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Convolution functions that are nowhere differentiable

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ABSTRACT

In 1951 V. Jarník constructed two continuous functions whose Volterra convolution is nowhere differentiable. We generalize Jarník's results by proving that the set of such functions is maximal lineable. This would shed some light on a question posed in 1973 on the structure of the set of continuous functions whose Volterra convolution is nowhere differentiable.

1. Introduction and preliminaries

Let us consider the space of real valued continuous mappings on an interval I and denote it by $\mathcal{C}(I)$. In this paper we study the Volterra convolution operator defined on $\mathcal{C}([0,\infty)) \times \mathcal{C}([0,\infty))$ by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

An interesting property of the convolution operator considered above is that it maximizes the differentiability of the operands f and g. Being more specific, it is easily seen that if any of the functions f or g are differentiable and have a bounded derivative, then $f * g$ is also differentiable. However, the differentiability of f or g is by no means a necessary condition for the differentiability of the convolution $f * g$. In this paper we provide examples of continuous nowhere differentiable functions whose convolutions are differentiable. We also study a problem which is at the other end of the scale. Already in 1951, Jarník [17] provided two functions in $\mathcal{C}([0,1])$ whose convolution was nowhere differentiable. We also prove that the set of functions in $\mathcal{C}([0,\infty))$ whose convolutions are nowhere differentiable is maximal lineable (see, [8]), i.e., it contains (except for the zero function) an infinite dimensional subspace of the *largest* possible dimension. This would shed some light on a question posed in 1973 on the structure of the set of continuous functions whose Volterra convolution is nowhere differentiable (see the final remark of [18]).

The word *lineability* was coined by Gurariy and used for the first time in [3]. The exact definition of that concept is the following: Given a property, we say that the subset M of functions on $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$) satisfying it is α*-lineable* if $M \cup \{0\}$ contains a vector space of dimension α (finite or infinite). If M contains an infinite-dimensional vector space, it will be called *lineable* for short. One of the first results related to the notion of lineability is tightly related to the questions studied

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in this paper. Gurariy [15,16] proved that the subset of $\mathcal{C}([0,1])$ consisting of nowhere differentiable functions is lineable. His proof, however, was not constructive. Jiménez-Rodríguez, Muñoz-Fernández and Seoane-Sepúlveda show in [19] that there is a $\mathfrak{c}$-dimensional linear space of continuous nowhere differentiable functions generated by instances of the so called Weierstrass' function (also called the Monster of Weierstrass). Observe that here $\mathfrak{c}$ is the cardinality of $\mathbb{R}$.

As the reader may already know, Weierstrass' function shocked the mathematical community of the 19th century by representing a continuous nowhere differentiable function. Actually it was the first example of such a function to be published (which happened in 1872), although not the only one nor the first to be discovered. An excellent account on nowhere differentiable functions and their historical evolution can be found in [21]. Another example of nowhere differentiable function was given by Knopp [20] in 1918 (this example plays an important role in Section 3).

The study of the algebraic properties of sets of functions satisfying *strange* or *striking* properties has become a fruitful field in the last decade. Words such as *lineability* (already defined above), *algebrability* or *spaceability* appear nowadays in the titles and abstracts of numerous publications concerning many different problems in several different areas of Mathematics and have attracted the interest of many mathematicians, among whom we have R. Aron, L. Bernal-González, G. Botelho, P. Enflo, G. Godefroy, V. Fonf, V. Gurariy, V. Kadets, D. Pellegrino, or J.B. Seoane-Sepúlveda (see, e.g., [1,2,4,6–8,10–14]). The present work is yet another contribution to the field of lineability. The reader can find a very recent and excellent expository paper on lineability, spaceability, and algebrability in [9].

2. Lineability of continuous functions whose convolutions are nowhere differentiable

As mentioned in the introduction, the construction of a vector space of maximal dimension of continuous functions whose convolutions are nowhere differentiable is based on the example given by Weierstrass in the 19th century of a continuous nowhere differentiable function, also known as the Monster of Weierstrass:

Definition 2.1. We shall define, for $\frac{7}{9} < a < 1$, the Monster of Weierstrass associated to the parameter a as follows:

$$W_a(t) = \sum_{k=0}^{\infty} a^k \cos\left(9^k \pi t\right) =: \sum_{k=0}^{\infty} y_{a,k}(t)$$

(in what follows we shall assume those functions are only defined over $[0,\infty)$).

