let $0 < T \leq \infty$. If the formal power series solution (4) of the initial value problem (3) is convergent for $|t| < T$ locally uniformly in Ω , then $M^a(w_0; x, R)$ and $N^a(w_0; x, R)$ as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω . Conversely, if $M^a(w_0; x, R)$ or $N^a(w_0; x, R)$ can be holomorphically extended to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω , then the solution (4) of (3) is convergent for $|t| < T$ locally uniformly in Ω .

2.4. Heat type equations

Let \mathcal{M} be a real analytic manifold of dimension n and X_1, \dots, X_n real analytic linearly independent vector fields on \mathcal{M} .¹ Define a Laplace type operator on \mathcal{M} by

$$\tilde{\Delta} = X_1^2 + \dots + X_n^2.$$

(In the case $\mathcal{M} = \Omega \subset \mathbb{R}^n$ and $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, $\tilde{\Delta}$ is the usual Laplace operator Δ .) Let us consider the initial value problem

$$\begin{cases} \partial_t v - \tilde{\Delta} v = 0, \\ v|_{t=0} = v_0, \end{cases} \quad (5)$$

where $v_0 \in \mathcal{A}(\mathcal{M})$. Then the formal power series solution of (5) is given by

$$\hat{v}(t, y) = \sum_{k=0}^{\infty} \frac{\tilde{\Delta}^k v_0(y)}{k!} t^k. \quad (6)$$

It is well known (see [11, Theorem 2.113]) that if real analytic linearly independent vector fields X_1, \dots, X_n commute on \mathcal{M} ,² i.e., they satisfy

$$X_i \circ X_j = X_j \circ X_i \quad \text{for } i, j = 1, \dots, n, \quad (7)$$

then for any fixed $\dot{y} \in \mathcal{M}$ one can find a real analytic diffeomorphism $\Phi : \Omega \subset \mathbb{R}^n \rightarrow \mathcal{M}$ such that $\dot{y} \in V = \Phi(\Omega)$ and $\Phi_*^{-1}(X_i) = \frac{\partial}{\partial x_i}$ in Ω for $i = 1, \dots, n$. Set $B_\Phi(y, R) = \Phi(B(x, R))$ and $S_\Phi(y, R) = \Phi(S(x, R))$ for $y \in V$ and $0 < R < \text{dist}(x, \partial\Omega)$ where $x = \Phi^{-1}(y)$. Let μ_Φ (respectively, dS_Φ) be a measure on V (respectively, on S_Φ) defined by $\mu_\Phi(A) = \int_{\Phi^{-1}(A)} d\xi$ for a Borel measurable set $A \subset V$ (respectively, $dS_\Phi(A) = \int_{\Phi^{-1}(A)} dS(\xi)$ for a Borel measurable set $A \subset S_\Phi$).

¹ In the case of a compact real analytic manifold of dimension n the condition of existence of n globally linearly independent vector fields holds for flat n -dimensional tori and for the spheres S^n for $n = 3$ and $n = 7$, while not for $n = 5$ or n even, see [1] and references therein.

² This is a quite restrictive condition, even locally.

Theorem 2. Let X_1, \dots, X_n be real analytic linearly independent vector fields on \mathcal{M} satisfying (7) and $0 < T \leq \infty$. Fix $\hat{y} \in \mathcal{M}$ and let $\Phi, V, B_\Phi, \mu_\Phi$ and dS_Φ be as above. The formal power series solution (6) of (5) is convergent for $|t| < T$ locally uniformly in V if and only if the solid integral mean

$$M_\Phi(v_0; y, R) = \frac{1}{\mu_\Phi(B_\Phi(y, R))} \int_{B_\Phi(y, R)} v_0(\eta) d\mu_\Phi(\eta)$$

and/or the spherical integral mean

$$N_\Phi(v_0; y, R) = \frac{1}{dS_\Phi(S_\Phi(y, R))} \int_{S_\Phi(y, R)} v_0(\eta) dS_\Phi(\eta)$$

as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in V .

Proof. Assume that the formal power series solution (6) of (5) is convergent for $|t| < T$ locally uniformly in V . Denote its sum as $v(t, y)$ and set $u(t, x) = v(t, \Phi(x))$, $u_0(x) = v_0(\Phi(x))$ for $|t| < T$, $x \in \Omega$. Then u satisfies (1) and is given by (2) with the series convergent for $|t| < T$ locally uniformly in Ω . Hence by Theorem 1, $M(u_0; x, R)$ and $N(u_0; x, R)$ as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω . But for $x \in \Omega$ and $y = \Phi(x)$ we have

