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Spectral results for mixed problems and fractional elliptic operators

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ABSTRACT

One purpose of the paper is to show Weyl type spectral asymptotic formulas for pseudodifferential operators P_a of order $2a$, with type and factorization index $a \in \mathbb{R}_+$ when restricted to a compact set with smooth boundary. The P_a include fractional powers of the Laplace operator and of variable-coefficient strongly elliptic differential operators. Also the regularity of eigenfunctions is described. The other purpose is to improve the knowledge of realizations A_{χ,Σ_+} in $L_2(\Omega)$ of mixed problems for second-order strongly elliptic symmetric differential operators A on a bounded smooth set $\Omega \subset \mathbb{R}^n$. Here the boundary $\partial\Omega = \Sigma$ is partitioned smoothly into $\Sigma = \Sigma_- \cup \Sigma_+$, the Dirichlet condition $\gamma_0 u = 0$ is imposed on Σ_-, and a Neumann or Robin condition $\chi u = 0$ is imposed on Σ_+. It is shown that the Dirichlet-to-Neumann operator $P_{\gamma,\chi}$ is principally of type $\frac{1}{2}$ with factorization index $\frac{1}{2}$, relative to Σ_+. The above theory allows a detailed description of $D(A_{\chi,\Sigma_+})$ with singular elements outside of $\overline{H}^{\frac{3}{2}}(\Omega)$, and leads to a spectral asymptotic formula for the Krein resolvent difference $A^{-1}_{\chi,\Sigma_+} - A^{-1}_{\gamma}$.

0. Introduction

This paper has two parts. After a section with preliminaries, we establish in the first part (Section 2) spectral asymptotic formulas of Weyl type for general Dirichlet realizations of pseudodifferential operators (ψdo's) of type $a > 0$, as defined in Grubb [16,18], and discuss the regularity of eigenfunctions.

In the second part (Section 3) we consider mixed boundary value problems for second-order symmetric strongly elliptic differential operators, characterize the domain, and find the spectral asymptotics of the Krein term (the difference of the resolvent from the Dirichlet resolvent) in general variable-coefficient situations, extending the result of [13] for the principally Laplacian case. This includes showing that the relevant Dirichlet-to-Neumann operator fits into the calculus of the first part.

In Section 2: A typical example of the ψdo's P_a of type $a > 0$ and order $2a$ that we treat is the a-th power of the Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, which is currently of great interest in probability and finance,

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mathematical physics and geometry. Also powers of variable coefficient-operators and much more general ψdo's are included. For the Dirichlet realization $P_{a,\mathrm{Dir}}$ on a bounded open set $\Omega \subset \mathbb{R}^n$, spectral studies have mainly been aimed at the fractional Laplacian $(-\Delta)^a$. In the case of $(-\Delta)^a$, a Weyl asymptotic formula was shown already by Blumenthal and Getoor in [3]; recently a refined asymptotic formula was shown by Frank and Geisinger [7], and Geisinger gave an extension to certain other constant-coefficient operators [8]. The exact domain $D(P_{a,\mathrm{Dir}})$ has not been well described for $a \geq \frac{1}{2}$, except in integer cases where the operator belongs to the calculus of Boutet de Monvel [5]. Based on a recently published systematic theory [16] of ψdo's of type $\mu \in \mathbb{C}$ (where those in the Boutet de Monvel calculus are of type 0), it is now possible to describe domains and parametrices of operators $D(P_{a,\mathrm{Dir}})$ in an exact way, when Ω is smooth. We analyze the sequence of eigenvalues λ_j (singular values s_j when the operator is not selfadjoint), showing that a Weyl asymptotic formula holds in general:

$$s_j(P_{a,\mathrm{Dir}}) \sim C(P_a, \Omega) j^{2a/n} \quad \text{for } j \to \infty; \tag{0.1}$$

moreover we show that the possible eigenfunctions are in $d^a C^{2a}(\overline{\Omega})$ (in $d^a C^{2a-\varepsilon}(\overline{\Omega})$ if $2a \in \mathbb{N}$), where $d(x) \sim \operatorname{dist}(x, \partial\Omega)$. The results are generalized to operators P of order $m = a+b$ with type and factorization index a $(a, b \in \mathbb{R}_+)$.

In Section 3: The detailed knowledge of ψdo's of type a has an application to the classical "mixed" boundary value problems for a second-order strongly elliptic symmetric differential operator A on a smooth bounded set $\Omega \subset \mathbb{R}^n$. Here the boundary condition jumps from a Dirichlet to a Neumann (or Robin) condition at the interface of a smooth partition $\Sigma = \Sigma_- \cup \Sigma_+$ of the boundary $\Sigma = \partial\Omega$; it is also called the Zaremba problem when A is the Laplacian. The L_2-realization A_{χ,Σ_+} it defines is less regular than standard realizations such as the Dirichlet realization A_γ, but the domain has just been somewhat abstractly described; it is contained in $\overline{H}^{\frac{3}{2}-\varepsilon}(\Omega)$ only (observed by Shamir [23]), whereas $D(A_\gamma) \subset \overline{H}^2(\Omega)$. The resolvent difference $M = A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1}$ was shown by Birman [1] to have eigenvalues satisfying $\mu_j(M) = O(j^{-2/(n-1)})$. The present author studied A_{χ,Σ_+} from the point of view of extension theory for elliptic operators in [13] (to which we refer for more references to the literature); here we obtained the asymptotic estimate

$$\mu_j(M) \sim c(M) j^{-2/(n-1)} \quad \text{for } j \to \infty, \tag{0.2}$$

in the case where A is principally Laplacian. This was drawing on the theories of Vishik and Eskin [6] and Birman and Solomyak [2], and other traditional pseudodifferential methods.

We now show that the Dirichlet-to-Neumann operator $P_{\gamma,\chi}$ of order 1 on Σ associated with A is principally of type $\frac{1}{2}$ with factorization index $\frac{1}{2}$ relative to Σ_+. In the formulas connected with the mixed problem, $P_{\gamma,\chi}$ enters as truncated to Σ_+. Therefore we can now use the detailed information on type $\frac{1}{2}$ ψdo's to describe the domain of A_{χ,Σ_+} more precisely, showing how functions $\notin \overline{H}^{\frac{3}{2}}(\Omega)$ occur. Moreover, using Section 2 we can extend the spectral asymptotic formula (0.2) to the general case where A has variable coefficients.

1. Preliminaries

The notations of [16,18] will be used; we shall just give a brief summary here.

We consider a Riemannian n-dimensional C^∞ manifold Ω_1 (it can be $\mathbb{R}^n$) and an embedded smooth n-dimensional manifold $\overline{\Omega}$ with boundary $\partial\Omega$ and interior Ω. For $\Omega_1 = \mathbb{R}^n$, Ω can be $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$; here $(x_1, \dots, x_{n-1}) = x'$. In the general manifold case, $\overline{\Omega}$ is taken compact. For $\xi \in \mathbb{R}^n$, we denote $(1+|\xi|^2)^{\frac{1}{2}} = \langle\xi\rangle$. Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ resp. $\mathbb{R}^n_-$ (or from Ω_1 to Ω resp. $\complement\overline{\Omega}$) is denoted by r^+ resp. r^-, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from Ω resp. $\complement\overline{\Omega}$ to Ω_1) is denoted by $e^\pm$. In Section 3, the notation is used for a smooth subset Σ_+ of an $(n-1)$-dimensional manifold Σ.