It turns out that the convolution of Monsters of Weierstrass is nowhere differentiable:

Proposition 2.2. *For $\frac{7}{9} < a_1, a_2 < 1$, we have that $W_{a_1} * W_{a_2}$ is nowhere differentiable.*

Proof. First, we can see that

$$\left|y_{a,k}(s)\right| \leqslant a^k \quad \text{and} \quad \left|y_{a,k}(s) - y_{a,k}(t)\right| \leqslant (9a)^k \pi.$$

Let now $t_0 > 0$ and, for every $n \in \mathbb{N}$, find an even number p_n such that

$$r_n := \frac{p_n}{9^n} \leqslant t_0 < \frac{p_n + 2}{9^n}$$

and define also $s_n := \frac{p_n+3}{9^n}$.

Then, for $k \geqslant n$

$$\begin{aligned}
y_{a_1,k} * y_{a_2,k}(r_n) &= \int_0^{r_n} a_1^k \cos\left(9^k \pi \tau\right) a_2^k \cos\left(9^k \pi (r_n - \tau)\right) d\tau \\
&= (a_1 a_2)^k \int_0^{r_n} \cos\left(9^k \pi \tau\right)\left[\cos\left(9^k \pi r_n\right) \cos\left(9^k \pi \tau\right) + \sin\left(9^k \pi r_n\right) \sin\left(9^k \pi \tau\right)\right] d\tau \\
&= (a_1 a_2)^k \int_0^{r_n} \cos^2\left(9^k \pi \tau\right) d\tau = \frac{(a_1 a_2)^k}{2}\left[\tau + \frac{\sin(9^k \pi \tau)}{9^k \pi}\right]_{\tau=0}^{\tau=r_n} \\
&= \frac{(a_1 a_2)^k}{2} r_n \geqslant \frac{(a_1 a_2)^k}{2}\left(t_0 - \frac{2}{9^n}\right).
\end{aligned}$$

Similarly, we get

$$y_{a_1,k} * y_{a_2,k}(s_n) = \frac{-(a_1a_2)^k}{2} s_n \leqslant \frac{-(a_1a_2)^k}{2} t_0,$$

from which

$$\frac{y_{a_1,k} * y_{a_2,k}(s_n) - y_{a_1,n} * y_{a_2,n}(r_n)}{s_n - r_n} \leqslant -\frac{(a_1a_2)^k}{3}\big(9^n t_0 - 1\big),$$

and hence

$$\sum_{k\geqslant n} \frac{y_{a_1,k} * y_{a_2,k}(s_n) - y_{a_1,n} * y_{a_2,n}(r_n)}{s_n - r_n} \leqslant -\frac{9^n t_0 - 1}{3} \cdot \frac{(a_1a_2)^n}{1 - a_1a_2}.$$

Then, there exists $\eta_n(a_1, a_2) \leqslant -1$ such that

$$\sum_{k\geqslant n} \frac{y_{a_1,k} * y_{a_2,k}(s_n) - y_{a_1,n} * y_{a_2,n}(r_n)}{s_n - r_n} = \eta_n(a_1, a_2)\frac{9^n t_0 - 1}{3} \cdot \frac{(a_1a_2)^n}{1 - a_1a_2}. \tag{2.1}$$

Let now $k \neq m \geqslant 0$. Then,

$$\begin{aligned}
y_{a_1,k} * y_{a_2,m}(s_n) &= a_1^k a_2^m \int_0^{s_n} \cos\big(9^k\pi s_n \tau\big)\big[\cos\big(9^m\pi s_n\big)\cos\big(9^m\pi\tau\big) + \sin\big(9^m\pi s_n\big)\sin\big(9^m\pi\tau\big)\big]\,d\tau \\
&= \frac{a_1^k a_2^m}{2}\Bigg[\cos\big(9^m\pi s_n\big)\bigg(\frac{\sin[(9^k+9^m)\pi\tau]}{(9^k+9^m)\pi} + \frac{\sin[(9^k-9^m)\pi\tau]}{(9^k-9^m)\pi}\bigg)_0^{s_n} \\
&\quad - \sin\big(9^m\pi s_n\big)\bigg(\frac{\cos[(9^k+9^m)\pi\tau]}{(9^k+9^m)\pi} - \frac{\cos[(9^k-9^m)\pi\tau]}{(9^k-9^m)\pi}\bigg)_0^{s_n}\Bigg] \\
&= \frac{a_1^k a_2^m}{2\pi}\bigg[\frac{1}{9^k+9^m}\big(\cos\big(9^m\pi s_n\big)\sin\big[\big(9^k+9^m\big)\pi s_n\big] - \sin\big(9^m\pi s_n\big)\cos\big[\big(9^k+9^m\big)\pi s_n\big]\big) \\
&\quad + \frac{1}{9^k-9^m}\big(\cos\big(9^m\pi s_n\big)\sin\big[\big(9^k-9^m\big)\pi s_n\big] + \sin\big(9^m\pi s_n\big)\cos\big[\big(9^k-9^m\big)\pi s_n\big]\big) \\
&\quad - \frac{2\cdot 9^m \sin(9^m\pi s_n)}{9^{2k}-9^{2m}}\bigg] \\
&= \frac{a_1^k a_2^m}{2\pi}\bigg(\frac{1}{9^k+9^m}\sin\big(9^k\pi s_n\big) + \frac{1}{9^k-9^m}\sin\big(9^k\pi s_n\big) - \frac{2\cdot 9^m \sin(9^m\pi s_n)}{9^{2k}-9^{2m}}\bigg).
\end{aligned}$$