$$\begin{aligned} \sigma(n)R^n &= \int_{\Phi^{-1}(B_\Phi(y, R))} d\xi = \mu_\Phi(B_\Phi(y, R)), \\ \int_{B(x, R)} u_0(\xi) d\xi &= \int_{\Phi^{-1}(B_\Phi(y, R))} v_0(\Phi(\xi)) d\xi = \int_{B_\Phi(y, R)} v_0(\eta) d\mu_\Phi(\eta) \end{aligned}$$

and

$$\begin{aligned} n\sigma(n)R^{n-1} &= \int_{\Phi^{-1}(S_\Phi(y, R))} dS(\xi) = dS_\Phi(S_\Phi(y, R)), \\ \int_{S(x, R)} u_0(\xi) dS(\xi) &= \int_{\Phi^{-1}(S_\Phi(y, R))} v_0(\Phi(\xi)) dS(\xi) = \int_{S_\Phi(y, R)} v_0(\eta) dS_\Phi(\eta). \end{aligned}$$

So

$$M(u_0; x, R) = M_\Phi(v_0; y, R) \quad \text{and} \quad N(u_0; x, R) = N_\Phi(v_0; y, R).$$

Hence $M_\Phi(v_0; y, R)$ and $N_\Phi(v_0; y, R)$ as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in V .

The proof of the converse statement is done in the same way with *and* replaced by *or*. \square

Using Corollary 1 instead of Theorem 1 we get a characterization of convergent solutions for a heat type equation with an affine perturbation of $\tilde{\Delta}$. To this end for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ set $\tilde{\Delta}^{a,b} = \tilde{\Delta} + \sum_{i=1}^n a_i X_i + b$ and consider the initial value problem

$$\begin{cases} \partial_t w - \tilde{\Delta}^{a,b} w = 0, \\ w|_{t=0} = w_0, \end{cases} \quad (8)$$

where $w_0 \in \mathcal{A}(\mathcal{M})$. Then the formal power series solution of (8) is given by

$$\hat{w}(t, y) = \sum_{k=0}^{\infty} \frac{(\tilde{\Delta}^{a,b})^k w_0(y)}{k!} t^k. \quad (9)$$

Corollary 2. Under the assumptions of Theorem 2 let μ_Φ^a be a measure on V defined by $\mu_\Phi^a(A) = \int_{\Phi^{-1}(A)} \exp\{\frac{1}{2}\langle a, \xi \rangle\} d\xi$ for a Borel measurable set $A \subset V$ and let dS_Φ^a be a respective measure on S_Φ . Then the formal power series solution (9) of (8) is convergent for $|t| < T$ locally uniformly in V if and only if the solid integral mean

$$M_\Phi^a(w_0; y, R) = \frac{1}{\mu_\Phi^a(B_\Phi(y, R))} \int_{B_\Phi(y, R)} w_0(\eta) d\mu_\Phi^a(\eta)$$

and/or the spherical integral mean

$$N_{\phi}^a(w_0; y, R) = \frac{1}{dS_{\phi}^a(S_{\phi}(y, R))} \int_{S_{\phi}(y, R)} w_0(\eta) dS_{\phi}^a(\eta)$$

as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in V .

3. Borel summability

Denote by D_r a complex disc in \mathbb{C} of radius $r > 0$ centered at the origin. For $d \in \mathbb{R}$, interpreted as a direction in the universal covering space $\tilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$, and $\varepsilon > 0$ define a sector $S(d, \varepsilon)$ in $\tilde{\mathbb{C}}$ of opening ε by

$$S(d, \varepsilon) = \{x = |x|e^{i\theta} \in \tilde{\mathbb{C}}: |d - \theta| < \varepsilon/2\}.$$

Definition 2. (See [17].) Let U be a domain in \mathbb{C}^n , $d \in \mathbb{R}$ and $\varphi_j \in \mathcal{O}(U)$ for $j \in \mathbb{N}_0$, i.e., φ_j are holomorphic on U . We say that a formal power series

$$\hat{\varphi}(t, x) = \sum_{j=0}^{\infty} \frac{\varphi_j(x)}{j!} t^j$$

is *Borel summable* (or *1-summable*) with respect to t in a direction d locally uniformly in U if its Borel transform defined on $D_r \times U$ with some $r > 0$ by

$$(\hat{B}\hat{\varphi})(s, x) = \sum_{j=0}^{\infty} \frac{\varphi_j(x)}{(j!)^2} s^j$$

extends holomorphically to a domain $(D_r \cup S(d, \varepsilon)) \times U$ with some small $\varepsilon > 0$ and the extension (also denoted by $\hat{B}\hat{\varphi}$) satisfies for any $U_1 \Subset U$ and $0 < \varepsilon_1 < \varepsilon$,

$$\max_{x \in U_1} |(\hat{B}\hat{\varphi})(s, x)| \leq A e^{B|s|} \quad \text{for } s \in S(d, \varepsilon_1)$$

with some $A, B < \infty$.