A pseudodifferential operator (ψdo) P on $\mathbb{R}^n$ is defined from a symbol $p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$Pu = p(x,D)u = \mathrm{OP}\big(p(x,\xi)\big)u = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi)\hat{u}\,d\xi = \mathcal{F}^{-1}_{\xi\to x}\big(p(x,\xi)\hat{u}(\xi)\big); \tag{1.1}$$

here $\mathcal{F}$ is the Fourier transform $(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx$. The symbol p is assumed to be such that $\partial_x^\beta \partial_\xi^\alpha p(x,\xi)$ is $O(\langle\xi\rangle^{r-|\alpha|})$ for all α, β, for some $r \in \mathbb{R}$ (defining the symbol class $S^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$); then it has order r. The definition of P is carried over to manifolds by use of local coordinates; there are many textbooks (e.g. [12]) describing this and other rules for operations with P, e.g. composition rules. When P is a ψdo on $\mathbb{R}^n$ or Ω_1, $P_+ = r^+Pe^+$ denotes its truncation to $\mathbb{R}^n_+$ resp. Ω.

Let $1 < p < \infty$ (with $1/p' = 1 - 1/p$), then we define for $s \in \mathbb{R}$ the Bessel-potential spaces

$$\begin{aligned}
&H^s_p(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n) \bigm| \mathcal{F}^{-1}\big(\langle\xi\rangle^s\hat{u}\big) \in L_p(\mathbb{R}^n)\big\},\\
&\dot{H}^s_p(\overline{\mathbb{R}}^n_+) = \big\{u \in H^s_p(\mathbb{R}^n) \bigm| \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+\big\},\\
&\overline{H}^s_p(\mathbb{R}^n_+) = \big\{u \in \mathcal{D}'(\mathbb{R}^n_+) \bigm| u = r^+U \text{ for some } U \in H^s_p(\mathbb{R}^n)\big\};
\end{aligned} \tag{1.2}$$

here $\operatorname{supp} u$ denotes the support of u. For $\overline{\Omega}$ compact $\subset \Omega_1$, the definition extends to define $\dot{H}^s_p(\overline{\Omega})$ and $\overline{H}^s_p(\Omega)$ by use of a finite system of local coordinates. When $p = 2$, we get the standard L_2-Sobolev spaces, here the lower index 2 is usually omitted. (These and other spaces are thoroughly described in Triebel's book [24]. He writes $\widetilde{H}$ instead of $\dot{H}$; the present notation stems from Hörmander's works.) We also need the Hölder spaces C^t for $t \in \mathbb{R}_+ \setminus \mathbb{N}$; when $t \in \mathbb{N}_0$, C^t stands for functions with continuous derivatives up to order t. $\dot{C}^t(\overline{\Omega})$ denotes the C^t-functions on Ω_1 supported in $\overline{\Omega}$. Occasionally, we shall also formulate results in the Hölder–Zygmund spaces C^t_* for $t \geq 0$, that allow some statements to be valid for all t; they equal C^t when $t \notin \mathbb{N}_0$ and contain C^t in the integer cases (more details in [18]). The conventions $\bigcup_{\varepsilon>0} H^{s+\varepsilon}_p = H^{s+0}_p$, $\bigcap_{\varepsilon>0} H^{s-\varepsilon}_p = H^{s-0}_p$, defined in a similar way for the other scales of spaces, will sometimes be used.

A ψdo P is called classical (or polyhomogeneous) when the symbol p has an asymptotic expansion $p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$ with p_j homogeneous in ξ of degree $m - j$ for all j. Then P has order m. One can even allow m to be complex; then $p \in S^{\operatorname{Re} m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$; the operator and symbol are still said to be of order m.

Here there is an additional definition: P satisfies *the μ-transmission condition* (in short: *is of type μ*) for some $\mu \in \mathbb{C}$ when, in local coordinates,

$$\partial_x^\beta \partial_\xi^\alpha p_j(x,-N) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x,N), \tag{1.3}$$

for all $x \in \partial\Omega$, all j, α, β, where N denotes the interior normal to $\partial\Omega$ at x. The implications of the μ-transmission property were a main subject of [16,18]; the mapping properties for such operators in C^∞-based spaces were shown in Hörmander [19, Sect. 18.2].

A special role in the theory is played by the *order-reducing operators*. There is a simple definition of operators $\Xi^\mu_\pm$ on $\mathbb{R}^n$

$$\Xi^\mu_\pm = \mathrm{OP}\big(\big(\langle\xi'\rangle \pm i\xi_n\big)^\mu\big);$$

they preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. Here the functions $(\langle\xi'\rangle \pm i\xi_n)^\mu$ do not satisfy all the estimates required for the class $S^{\operatorname{Re}\mu}(\mathbb{R}^n \times \mathbb{R}^n)$, but the operators are useful for some purposes. There is a more refined choice $\Lambda^\mu_\pm$ that does satisfy all the estimates, and there is a definition $\Lambda^{(\mu)}_\pm$ in the manifold situation. These operators define homeomorphisms for all $s \in \mathbb{R}$ such as

$$\begin{aligned}
&\Lambda^{(\mu)}_+ : \dot{H}^s_p(\overline{\Omega}) \xrightarrow{\sim} \dot{H}^{s-\operatorname{Re}\mu}_p(\overline{\Omega}),\\
&\Lambda^{(\mu)}_{-,+} : \overline{H}^s_p(\Omega) \xrightarrow{\sim} \overline{H}^{s-\operatorname{Re}\mu}_p(\Omega);
\end{aligned} \tag{1.4}$$

here $\Lambda_{-,+}^{(\mu)}$ is short for $r^+\Lambda_-^{(\mu)}e^+$, suitably extended to large negative s (cf. Remark 1.1 and Theorem 1.3 in [16]).

The following special spaces introduced by Hörmander are particularly adapted to μ-transmission operators P:

$$H_p^{\mu(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\mu} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}_+^n), \quad s > \operatorname{Re}\mu - 1/p',$$
$$H_p^{\mu(s)}(\overline{\Omega}) = \Lambda_+^{(-\mu)} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\Omega), \quad s > \operatorname{Re}\mu - 1/p',$$
$$\mathcal{E}_\mu(\overline{\Omega}) = e^+\{u(x) = d(x)^\mu v(x) \mid v \in C^\infty(\overline{\Omega})\}; \tag{1.5}$$

namely, r^+P (of order m) maps them into $\overline{H}_p^{s-\operatorname{Re}m}(\mathbb{R}_+^n)$, $\overline{H}_p^{s-\operatorname{Re}m}(\Omega)$ resp. $C^\infty(\overline{\Omega})$ (cf. [16, Sections 1.3, 2, 4]), and they appear as domains of elliptic realizations of P. In the third line, $\operatorname{Re}\mu > -1$ (for other μ, cf. [16]) and $d(x)$ is a C^∞-function positive on Ω and vanishing to order 1 at $\partial\Omega$, e.g. $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. One has that $H_p^{\mu(s)}(\overline{\Omega}) \supset \dot{H}_p^s(\overline{\Omega})$, and that the distributions are locally in H_p^s on Ω, but at the boundary they in general have a singular behavior. More details are given in [16,18].

2. Spectral results for Dirichlet realizations of type a operators

2.1. Dirichlet realizations of type a operators

Consider a Riemannian n-dimensional C^∞-manifold Ω_1 ($n \geq 2$) and an embedded compact n-dimensional C^∞-manifold $\overline{\Omega}$ with boundary $\partial\Omega$ and interior Ω. We consider an elliptic pseudodifferential operator on Ω_1 with the following properties explained in detail in [16]:

Assumption 2.1. Let $a \in \mathbb{R}_+$. P_a is a classical elliptic ψdo on Ω_1 of order $2a$, which relative to Ω satisfies the a-transmission condition and has the factorization index a.