Hence, we can put, for $m \neq k$,

$$\begin{aligned}
&y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n) \\
&\quad = \frac{a_1^k a_2^m}{2\pi}\bigg[\frac{2\cdot 9^k}{9^{2k}-9^{2m}}\big(\sin\big(9^k\pi s_n\big) - \sin\big(9^k\pi r_n\big)\big) + \frac{2\cdot 9^m}{9^{2k}-9^{2m}}\big(\sin\big(9^m\pi r_n\big) - \sin\big(9^m\pi s_n\big)\big)\bigg].
\end{aligned}$$

In a similar way (for $0 \leqslant k \leqslant n-1$),

$$\begin{aligned}
y_{a_1,k} * y_{a_2,k}(s_n) &= (a_1a_2)^k \int_0^{s_n} \cos\big(9^k\pi\tau\big)\big[\cos\big(9^k\pi s_n\big)\cos\big(9^k\pi\tau\big) + \sin\big(9^k\pi s_n\big)\sin\big(9^k\pi\tau\big)\big]\,d\tau \\
&= \frac{(a_1a_2)^k}{2}\bigg[\cos\big(9^k\pi s_n\big)\bigg(\tau + \frac{\sin(2\cdot 9^k\pi\tau)}{2\cdot 9^k\pi}\bigg)_0^{s_n} - \sin\big(9^k\pi s_n\big)\bigg(\frac{\cos(2\cdot 9^k\pi\tau)}{2\cdot 9^k\pi}\bigg)_0^{s_n}\bigg] \\
&= \frac{(a_1a_2)^k}{2}\bigg[\cos\big(9^k\pi s_n\big)\bigg(s_n + \frac{\sin(2\cdot 9^k\pi s_n)}{2\cdot 9^k\pi}\bigg) - \sin\big(9^k\pi s_n\big)\frac{\cos(2\cdot 9^k\pi s_n) - 1}{2\cdot 9^k\pi}\bigg] \\
&= \frac{(a_1a_2)^k}{2}\bigg(\frac{\sin(9^k\pi s_n)}{9^k\pi} + s_n\cos\big(9^k\pi s_n\big)\bigg),
\end{aligned}$$

and hence we can write (reaching the analogous expression for $y_{a_1,k} * y_{a_2,k}(r_n)$):

$$y_{a_1,k} * y_{a_2,k}(s_n) - y_{a_1,k} * y_{a_2,k}(r_n) = \frac{(a_1a_2)^k}{2}\bigg[\frac{\sin(9^k\pi s_n) - \sin(9^k\pi r_n)}{9^k\pi} + s_n\cos\big(9^k\pi s_n\big) - r_n\cos\big(9^k\pi r_n\big)\bigg].$$

Putting everything together, we can conclude the following inequalities:

(1) If $0 \leqslant m \neq k$,

$$\left|\frac{y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n)}{s_n - r_n}\right| \leqslant a_1^k a_2^m \frac{9^{2k} + 9^{2m}}{|9^{2k} - 9^{2m}|} \leqslant 2 \cdot a_1^k a_2^m. \tag{2.2}$$

(2) If $0 \leqslant k \leqslant n-1$,

$$\left|\frac{y_{a_1,k} * y_{a_2,k}(s_n) - y_{a_1,k} * y_{a_2,k}(r_n)}{s_n - r_n}\right| \leqslant \frac{(a_1a_2)^k}{2}\left[2 + 9^k\pi\left(t_0 + \frac{3}{9^n}\right)\right]. \tag{2.3}$$