For fundamental facts on Borel summability the reader is referred to the monograph by Balser [3]. Clearly the above definition can be naturally extended to the case when U is a domain in the complexification $\mathcal{M}^{\mathbb{C}}$ of a real analytic manifold \mathcal{M} .

Michalik obtained the following characterization of Borel summable solutions of the heat equation (1).

Theorem 3. (See [17, Theorem 4.1].) Let $U \subset \mathbb{C}^n$ be a domain, $u_0 \in \mathcal{O}(U)$, $d \in \mathbb{R}$ and let \hat{u} be the formal power series solution (2) of (1). Then the following conditions are equivalent:

1. \hat{u} is Borel summable in the direction d locally uniformly in U ;
2. $M(u_0; x, R)$ as a function of R extends to a holomorphic function on $D_{\varepsilon} \cup S(d/2, \varepsilon) \cup S(d/2 + \pi, \varepsilon)$ with some small $\varepsilon > 0$ and the extension satisfies for any $U_1 \Subset U$ and $0 < \varepsilon_1 < \varepsilon$,

$$\sup_{x \in U_1} |M(u_0; x, R)| \leq A e^{B|R|^2} \quad \text{for } R \in S(d/2, \varepsilon_1) \cup S(d/2 + \pi, \varepsilon_1)$$

with some $A, B < \infty$;

3. $N(u_0; x, R)$ as a function of R extends to a holomorphic function on $D_{\varepsilon} \cup S(d/2, \varepsilon) \cup S(d/2 + \pi, \varepsilon)$ with some small $\varepsilon > 0$ and the extension satisfies for any $U_1 \Subset U$ and $0 < \varepsilon_1 < \varepsilon$,

$$\sup_{x \in U_1} |N(u_0; x, R)| \leq A e^{B|R|^2} \quad \text{for } R \in S(d/2, \varepsilon_1) \cup S(d/2 + \pi, \varepsilon_1)$$

with some $A, B < \infty$.

Repeating the proof of Theorem 2, but now using Theorem 3 in place of Theorem 1, we get

Theorem 4. Let \mathcal{M} be a real analytic manifold, $v_0 \in \mathcal{A}(\mathcal{M})$ and X_1, \dots, X_n real analytic linearly independent vector fields on \mathcal{M} satisfying (7). Fix $\hat{y} \in \mathcal{M}$ and let $\Phi, \Omega, V, B_\Phi, \mu_\Phi$ and dS_Φ be as in Theorem 2. Set $u_0 = v_0 \circ \Phi$ and assume that u_0 and Φ extend to a complex neighborhood $U \subset \mathbb{C}^n$ of Ω . Then v_0 extends to the neighborhood $\Phi(U)$ of V in the complexification $\mathcal{M}^\mathbb{C}$ of \mathcal{M} . Let $d \in \mathbb{R}$ and let \hat{v} be the formal power series solution (6) of the heat type equation (5). Then the following conditions are equivalent:

1. \hat{v} is Borel summable in the direction d locally uniformly in $\Phi(U)$;
2. $M_\Phi(v_0; y, R)$ as a function of R extends to a holomorphic function on $D_\epsilon \cup S(d/2, \epsilon) \cup S(d/2 + \pi, \epsilon)$ with some small $\epsilon > 0$ and the extension satisfies for any $U_1 \Subset U$ and $0 < \epsilon_1 < \epsilon$,

$$\sup_{y \in \Phi(U_1)} |M_\Phi(v_0; y, R)| \leq A e^{B|R|^2} \quad \text{for } R \in S(d/2, \epsilon_1) \cup S(d/2 + \pi, \epsilon_1)$$

with some $A, B < \infty$;

3. $N_\Phi(v_0; y, R)$ as a function of R extends to a holomorphic function on $D_\epsilon \cup S(d/2, \epsilon) \cup S(d/2 + \pi, \epsilon)$ with some small $\epsilon > 0$ and the extension satisfies for any $U_1 \Subset U$ and $0 < \epsilon_1 < \epsilon$,

$$\sup_{y \in \Phi(U_1)} |N_\Phi(v_0; y, R)| \leq A e^{B|R|^2} \quad \text{for } R \in S(d/2, \epsilon_1) \cup S(d/2 + \pi, \epsilon_1)$$

with some $A, B < \infty$.

Remark 1. Replacing functions $M_\Phi(v_0; y, R)$ and $N_\Phi(v_0; y, R)$ in Theorem 4 by $M_\Phi^a(w_0; y, R)$ and $M_\Phi^a(w_0; y, R)$ in Corollary 2, respectively, one obtains an analogous characterization of Borel summable solutions to (8).