For example, P_a can be the a-th power of a strongly elliptic second-order differential operator on Ω_1, in particular $(-\Delta)^a$, or it can be the a/m-th power of a properly elliptic differential operator of even order $2m$, but also other operators are allowed. (We call such operators "fractional elliptic", because they share important properties with the fractional Laplacian.)

As in [16], we define the Dirichlet realization $P_{a,\mathrm{Dir}}$ in $L_2(\Omega)$ as the operator acting like r^+P_a with domain

$$D(P_{a,\mathrm{Dir}}) = \{u \in \dot{H}^a(\overline{\Omega}) \mid r^+P_a u \in L_2(\Omega)\}. \tag{2.1}$$

Then according to [16, Sections 4–5],

$$D(P_{a,\mathrm{Dir}}) = H^{a(2a)}(\overline{\Omega}) = \Lambda_+^{(-a)} e^+ \overline{H}^a(\Omega). \tag{2.2}$$

We recall from [16]:

Lemma 2.2. *For $1 < p < \infty$, $s > a - 1/p'$, the spaces $H_p^{a(s)}(\overline{\Omega})$ satisfy*

$$H_p^{a(s)}(\overline{\Omega}) = \Lambda_+^{(-a)} e^+ \overline{H}_p^{s-a}(\Omega) \begin{cases} = \dot{H}_p^s(\overline{\Omega}), & \text{if } s - a \in\,]-1/p', 1/p[, \\ \subset \dot{H}_p^{s-0}(\overline{\Omega}), & \text{if } s = a + 1/p, \\ \subset d^a e^+ \overline{H}_p^{s-a}(\Omega) + \dot{H}_p^s(\overline{\Omega}), & \text{if } s - a - 1/p \in \mathbb{R}_+ \setminus \mathbb{N}, \\ \subset d^a e^+ \overline{H}_p^{s-a}(\Omega) + \dot{H}_p^{s-0}(\overline{\Omega}), & \text{if } s - a - 1/p \in \mathbb{N}. \end{cases} \tag{2.3}$$

Moreover,

$$H_p^{a(s)}(\overline{\Omega}) \subset \dot{H}_p^a(\overline{\Omega}), \quad \textit{when } s-a \geq 0. \tag{2.4}$$

Proof. The equalities in (2.3) come from the definition of $H_p^{a(s)}(\overline{\Omega})$, and the inclusions are special cases of [16, Th. 5.4]. For the last statement, we note that when $s-a \geq 0$, $e^+\overline{H}_p^{s-a}(\Omega) \subset e^+L_p(\Omega)$, which is mapped into $\dot{H}_p^a(\overline{\Omega})$ by $\Lambda_+^{(-a)}$. $\square$

In the case where P_a is strongly elliptic, i.e., the principal symbol $p_{a,0}(x,\xi)$ satisfies

$$\operatorname{Re} p_{a,0}(x,\xi) \geq c|\xi|^{2a},$$

with $c > 0$, we can describe $D(P_{a,\mathrm{Dir}})$ in a different way:

Modifying Ω_1 at a distance from $\overline{\Omega}$ if necessary, we can assume Ω_1 to be compact without boundary. Then it is well-known that P_a satisfies a Gårding inequality for $u \in C^\infty(\Omega_1)$:

$$\operatorname{Re}(P_a u, u)_{L_2(\Omega_1)} \geq c_0\|u\|^2_{H^a(\Omega_1)} - k\|u\|^2_{L_2(\Omega_1)}, \tag{2.5}$$

with $c_0 > 0$, $k \in \mathbb{R}$ (cf. e.g. [12, Ch. 7]), besides the inequality

$$\left|(P_a u, v)_{L_2(\Omega_1)}\right| \leq C\|u\|_{H^a(\Omega_1)}\|v\|_{H^a(\Omega_1)}.$$

(In the case of $(-\Delta)^a$ on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, there is a slightly different formulation: For general P_a one would here require x-estimates of the symbol to be uniform on the noncompact set $\mathbb{R}^n$; see e.g. [15] for the appropriate version of the Gårding inequality. One can also include this case by replacing $\mathbb{R}^n \setminus \Omega$ by a suitable compact manifold.)

Define the sesquilinear form s_0 on $C_0^\infty(\Omega)$ by

$$s_0(u,v) = \left(r^+P_a u, v\right)_{L_2(\Omega)} = (P_a u, v)_{L_2(\Omega_1)}, \quad \text{for } u, v \in C_0^\infty(\Omega);$$

it extends by closure to a bounded sesquilinear form $s(u,v)$ on $\dot{H}^a(\overline{\Omega})$, to which the inequality (2.5) extends. The Lax–Milgram construction applied to $s(u,v)$ (cf. e.g. [12, Ch. 12]) leads to an operator S which acts like $r^+P_a\colon \dot{H}^a(\overline{\Omega}) \to \overline{H}^{-a}(\Omega)$, with domain consisting of the functions that are mapped into $L_2(\Omega)$; this is exactly $P_{a,\mathrm{Dir}}$ as in (2.1), (2.2). Here both S and S^* are lower bounded, with lower bound $> -k$ (they are in fact sectorial), hence have $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -k\}$ in their resolvent sets.

When P_a moreover is symmetric, $P_{a,\mathrm{Dir}}$ is the Friedrichs extension of $(r^+P_a)|_{C_0^\infty(\Omega)}$.

In the case of $P_a = (-\Delta)^a$, some authors for precision call this $P_{a,\mathrm{Dir}}$ the "restricted fractional Laplacian", see e.g. Bonforte, Sire and Vazquez [4], in order to distinguish it from the "spectral fractional Laplacian" defined as the a-th power of the Dirichlet realization of $-\Delta$.

2.2. Regularity of eigenfunctions

The possible eigenfunctions have a certain smoothness:

Theorem 2.3. *Let P_a satisfy Assumption 2.1.*

If 0 is an eigenvalue of $P_{a,\mathrm{Dir}}$, its associated eigenfunctions are in $\mathcal{E}_a(\overline{\Omega})$.

When $a \in \mathbb{R}_+ \setminus \mathbb{N}$, then the eigenfunctions u of $P_{a,\mathrm{Dir}}$ associated with nonzero eigenvalues λ lie in $d^aC^{2a}(\overline{\Omega})$ if $2a \notin \mathbb{N}$, in $d^aC^{2a-\varepsilon}(\overline{\Omega})$ (for any $\varepsilon > 0$) if $2a \in \mathbb{N}$; they are also in $C^\infty(\Omega)$.

When $a \in \mathbb{N}$, the eigenfunctions u of $P_{a,\mathrm{Dir}}$ associated with an eigenvalue λ lie in $\{u \in C^\infty(\overline{\Omega}) \mid \gamma_0 u = \gamma_1 u = \cdots = \gamma_{a-1}u = 0\}$ (equal to $\mathcal{E}_a(\overline{\Omega})$ in this case).

Proof. (In some of the formulas here, the extension by zero e^+ is tacitly understood.) When λ is an eigenvalue, the associated eigenfunctions u are nontrivial solutions of

$$r^+ P_a u = \lambda u. \tag{2.6}$$

If $\lambda = 0$, then $u \in \mathcal{E}_a(\overline{\Omega})$, since the right-hand side in (2.6) is in $C^\infty(\overline{\Omega})$, and we can apply [16, Th. 4.4].

Now let $\lambda \neq 0$. When $a \in \mathbb{N}$, we are in a well-known standard elliptic case (as treated e.g. in [11, Sect. 1.7]); the eigenfunctions are in $C^\infty(\overline{\Omega})$ as well as in $\mathcal{E}_a(\overline{\Omega})$, and $\mathcal{E}_a(\overline{\Omega})$ is the described subset of $C^\infty(\overline{\Omega})$.