With those inequalities, we may see that

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty}\sum_{m\neq k}\frac{y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n)}{s_n - r_n} + \sum_{k=0}^{n-1}\frac{y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n)}{s_n - r_n}\right| \\
&\quad\leqslant \sum_{k=0}^{\infty}\sum_{m\neq k} 2 \cdot a_1^k a_2^m + \sum_{k=0}^{n-1}\frac{(a_1a_2)^k}{2}\left[2 + 9^k\pi\left(t_0 + \frac{3}{9^n}\right)\right] \\
&\quad\leqslant \frac{2}{(1-a_1)(1-a_2)} + \frac{(a_1a_2)^n}{1-a_1a_2} + \frac{\pi}{2}\frac{(9a_1a_2)^n - 1}{9a_1a_2 - 1}\left(t_0 + \frac{3}{9^n}\right) \\
&\quad< \frac{2}{(1-a_1)(1-a_2)} + \frac{(a_1a_2)^n}{1-a_1a_2} + \frac{9^n t_0 - 1}{4} \cdot \frac{(a_1a_2)^n}{1-a_1a_2}.
\end{aligned}$$

After all these calculations, we can guarantee, for n large enough,

$$\left|\sum_{k=0}^{\infty}\sum_{m\neq k}\frac{y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n)}{s_n - r_n} + \sum_{k=0}^{n-1}\frac{y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n)}{s_n - r_n}\right| \leqslant \frac{9^n t_0 - 1}{4} \cdot \frac{(a_1a_2)^n}{1-a_1a_2},$$

and then, for n large enough, we can find the existence of a constant $\varepsilon_n(a_1, a_2) \in [-1, 1]$ such that

$$\sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\frac{y_{a_1,k} * y_{a_2,m}(s_n) - y_{a_1,k} * y_{a_2,m}(r_n)}{s_n - r_n} = \varepsilon_n(a_1, a_2)\frac{9^n t_0 - 1}{4} \cdot \frac{(a_1a_2)^n}{1-a_1a_2}. \tag{2.4}$$

In conclusion, using the identities in (2.1) and (2.4), we can say, for n large enough, that

$$\frac{W_{a_1} * W_{a_2}(s_n) - W_{a_1} * W_{a_2}(r_n)}{s_n - r_n} = \eta_n(a_1, a_2)\frac{9^n t_0 - 1}{3} \cdot \frac{(a_1a_2)^n}{1-a_1a_2} + \varepsilon_n(a_1, a_2)\frac{9^n t_0 - 1}{4} \cdot \frac{(a_1a_2)^n}{1-a_1a_2},$$

for some constants $\eta_n(a_1, a_2) \leqslant -1$, $\varepsilon_n(a_1, a_2) \in [-1, 1]$. Using this last expression, it is easy to see that

$$\frac{W_{a_1} * W_{a_2}(s_n) - W_{a_1} * W_{a_2}(r_n)}{s_n - r_n} \underset{n\to\infty}{\longrightarrow} -\infty. \qquad \square$$

Remark 2.3. If, instead of the choice of sequences $\{r_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ we had used the following definition of sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$:

$$v_n := \frac{q_n}{9^n} \leqslant t_0 < \frac{q_n + 2}{9^n}, \qquad w_n := \frac{q_n + 3}{9^n},$$

for a properly choice of odd numbers $q_n \in \mathbb{Z}$, we would have had that

$$\frac{W_{a_1} * W_{a_2}(w_n) - W_{a_1} * W_{a_2}(v_n)}{w_n - v_n} \underset{n\to\infty}{\longrightarrow} \infty.$$

Theorem 2.4. *The set of all continuous functions that, convoluting with themselves, give a nowhere differentiable function, is* $\mathfrak{c}$*-lineable.*