Remark 2. The convergence results of Theorem 2 and Corollary 2 and Borel summability results of Theorem 4 are local with respect to $\hat{y} \in \mathcal{M}$ (since vector fields $X_i, i = 1, \dots, n$, can be straighten only locally) and they depend on the real analytic structure of \mathcal{M} . The author does not know any results on global convergence related for example to the characterization of real analytic functions by means of eigenfunction expansions as in the paper of Seeley [18].

Open problem. It would be interesting to obtain conditions for convergence and Borel summability of formal solutions to (5) in cases when vector fields X_i do not satisfy (7). Of special interest here are cases when $\tilde{\Delta}$ is the Grushin operator $\partial_x^2 + x^2 \partial_y^2$ or the Laplace operator on the Heisenberg group.

4. An example

Example 1. Let $X_i = y_i \frac{\partial}{\partial y_i}, i = 1, \dots, n$, be vector fields on $\mathbb{R}_+^n = (\mathbb{R}_+)^n$. Define a singular Laplace operator on \mathbb{R}_+^n by $\tilde{\Delta} = \sum_{i=1}^n X_i^2$. Then $\Phi: \mathbb{R}_+^n \ni x \mapsto y = e^x = (e^{x_1}, \dots, e^{x_n}) \in \mathbb{R}_+^n$ is a diffeomorphism satisfying $\Phi_*^{-1}(X_i) = \frac{\partial}{\partial x_i}$ for $i = 1, \dots, n$. So for a function u smooth on \mathbb{R}^n if we put $v(y) = u(\ln y)$, where $\ln y = (\ln y_1, \dots, \ln y_n), y \in \mathbb{R}_+^n$, then $\Delta u(x) = \tilde{\Delta} v(y), x = \ln y$. Define a measure μ_Φ on \mathbb{R}_+^d (induced by Φ) by $\mu_\Phi(A) = \int_{\Phi^{-1}(A)} d\xi$ for a Borel measurable set $A \subset \mathbb{R}_+^n$. Note that $\mu_\Phi(A) = \int_A \eta^{-1} d\eta$, where $\eta^{-1} = (\eta_1 \cdots \eta_n)^{-1}$ is the Jacobian of $\Phi^{-1}(\eta)$. For $y \in \mathbb{R}_+^n$ and $R > 0$ set $B_\Phi(y, R) = \{\eta \in \mathbb{R}_+^n : \sum_{i=1}^n (\ln \eta_i - \ln y_i)^2 \leq R^2\}$. Then for $x = \ln y$ we have

$$\begin{aligned} |B(x, R)| &= \int_{\Phi^{-1}(B_\Phi(y, R))} d\xi = \mu_\Phi(B_\Phi(y, R)) = \int_{B_\Phi(y, R)} \eta^{-1} d\eta, \\ \int_{B(x, R)} u(\xi) d\xi &= \int_{\Phi^{-1}(B_\Phi(y, R))} v(\Phi(\xi)) d\xi = \int_{B_\Phi(y, R)} v(\eta) d\mu_\Phi(\eta) = \int_{B_\Phi(y, R)} v(\eta) \eta^{-1} d\eta. \end{aligned}$$

So

$$M(u; x, R) = \frac{1}{\int_{B_\Phi(y, R)} \eta^{-1} d\eta} \cdot \int_{B_\Phi(y, R)} v(\eta) \eta^{-1} d\eta = M_\Phi(v; y, R).$$

Hence by Theorem 2 we get

Corollary 3. Let V be a domain in $\mathbb{R}_+^d, v_0 \in \mathcal{A}(V)$ and $0 < T \leq \infty$. Then the formal power series solution $\hat{v}(t, y) = \sum_{k=0}^\infty \frac{\tilde{\Delta}^k v_0(y)}{k!} t^k$ of the singular heat equation

$$\begin{cases} \partial_t v - \tilde{\Delta} v = 0, \\ v|_{t=0} = v_0, \end{cases} \quad (10)$$

is convergent for $|t| < T$ locally uniformly in V if and only if the integral mean

$$M_{\Phi}(v_0; y, R) = \frac{1}{\int_{B_{\Phi}(y, R)} \eta^{-1} d\eta} \cdot \int_{B_{\Phi}(y, R)} v_0(\eta) \eta^{-1} d\eta$$

as a function of R extends holomorphically to an entire function of exponential growth $(2, 1/(4T))$ locally uniformly in V .

Analogously, from Theorem 4 one can derive a characterization of Borel summable solutions of (10) in terms of holomorphicity of $M_{\Phi}(v_0; y, R)$ as a function of R and its growth as $R \rightarrow \infty$.

Acknowledgments

The author would like to express his gratitude to an anonymous referee for his/her remark that the main results of the paper hold for affine perturbations of $\tilde{\Delta}$ and for pointing out Refs. [1,9,18].

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