Next, consider the case $a \in \mathbb{R}_+ \setminus \mathbb{N}$.

To begin with, we know that $u \in \dot{H}^a(\overline{\Omega})$ (from (2.1)). We shall use the well-known general embedding properties for $p, p_1 \in]1, \infty[$:

$$\dot{H}_p^a(\overline{\Omega}) \subset e^+ L_{p_1}(\Omega), \quad \text{when } \frac{n}{p_1} \geq \frac{n}{p} - a, \qquad \dot{H}_p^a(\overline{\Omega}) \subset \dot{C}^0(\overline{\Omega}) \quad \text{when } a > \frac{n}{p}. \tag{2.7}$$

If $a > \frac{n}{2}$, we have already that $\dot{H}^a(\overline{\Omega}) \subset \dot{C}^0(\overline{\Omega})$, so (2.6) has right-hand side in $C^0(\overline{\Omega})$, and we can go on with solution results in Hölder spaces; this will be done further below.

If $a \leq \frac{n}{2}$, we make a finite number of iterative steps to reach the information $u \in C^0(\overline{\Omega})$, as follows: Define $p_0, p_1, p_2, \dots$, with $p_0 = 2$ and $q_j = \frac{n}{p_j}$ for all the relevant j, such that

$$q_j = q_{j-1} - a \quad \text{for } j \geq 1.$$

This means that $q_j = q_0 - ja$; we stop the sequence at j_0 the first time we reach a $q_{j_0} \leq 0$. As a first step, we note that $u \in \dot{H}^a(\overline{\Omega}) \subset e^+ L_{p_1}(\Omega)$ implies $u \in H_{p_1}^{a(2a)}(\overline{\Omega})$ by [16, Th. 4.4], and then by (2.4), $u \in \dot{H}_{p_1}^a(\overline{\Omega})$. In the next step we use the embedding $\dot{H}_{p_1}^a(\overline{\Omega}) \subset e^+ L_{p_2}(\Omega)$ to conclude in a similar way that $u \in \dot{H}_{p_2}^a(\overline{\Omega})$, and so on. If $q_{j_0} < 0$, we have that $\frac{n}{p_{j_0}} < a$, so $u \in \dot{H}_{p_{j_0}}^a(\overline{\Omega}) \subset \dot{C}^0(\overline{\Omega})$. If $q_{j_0} = 0$, the corresponding p_{j_0} would be $+\infty$, and we see at least that $u \in e^+ L_p(\Omega)$ for any large p; then one step more gives that $u \in \dot{C}^0(\overline{\Omega})$.

The rest of the argumentation relies on Hölder estimates, as in [16, Sect. 7], or still more efficiently by [18, Sect. 3]. By the regularity results there,

$$u \in C^0(\overline{\Omega}) \quad \Longrightarrow \quad u \in C_*^{a(2a)}(\overline{\Omega}) \subset e^+ d^a C^a(\overline{\Omega}) + \dot{C}^{2a-0}(\overline{\Omega}) \subset e^+ C^a(\overline{\Omega}).$$

Next, $u \in C^a(\overline{\Omega})$ implies

$$u \in C_*^{a(3a)}(\overline{\Omega}) \subset e^+ d^a C_*^{2a}(\overline{\Omega}) + \dot{C}^{3a(-\varepsilon)}(\overline{\Omega}) \subset e^+ d^a C^{2a(-\varepsilon)}(\overline{\Omega})$$

where $(-\varepsilon)$ is active if $2a \in \mathbb{N}$. Moreover, by the ellipticity of $P_a - \lambda$ on Ω_1, u is C^∞ on the interior Ω. $\square$

The fact that an eigenfunction in $\dot{H}^a(\overline{\Omega})$ is in $L_\infty(\Omega)$ was shown for $P_a = (-\Delta)^a$ with $0 < a < 1$ by Servadei and Valdinoci [22] by a completely different method.

Remark 2.4. For $P_a = (-\Delta)^a$ it has been shown by Ros-Oton and Serra (see [21]) that an eigenfunction u cannot have u/d^a vanishing identically on $\partial\Omega$. This implies that the regularity of u cannot be improved all the way up to $\mathcal{E}_a(\overline{\Omega})$, when $\lambda \neq 0$, $a \in \mathbb{R}_+ \setminus \mathbb{N}$. For if u were in $\mathcal{E}_a(\overline{\Omega})$, it would also lie in $C^\infty(\overline{\Omega})$ (since $r^+ P_a u = \lambda u$ would lie there). Now it is easily checked that $C^\infty(\overline{\Omega}) \cap \mathcal{E}_a(\overline{\Omega}) = \dot{C}^\infty(\overline{\Omega})$ when $a \in \mathbb{R}_+ \setminus \mathbb{N}$, where the functions vanish to order ∞ at the boundary. In particular, u/d^a would be zero on $\partial\Omega$, contradicting $u \neq 0$.

The theorem extends without difficulty to operators of order $m = a + b$ considered in H_p^s-spaces:

Theorem 2.5. *Let P be of type $a > 0$ with factorization index a, and of order $m = a + b$, $b > 0$. Let $1 < p < \infty$, and define P_{Dir} as the operator from $H_p^{a(m)}(\overline{\Omega})$ to $L_p(\Omega)$ acting like r^+P. If 0 is an eigenvalue, the associated eigenfunctions are in $\mathcal{E}_a(\overline{\Omega})$. If $\lambda \neq 0$ is an eigenvalue, the associated eigenfunctions are in $d^aC^m(\overline{\Omega})$ (in $d^aC^{m-\varepsilon}(\overline{\Omega})$ if m is integer).*

Proof. The zero eigenfunctions are solutions with a C^∞ right-hand side, hence lie in $\mathcal{E}_a(\overline{\Omega})$ by [16, Th. 4.4].

Now let u be an eigenfunction associated with an eigenvalue $\lambda \neq 0$. In view of (2.4), we have $u \in \dot{H}_p^a(\overline{\Omega})$. Using (2.7), we find by application of the regularity result of [16, Th. 4.4], by a finite number of iterative steps as in the proof of Theorem 2.3, that $u \in \dot{H}_{p_1}^a, \dot{H}_{p_2}^a, \dots$ with increasing p_j's, until we reach $u \in C^0(\overline{\Omega})$.

Now we can apply the Hölder results from [16,18]; this goes most efficiently by [18, Th. 3.2 2° and Th. 3.3] for Hölder–Zygmund spaces:

$$r^+Pu \in \overline{C}_*^t(\Omega) \quad \Longrightarrow \quad u \in C_*^{a(m+t)}(\overline{\Omega}) \subset d^ae^+\overline{C}_*^{m+t-a}(\Omega) + \dot{C}_*^{m+t(-\varepsilon)}(\overline{\Omega}), \tag{2.8}$$

$t \geq 0$, where $(-\varepsilon)$ is active if $m + t - a \in \mathbb{N}$.

If $b > a$, there are two steps:

$$u \in C^0(\overline{\Omega}) \quad \Longrightarrow \quad u \in C_*^{a(a+b)}(\overline{\Omega}) \subset e^+d^a\overline{C}_*^b(\Omega) + \dot{C}_*^{a+b(-\varepsilon)}(\overline{\Omega}) \subset e^+\overline{C}_*^a(\Omega).$$

Next, $u \in \overline{C}_*^a(\Omega)$ implies

$$u \in C_*^{a(m+a)}(\overline{\Omega}) \subset e^+d^a\overline{C}_*^m(\Omega) + \dot{C}_*^{m+a(-\varepsilon)}(\overline{\Omega}) \subset e^+d^aC^{m(-\varepsilon)}(\overline{\Omega}),$$

where $(-\varepsilon)$ is active if $m \in \mathbb{N}$.