Proof. Consider the set

$$\left\{W_a(x)\colon \frac{7}{9} < a < 1\right\}.$$

Assume $\frac{7}{9} < a_1 < a_2 < \cdots < a_k < 1$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R} \setminus \{0\}$ and consider the function

$$g(x)=\sum_{i=1}^{k}\alpha_i W_{a_i}(x).$$

Following the steps of Proposition 2.2 and with the same definitions of sequences $\{s_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$, we find that, for n large enough,

$$\begin{aligned}\frac{g*g(s_n)-g*g(r_n)}{s_n-r_n}&=\sum_{i,j=1}^{k}\alpha_i\alpha_j\left\{\left(\frac{\eta_n(a_i,a_j)}{3}+\frac{\varepsilon_n(a_i,a_j)}{4}\right)\cdot\frac{(a_ia_j)^n(9^nt_0-1)}{1-a_1a_2}\right\}\\&=\left(9a_k^2\right)^n\sum_{i,j=1}^{k}\alpha_i\alpha_j\left\{\left(\frac{\eta_n(a_i,a_j)}{3}+\frac{\varepsilon_n(a_i,a_j)}{4}\right)\left(\frac{a_ia_j}{a_k^2}\right)^n\frac{9^nt_0-1}{9^n(1-a_1a_2)}\right\}.\end{aligned}$$

Now, if $(i,j)\neq(k,k)$, we get that

$$\alpha_i\alpha_j\left\{\left(\frac{\eta_n(a_i,a_j)}{3}+\frac{\varepsilon_n(a_i,a_j)}{4}\right)\left(\frac{a_ia_j}{a_k^2}\right)^n\frac{9^nt_0-1}{9^n(1-a_1a_2)}\right\}\underset{n\to\infty}{\longrightarrow}0,$$

and if $(i,j)=(k,k)$, we get

$$\alpha_k^2\left\{\left(\frac{\eta_n(a_k,a_k)}{3}+\frac{\varepsilon_n(a_k,a_k)}{4}\right)\frac{9^nt_0-1}{9^n(1-a_1a_2)}\right\}\underset{n\to\infty}{\longrightarrow}x<0.$$

Hence, we conclude

$$\frac{g*g(s_n)-g*g(r_n)}{s_n-r_n}\underset{n\to\infty}{\longrightarrow}-\infty. \qquad\square$$

Remark 2.5. If, instead of the choice of sequences $\{r_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ we had used the definition of sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as in Remark 2.3, we would have had that

$$\frac{g*g(w_n)-g*g(v_n)}{w_n-v_n}\underset{n\to\infty}{\longrightarrow}\infty.$$

3. Final remarks

After the previous results and constructions, one might think that whenever we have any two continuous nowhere differentiable functions then their convolution is also nowhere differentiable. Of course, as it is well known, the convolution of two continuous functions f and g is differentiable if any of the functions f and g are differentiable and have a bounded derivative. The convolution operator acts then as a smoothing transformation. This property may still hold even when f and g are highly nondifferentiable. Here we give an example of a function which is continuous nowhere differentiable function and whose convolution with itself is everywhere differentiable. Our construction is based on Knopp's example mentioned in the Introduction and the following consequence of the so called Weierstrass M-Test.

Proposition 3.1. *Let $(f_n)_{n=0}^{\infty}$ be a sequence of differentiable functions on an interval $I=[a,b]$ and let $(a_n)_{n=0}^{\infty}$ be a sequence of numbers such that $\sum_{n=0}^{\infty}|a_n|<\infty$. Assume that $\|f_n'\|_\infty\leqslant K<\infty$ for all $n\geqslant 0$ and that $\sum_{n=0}^{\infty}a_nf_n(x)$ converges for at least one $x\in I$. Then, $\sum_{n=0}^{\infty}a_nf_n$ converges uniformly on I to a differentiable function f such that $f'=\sum_{n=0}^{\infty}a_nf_n'$.*

Proof. We just need to apply Weierstrass M-test to show that $\sum_{n=0}^{\infty}a_nf_n'$ converges uniformly on I. Since $\sum_{n=0}^{\infty}a_nf_n(x)$ converges for some $x\in I$, according to a basic result on functions of one real variable, $\sum_{n=0}^{\infty}a_nf_n$ converges uniformly to a differentiable function and $(\sum_{n=0}^{\infty}a_nf_n)'=\sum_{n=0}^{\infty}a_nf_n'$. $\square$

Next we reproduce the definition of Knopp's function. Let $0<a<1$, $b>1$ with $1/a>ab>1$ and $\Phi(z):=\operatorname{dist}(z,\mathbb{Z})$. Observe that $\operatorname{dist}(z,\mathbb{Z})$ is the distance from z to $\mathbb{Z}$, i.e.,

$$\operatorname{dist}(z,\mathbb{Z})=\inf\bigl\{|z-m|:\ m\in\mathbb{Z}\bigr\}.$$

If $f_k(x)=\Phi(b^kx)$ is defined over a bounded interval, say $[0,M]$, then Knopp's example is defined as $f(x)=\sum_{k=0}^{\infty}a^kf_k(x)$ for $x\in[0,M]$.