If $b \leq a$, we need a finite number of steps, such as

$$u \in C^0(\overline{\Omega}) \quad \Longrightarrow \quad u \in C_*^{a(a+b)}(\overline{\Omega}) \subset e^+d^a\overline{C}_*^b(\Omega) + \dot{C}_*^{a+b(-\varepsilon)}(\overline{\Omega}) \subset e^+\overline{C}_*^b(\Omega),$$

where we use that $a + b - \varepsilon > b$ for small ε. Next, $u \in \overline{C}_*^b(\Omega)$ implies

$$u \in C_*^{a(m+b)}(\overline{\Omega}) \subset e^+d^a\overline{C}_*^{2b}(\Omega) + \dot{C}_*^{a+2b(-\varepsilon)}(\overline{\Omega}) \subset e^+\overline{C}_*^{\min\{2b,a\}}(\Omega),$$

where we use that $a + 2b - \varepsilon > \min\{2b, a\}$ for small ε. If $2b \geq a$, we end the proof as above. If not, we estimate again, now arriving at the exponent $\min\{3b, a\}$, etc., continuing until we reach $kb \geq a$; then the proof is completed as above. □

2.3. Spectral asymptotics

We shall now study spectral asymptotic estimates for our operators. We first recall some notation and basic rules.

As in [10] we denote by $\mathfrak{C}_p(H, H_1)$ the p-th Schatten class consisting of the compact operators B from a Hilbert space H to another H_1 such that $(s_j(B))_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$. Here the s-numbers, or singular values, are defined as $s_j(B) = \mu_j(B^*B)^{\frac{1}{2}}$, where $\mu_j(B^*B)$ denotes the j-th positive eigenvalue of B^*B, arranged nonincreasingly and repeated according to multiplicities. The so-called weak Schatten class consists of the compact operators B such that

$$s_j(B) \leq Cj^{-1/p} \quad \text{for all } j; \qquad \text{we set } \mathbf{N}_p(B) = \sup_{j\in\mathbb{N}} s_j(B)j^{1/p}. \tag{2.9}$$

The notation $\mathfrak{S}_{(p)}(H, H_1)$ was used in [10] for this space; instead we here use the name $\mathfrak{S}_{p,\infty}(H, H_1)$ (as in [17] and in other works). The indication (H, H_1) is replaced by (H) if $H = H_1$; it can be omitted when it is clear from the context. One has that $\mathfrak{S}_{p,\infty} \subset \mathfrak{C}_{p+\varepsilon}$ for any $\varepsilon > 0$. They are linear spaces.

We recall (cf. e.g. [10] for details and references) that $\mathbf{N}_p(B)$ is a quasinorm on $\mathfrak{S}_{p,\infty}$, with a good control over the behavior under summation. Recall also that

$$\mathfrak{S}_{p,\infty}\cdot\mathfrak{S}_{q,\infty}\subset\mathfrak{S}_{r,\infty},\quad\text{where } r^{-1}=p^{-1}+q^{-1},\tag{2.10}$$

and

$$s_j\big(B^*\big)=s_j(B),\qquad s_j(EBF)\le\|E\|s_j(B)\|F\|,\tag{2.11}$$

when $E\colon H_1\to H_3$ and $F\colon H_2\to H$ are bounded linear maps between Hilbert spaces.

Moreover, we recall that when $\varXi$ is a bounded open subset of $\mathbb{R}^m$ and reasonably regular, or is a compact smooth m-dimensional manifold with boundary, then the injection $H^t(\varXi)\hookrightarrow L_2(\varXi)$ is in $\mathfrak{S}_{m/t,\infty}$ when $t>0$. It follows that when B is a linear operator in $L_2(\varXi)$ that is bounded from $L_2(\varXi)$ to $H^t(\varXi)$, then $B\in\mathfrak{S}_{m/t,\infty}$, with

$$\mathbf{N}_{m/t}(B)\le C\|B\|_{\mathcal{L}(L_2(\varXi),H^t(\varXi))}.\tag{2.12}$$

Recall also the Weyl–Ky Fan perturbation result:

$$s_j(B)j^{1/p}\to C_0,\qquad s_j\big(B'\big)j^{1/p}\to 0\quad\Longrightarrow\quad s_j\big(B+B'\big)j^{1/p}\to C_0,\quad\text{for } j\to\infty.\tag{2.13}$$

We shall moreover use Laptev's result [20]: When P is a classical ψdo of order $t<0$ on a closed m-dimensional manifold $\varXi_1$ with a smooth subset $\varXi$, $m\ge 2$, then

$$1_{\varXi_1\setminus\varXi}P1_{\varXi}\in\mathfrak{S}_{(m-1)/t,\infty};\tag{2.14}$$

in fact it has a Weyl-type asymptotic formula of that order.

Results on the spectral behavior of compositions of ψdo's of negative order interspersed with functions with jumps were shown in [14], see in particular Theorem 4.3 there. We need to supply this result with a statement allowing a zero-order factor of the form of a sum of a pseudodifferential and a singular Green operator (in the Boutet de Monvel calculus); as functions with jumps we here just take 1_Ω.

Theorem 2.6. *Let M_Ω be an operator on $\overline{\Omega}$ composed of $l\ge 1$ factors $P_{j,+}$ formed of classical pseudodifferential operators P_j on Ω_1 of negative orders $-t_j$ and truncated to $\overline{\Omega}$, $j=1,\dots,l$, and one factor Q_++G (placed somewhere between them), where Q is classical of order 0 and G is a singular Green operator on $\overline{\Omega}$ of order and class 0:*

$$M_\Omega=P_{1,+}\dots P_{l_0,+}(Q_++G)P_{l_0+1,+}\dots P_{l,+}.\tag{2.15}$$

Let $t=t_1+\dots+t_l$, and let $m(x,\xi)$ be the product of the principal ψdo symbols on Ω_1:

$$m(x,\xi)=p_{1,0}(x,\xi)\dots q_0(x,\xi)\dots p_{l,0}(x,\xi).$$

Then M_Ω has the spectral behavior:

$$s_j(M_\Omega)j^{t/n}\to c(M_\Omega)^{t/n}\quad\text{for } j\to\infty,\tag{2.16}$$

where

$$c(M_\Omega)=\frac{1}{n(2\pi)^n}\int\limits_\Omega\int\limits_{|\xi|=1}\big(m(x,\xi)^*m(x,\xi)\big)^{n/2t}\,d\omega(\xi)dx.\tag{2.17}$$

Proof. By Theorem 4.3 of [14] with interspersed functions of the form 1_Ω, the statement holds if $Q = 1$ and $G = 0$, so the new thing is to include nontrivial cases of Q and G. We can assume that $l_0 \geq 1$. For the contribution from Q we write

$$P_{l_0,+}Q_+ = r^+P_{l_0}e^+r^+Qe^+ = r^+P_{l_0}Qe^+ - r^+P_{l_0}e^-r^-Qe^+. \tag{2.18}$$

Here $P_{l_0}Q$ is a ψdo of order $-t_{l_0} < 0$ with principal symbol $p_{l_0,0}q_0$, and when $r^+P_{l_0}Qe^+$ is taken into the original expression, we get an operator of the type treated by Theorem 4.3 of [14],

$$P_{1,+}\dots(P_{l_0}Q)_+P_{l_0+1,+}\dots P_{l,+}, \tag{2.19}$$

for which the statement (2.16), (2.17) holds. For the other term in (2.18), we use that $r^+P_{l_0}e^-$ is the type of operator covered by the theorem of Laptev [20] (cf. (2.14)), belonging to $\mathfrak{S}_{(n-1)/t_{l_0},\infty}$, and r^-Qe^+ is bounded in L_2, so in view of the rules (2.10) and (2.11) for compositions, the full expression with this term inserted is in $\mathfrak{S}_{n/(t+\theta),\infty}$ for a certain $\theta > 0$. The spectral asymptotic estimate obtained for the term (2.19) is preserved when we add this term of a better weak Schatten class, in view of (2.13).

The contribution from G will likewise be shown to be in a better weak Schatten class that the main ψdo term; this requires a deeper effort. Actually, the strategy can be copied from some proofs in [17], as follows: Consider first the composition of G with just one operator:

$$M = P_+G,$$

where P is of order $-t < 0$. In local coordinates, we can extend Theorem 4.1 in [17] to this operator, writing

$$\psi P_+G\psi_1 = \sum_{k\in\mathbb{N}_0}\psi P_+K_k\Phi_k^*\psi_1 = \sum_{k\in\mathbb{N}_0}\psi P_+\zeta K_k\Phi_k^*\psi_1 + \sum_{k\in\mathbb{N}_0}\psi P_+K_k(1-\zeta)\Phi_k^*\psi_1,$$

with Poisson and trace operators K_k and Φ_k^* as explained in [17], and letting P_+K_k play the role of K_k in the proof there. Here $(\psi P_+K_k\zeta)^*$ is bounded from $L_2(B_{R,+})$ to $\overline{H}^t(B'_{R'})$ for a large R', hence lies in $\mathfrak{S}_{(n-1)/t,\infty}$ (by the property of the injection of $\overline{H}^t(B'_{R'})$ into $L_2(B'_{R'})$, $B'_{R'} = \{x' \in \mathbb{R}^{n-1} \mid |x'| < R'\}$). The proof that the full series P_+G lies in $\mathfrak{S}_{(n-1)/t,\infty}$ goes as in [17] (using also that the terms with $1-\zeta$ have a smoothing component, and that the series is rapidly convergent). Moreover, Corollary 4.2 there shows how the result is extended to the manifold situation.

When there are several factors in M, we need only use that $P_{j,+} \in \mathfrak{S}_{n/t_j,\infty}$ for the other factors and apply the product rule (2.10), and we end with the information that the full product is in $\mathfrak{S}_{n/(t+\theta),\infty}$ for some $\theta > 0$, so that the spectral asymptotics remains as that of (2.19), when this term is added on. □

The result extends easily to matrix-formed operators.

Now we can show a spectral asymptotic estimate for $P_{a,\mathrm{Dir}}$.

Theorem 2.7. *Let P_a satisfy Assumption 2.1. Assume that $P_{a,\mathrm{Dir}}$ is invertible, or more generally that $P_{a,\mathrm{Dir}}+c$ is invertible from $D(P_{a,\mathrm{Dir}})$ to $L_2(\Omega)$ for some $c \in \mathbb{C}$ (this holds if P_a is strongly elliptic).*

The singular values $s_j(P_{a,\mathrm{Dir}})$ (eigenvalues of $(P_{a,\mathrm{Dir}}^*P_{a,\mathrm{Dir}})^{\frac{1}{2}}$) have the asymptotic behavior:

$$s_j(P_{a,\mathrm{Dir}}) = C(P_{a,\mathrm{Dir}})j^{2a/n} + o\big(j^{2a/n}\big), \quad \text{for } j \to \infty, \tag{2.20}$$

where $C(P_{a,\mathrm{Dir}}) = C'(P_{a,\mathrm{Dir}})^{-2a/n}$, defined from the principal symbol $p_{a,0}(x,\xi)$ by

$$C'(P_{a,\mathrm{Dir}}) = \frac{1}{n(2\pi)^n}\int_\Omega\int_{|\xi|=1}\big|p_{a,0}(x,\xi)\big|^{-n/2a}\,d\omega(\xi)dx. \tag{2.21}$$

Proof. By Theorem 4.4 of [16], $P_{a,\mathrm{Dir}}$, acting like r^+P_a, has a parametrix of order $-2a$,

$$R = \Lambda_{+,+}^{(-a)}(\widetilde{Q}_+ + G)\Lambda_{-,+}^{(-a)} = r^+\Lambda_+^{(-a)}e^+\big(r^+\widetilde{Q}e^+ + G\big)r^+\Lambda_-^{(-a)}e^+; \tag{2.22}$$

in the last expression, we have written the restriction- and extension-operators out in detail. In comparison with the formula for R in [16, Th. 4.4], we have moreover placed an r^+ in front, which is allowed since R maps into a space of functions supported in $\overline{\Omega}$. (The singular Green operator component G was missing in some preliminary versions of [16].) The operator is of the form treated in Theorem 2.6, which gives the asymptotic behavior of the s-numbers of R:

$$s_j(R)j^{2a/n} \to c(R)^{2a/n} \quad \text{for } j \to \infty; \tag{2.23}$$

here $c(R) = C'(P_{a,\mathrm{Dir}})$ defined in (2.21), since the principal symbol of $\Lambda_+^{(-a)}\widetilde{Q}\Lambda_-^{(-a)}$ is the inverse of the principal symbol of P_a.

That R is parametrix of $r^+P_a = P_{a,\mathrm{Dir}}$ implies that

$$P_{a,\mathrm{Dir}}R = I - S_1, \quad \text{where } S_1 \colon L_2(\Omega) \to C^\infty(\overline{\Omega}). \tag{2.24}$$

Consider the case where $P_{a,\mathrm{Dir}}$ is invertible; it is clearly compact since it maps $L_2(\Omega)$ into $\dot{H}^a(\overline{\Omega})$. It follows from (2.24) that

$$P_{a,\mathrm{Dir}}^{-1} = P_{a,\mathrm{Dir}}^{-1}(P_{a,\mathrm{Dir}}R + S_1) = R + S_2, \quad S_2 = P_{a,\mathrm{Dir}}^{-1}S_1,$$

where $P_{a,\mathrm{Dir}}^{-1} \in \mathfrak{S}_{n/a,\infty}$ (since it maps $L_2(\Omega)$ into $\dot{H}^a(\overline{\Omega})$), and $S_1 \in \bigcap_{p>0}\mathfrak{S}_{p,\infty}$, so $S_2 \in \bigcap_{p>0}\mathfrak{S}_{p,\infty}$ by (2.10). By (2.13), the spectral asymptotic formula (2.23) for R will therefore imply the same spectral asymptotic formula for $P_{a,\mathrm{Dir}}^{-1}$, so

$$s_j\big(P_{a,\mathrm{Dir}}^{-1}\big)j^{2a/n} \to C'\big(P_{a,\mathrm{Dir}}^{-1}\big)^{2a/n}.$$

The asymptotic formula can also be written as the formula (2.20) for the s-numbers of $P_{a,\mathrm{Dir}}$.