Although f is a continuous nowhere differentiable function (see [20] for the original work by Knopp or [5] for a more modern exposition), it can be proved that $f*f$ is differentiable. Indeed, we have that

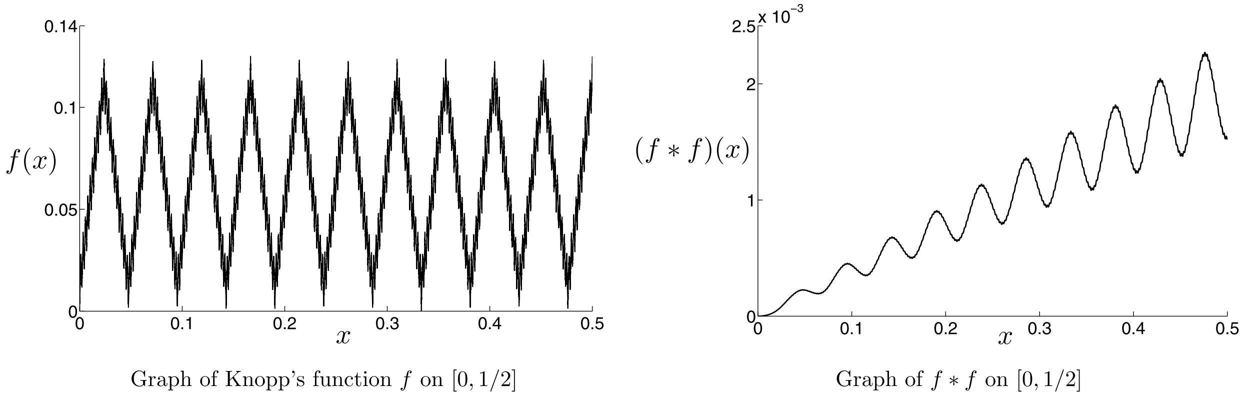

Fig. 1. We have considered f with the parameters $a = 0.2$ and $b = 21$. We have truncated the series appearing in the definition of f up to 10 terms in order to sketch the graph of f and $f * f$.

$$\begin{aligned} f * f(x) &= \int_0^x f(\tau) f(x-\tau)\,d\tau = \int_0^x \left(\sum_{k=0}^{\infty} a^k f_k(\tau)\right)\left(\sum_{j=0}^{\infty} a^j f_j(x-\tau)\right) d\tau \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \int_0^x a^k \Phi\big(b^k\tau\big) a^{k-j} \Phi\big(b^{k-j}(x-\tau)\big)\,d\tau \\ &= \sum_{k=0}^{\infty}\left(\frac{a^2b+1}{2}\right)^k \left(\frac{2}{a^2b+1}\right)^k \sum_{j=0}^{k} a^{2k-j} \int_0^x \Phi\big(b^k\tau\big)\Phi\big(b^{k-j}(x-\tau)\big)\,d\tau \\ &= \sum_{k=0}^{\infty}\left(\frac{a^2b+1}{2}\right)^k g_k(x), \end{aligned}$$

with

$$g_k(x) = \left(\frac{2}{a^2b+1}\right)^k \sum_{j=0}^{k} a^{2k-j} \int_0^x \Phi\big(b^k\tau\big)\Phi\big(b^{k-j}(x-\tau)\big)\,d\tau.$$

Each of the functions g_k is differentiable, with

$$g_k'(x) = \left(\frac{2}{a^2b+1}\right)^k \sum_{j=0}^{k} a^{2k-j} \int_0^x \Phi\big(b^k\tau\big) b^{k-j} \Phi'\big(b^{k-j}(x-\tau)\big)\,d\tau,$$

and hence,

$$\left|g_k'(x)\right| \leqslant \left(\frac{2a^2b}{a^2b+1}\right)^k \sum_{j=0}^{k}\left(\frac{1}{ab}\right)^j \frac{M}{2} \leqslant \frac{Mab}{2(1-ab)}.$$

In conclusion, $|g_k'(x)| \leqslant \frac{Mab}{2(1-ab)}$, for all x in $[0, M]$ and for all k. Applying Proposition 3.1 it follows that f is differentiable. The reader may find of interest Fig. 1, where we have a sketch of the graph of $f * f$ in a small interval.

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