If instead $P_{a,\mathrm{Dir}} + c$ is invertible, we can write

$$(P_{a,\mathrm{Dir}} + c)R = I - S_1 + cR,$$

with S_1 as in (2.24), and hence

$$\begin{aligned}(P_{a,\mathrm{Dir}} + c)^{-1} &= (P_{a,\mathrm{Dir}} + c)^{-1}\big((P_{a,\mathrm{Dir}} + c)R + S_1 - cR\big)\\ &= R + (P_{a,\mathrm{Dir}} + c)^{-1}S_1 - c(P_{a,\mathrm{Dir}} + c)^{-1}R.\end{aligned}$$

Here $(P_{a,\mathrm{Dir}} + c)^{-1}S_1 \in \bigcap_{p>0}\mathfrak{S}_{p,\infty}$ and $c(P_{a,\mathrm{Dir}} + c)^{-1}R \in \mathfrak{S}_{n/3a,\infty}$, since $(P_{a,\mathrm{Dir}} + c)^{-1} \in \mathfrak{S}_{n/a,\infty}$, and $R \in \mathfrak{S}_{n/2a,\infty}$ in view of its spectral behavior shown above. Thus $(P_{a,\mathrm{Dir}} + c)^{-1}$ is a perturbation of R by operators in better weak Schatten classes, and the desired spectral results follow for $(P_{a,\mathrm{Dir}} + c)^{-1}$ and its inverse $P_{a,\mathrm{Dir}} + c$. □

When $P_{a,\mathrm{Dir}}$ is selfadjoint ≥ 0, its eigenvalue sequence λ_j, $j \in \mathbb{N}$, coincides with the sequence of s_j-values, and Theorem 2.7 gives an asymptotic estimate of the eigenvalues.

In this case, the asymptotic estimate extends to arbitrary open sets Ω (assumed bounded when $\Omega_1 = \mathbb{R}^n$), with the Dirichlet realization defined by Friedrichs extension of r^+P_a from $C_0^\infty(\Omega)$, since the eigenvalues can be characterized by the minimax principle, which gives a monotonicity property in terms of nested open sets.

As mentioned in the introduction, the estimate (2.20) was shown for the case $P_a = (-\Delta)^a$ by Blumenthal and Getoor in [3]. In this case, a two-terms asymptotic formula for the N-th average of eigenvalues as $N \to \infty$ was obtained by Frank and Geisinger in [7], and Geisinger extended the estimate (2.20) to a larger class of constant-coefficient ψdo's in [8].

Remark 2.8. Theorem 2.7 extends straightforwardly to Dirichlet realizations of operators P as in Theorem 2.5; in the proof, the factor $\Lambda_{-,+}^{(-a)}$ is replaced by $\Lambda_{-,+}^{(-b)}$, and $2a$ in the asymptotic formula is replaced by $m = a + b$.

3. Mixed problems for second-order symmetric strongly elliptic differential operators

3.1. The Krein resolvent formula

We shall now apply the knowledge of the operators of type $\frac{1}{2}$ to the mixed boundary value problem for second-order elliptic differential operators. The setting is the following:

On a bounded C^∞-smooth open subset Ω of $\mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$ we consider a second-order symmetric differential operator with real coefficients in $C^\infty(\overline{\Omega})$:

$$Au = -\sum_{j,k=1}^{n} \partial_j \big(a_{jk}(x)\partial_k u\big) + a_0(x)u, \tag{3.1}$$

here $a_{jk} = a_{kj}$ for all j, k. A is assumed strongly elliptic, i.e., $\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2$ for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$, with $c_0 > 0$. We denote as usual $u|_\Sigma = \gamma_0 u$, and consider moreover the conormal derivative ν and a Robin variant χ (both are Neumann-type boundary operators)

$$\nu u = \sum_{j,k=1}^{n} n_j \gamma_0(a_{jk}\partial_k u), \qquad \chi u = \nu u - \sigma\gamma_0 u; \tag{3.2}$$

here $\vec{n} = (n_1, \dots, n_n)$ denotes the interior unit normal to the boundary, and σ is a real C^∞-function on Σ. With Σ_+ denoting a closed C^∞-subset of Σ, we define $L_2(\Omega)$-realizations A_γ and A_{χ,Σ_+} of A determined respectively by the boundary conditions:

$$\begin{aligned} \gamma_0 u = 0 \quad &\text{on } \Sigma, \quad \text{the Dirichlet condition,} \\ \chi u = 0 \quad &\text{on } \Sigma_+, \qquad \gamma_0 u = 0 \quad \text{on } \Sigma \setminus \Sigma_+, \quad \text{a mixed condition.} \end{aligned} \tag{3.3}$$

It is accounted for in [13] that with the domains defined more precisely by

$$\begin{aligned} D(A_\gamma) &= \big\{u \in \overline{H}^2(\Omega) \bigm| \gamma_0 u = 0\big\}, \\ D(A_{\chi,\Sigma_+}) &= \big\{u \in \overline{H}^1(\Omega) \cap D(A_{\max}) \bigm| \gamma_0 u \in \dot{H}^{\frac{1}{2}}(\Sigma_+),\ \chi u = 0 \text{ on } \Sigma_+^\circ\big\}, \end{aligned} \tag{3.4}$$

where $A_{\max}$ denotes the operator acting like A with domain $D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$, the operators A_γ and A_{χ,Σ_+} are selfadjoint lower bounded. We can and shall assume that a sufficiently large constant has been added to A such that both operators have a positive lower bound.

Let

$$X = \dot{H}^{-\frac{1}{2}}(\Sigma_+); \quad \text{then } X^* = \overline{H}^{\frac{1}{2}}\big(\Sigma_+^\circ\big), \tag{3.5}$$

with respect to a duality consistent with the L_2-scalar product on Σ_+. The injection $\mathrm{i}_X\colon X \hookrightarrow H^{-\frac{1}{2}}(\Sigma)$ can be viewed as an "extension by zero" e^+ (often tacitly understood), and its adjoint $(\mathrm{i}_X)^*\colon H^{\frac{1}{2}}(\Sigma) \to \overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ is the restriction r^+.

Recalling that γ_0 defines a homeomorphism from $Z = \ker(A_{\max}) = \{u \in L_2(\Omega) \mid Au = 0\}$ to $H^{-\frac{1}{2}}(\Sigma)$ with inverse K_γ (a Poisson operator), we define

$$V = K_\gamma X, \quad \gamma_V : V \xrightarrow{\sim} X; \tag{3.6}$$

here V is a closed subspace of Z (both closed in the $L_2(\Omega)$-norm), and γ_V denotes the restriction of γ_0 to V. Note that γ_V^{-1} acts like K_γ on X; it is also denoted by $K_{\gamma,X}$ in [13]. We denote by i_V the injection of V into Z, its adjoint is the orthogonal projection pr_V of Z onto V. Let us moreover introduce the relevant Dirichlet-to-Neumann operators

$$P_{\gamma,\nu} = \nu K_\gamma, \qquad P_{\gamma,\chi} = \chi K_\gamma = P_{\gamma,\nu} - \sigma; \tag{3.7}$$

they are pseudodifferential operators of order 1 on Σ, both formally selfadjoint.

The following Krein resolvent formula was shown in [13, Sect. 4.1]:

Proposition 3.1. *For the realizations of A defined above,*

$$A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma = \mathrm{i}_V \gamma_V^{-1} L^{-1} \big(\gamma_V^{-1}\big)^* \mathrm{pr}_V. \tag{3.8}$$

Here L is the (selfadjoint invertible) operator from X to X^* acting like $-r^+ P_{\gamma,\chi} e^+$ and with domain

$$D(L) = \gamma_0 D(A_{\chi,\Sigma_+}).$$

It was shown in [13] that $D(L) \subset \dot{H}^{1-\varepsilon}(\Sigma_+)$ for all $\varepsilon > 0$, but that the inclusion does not hold with $\varepsilon = 0$.

Since L acts like $-P_{\gamma,\chi,+}$ and is surjective onto $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$, we also have

$$D(L) = \big\{\varphi \in \dot{H}^{1-\varepsilon}(\Sigma_+) \bigm| r^+ P_{\gamma,\chi}\varphi \in \overline{H}^{\frac{1}{2}}\big(\Sigma_+^\circ\big)\big\}. \tag{3.9}$$

Below we shall improve the knowledge of the domain by setting $P_{\gamma,\chi}$ in relation to the types of operators studied in Section 2.

3.2. Structure of the Dirichlet-to-Neumann operator

To study the symbol of $P_{\gamma,\chi}$ we consider the operators in a neighborhood U of a point $x_0 \in \partial\Omega = \Sigma$, where local coordinates $x = (x_1 \dots, x_n) = (x', x_n)$ are chosen such that $U \cap \Omega = \{(x', x_n) \mid x' \in B_1,\ 0 < x_n < 1\}$ and $U \cap \partial\Omega = \{(x', x_n) \mid x' \in B_1,\ x_n = 0\}$; $B_1 = \{x' \in \mathbb{R}^{n-1} \mid |\xi'| < 1\}$. In these coordinates, the principal symbol of A at the boundary is a polynomial

$$\begin{aligned} \underline{a}(x', 0, \xi) &= \sum_{j,k=1}^{n} \underline{a}_{jk}(x', 0)\xi_j\xi_k = \underline{a}_{nn}(x', 0)\xi_n^2 + 2b(x', \xi')\xi_n + c(x', \xi'), \\ &\text{with } b = \sum_{j=1}^{n-1} \underline{a}_{jn}(x')\xi_j,\ \ c = \sum_{j,k=1}^{n-1} \underline{a}_{jk}(x')\xi_j\xi_k; \end{aligned} \tag{3.10}$$

the coefficients are real with $\underline{a}_{jk} = \underline{a}_{kj}$. We often write $(x', 0)$ as x'. Since A is strongly elliptic, $\underline{a}(x', \xi', \xi_n) > 0$ when $\xi' \neq 0$, so the polynomial $\underline{a}(x', \xi', \lambda)$ in λ has no real roots when $\xi' \neq 0$. When we set

$$a'(x', \xi') = \underline{a}_{nn}(x')c(x', \xi') - b(x', \xi')^2 = \sum_{j,k=1}^{n-1} a'_{jk}(x')\xi_j\xi_k,$$

we therefore have that $a'(x',\xi') > 0$ for $\xi' \in \mathbb{R}^{n-1} \setminus 0$. The roots of $\underline{a}(x',\xi',\lambda)$ equal $\lambda_\pm = \underline{a}_{nn}^{-1}(-b \pm i\kappa_0)$, lying respectively in $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda \gtrless 0\}$, where we have set

$$\kappa_0(x',\xi') = a'(x',\xi')^{\frac{1}{2}} > 0. \tag{3.11}$$

Denote

$$\kappa_\pm(x',\xi') = \mp i\lambda_\pm = \underline{a}_{nn}^{-1}(\kappa_0 \pm ib); \tag{3.12}$$

then $\underline{a}$ has the factorization

$$\underline{a}(x',\xi',\xi_n) = \underline{a}_{nn}(x')\big(\kappa_+(x',\xi') + i\xi_n\big)\big(\kappa_-(x',\xi') - i\xi_n\big), \tag{3.13}$$

where κ_+ and κ_- both have positive real part ($= \kappa_0$). This plays a role in standard investigations of boundary problems. We go on to study the Dirichlet-to-Neumann operators.

The principal symbol-kernel $\tilde{k}_\gamma(x',x_n,\xi')$ of K_γ is the solution operator for the semi-homogeneous model problem (with φ given in $\mathbb{C}$):

$$\underline{a}(x',\xi',D_n)u(x_n) = 0 \quad \text{on } \mathbb{R}_+, \qquad u(0) = \varphi;$$

it is seen from (3.13) that the solution in $L_2(\mathbb{R}_+)$ is $\varphi e^{-\kappa_+ x_n}$, so

$$\tilde{k}_\gamma(x',x_n,\xi') = e^{-\kappa_+ x_n}. \tag{3.14}$$

The conormal derivative for the model problem is

$$\nu u = \gamma_0\left(\underline{a}_{nn}\partial_{x_n}u(x_n) + \sum_{k=1}^{n-1}\underline{a}_{nk}i\xi_k u(x_n)\right).$$

Then the principal symbol of $P_{\gamma,\nu}$ is

$$\begin{aligned} p_{\gamma,\nu}(x',\xi')_0 &= \gamma_0\left(\underline{a}_{nn}\partial_{x_n} + \sum_{k=1}^{n-1}\underline{a}_{nk}i\xi_k\right)e^{\kappa_+ x_n} \\ &= -\underline{a}_{nn}\kappa_+ + \sum_{k=1}^{n-1}\underline{a}_{nk}i\xi_k \\ &= -\underline{a}_{nn}(-i)\underline{a}_{nn}^{-1}(-b + i\kappa_0) + ib \\ &= -\kappa_0. \end{aligned}$$

Since $P_{\gamma,\chi} = P_{\gamma,\nu} - \sigma$ with σ of order 0, $P_{\gamma,\chi}$ likewise has the principal symbol $-\kappa_0$.

The important fact that we observe here is that $\kappa_0(x',\xi')$ is *even* in ξ';

$$\kappa_0(x',-\xi') = \kappa_0(x',\xi'), \quad \text{with } \partial_{x'}^\beta\partial_{\xi'}^\alpha\kappa_0(x',-\xi') = (-1)^{|\alpha|}\partial_{x'}^\beta\partial_{\xi'}^\alpha\kappa_0(x',\xi') \text{ for all } \alpha,\beta \tag{3.15}$$

(since $c(x',\xi')$ and $b(x',\xi')^2$ are clearly even in ξ'). Since κ_0 is homogeneous of degree 1, it therefore has the $\frac{1}{2}$-transmission property with respect to any smooth subset of B_1, satisfying (1.3) with $m = 1$, $\mu = \frac{1}{2}$.

Moreover, we shall show that it has factorization index $\frac{1}{2}$ with respect to any smooth subset of B_1: We can take the subset as $B_{1,+} = \{x' \in \mathbb{R}^{n-1} \mid |x'| < 1,\ x_{n-1} > 0\}$, with $(x_1,\dots,x_{n-2})$ denoted by x''. Now we apply the same procedure as above to the polynomial $a'(x'', 0, \xi') = \kappa_0(x'', 0, \xi'', \xi_{n-1})^2$ in ξ_{n-1}. It has a factorization analogously to (3.13):

$$\kappa_0\big(x'', 0, \xi'\big)^2 = a'_{n-1,n-1}\big(x''\big)\big(\kappa'_+\big(x'', \xi''\big) + i\xi_{n-1}\big)\big(\kappa'_-\big(x'', \xi''\big) - i\xi_{n-1}\big),$$

where $a'_{n-1,n-1} > 0$ and $\kappa'_\pm$ have positive real part; here $\kappa'_\pm = \mp i\lambda'_\pm$, where $\lambda'_\pm$ are the roots of $a'(x'', 0, \xi'', \lambda)$ lying in $\mathbb{C}_\pm$, respectively. It follows that

$$\kappa_0\big(x'', 0, \xi'\big) = a'_{n-1,n-1}\big(x''\big)^{\frac{1}{2}}\big(\kappa'_+\big(x'', \xi''\big) + i\xi_{n-1}\big)^{\frac{1}{2}}\big(\kappa'_-\big(x'', \xi''\big) - i\xi_{n-1}\big)^{\frac{1}{2}}, \tag{3.16}$$

where $(\kappa'_+(x'', \xi'') + i\xi_{n-1})^{\frac{1}{2}}$ extends analytically in ξ_{n-1} into $\mathbb{C}_-$ and $(\kappa'_-(x'', \xi'') - i\xi_{n-1})^{\frac{1}{2}}$ extends analytically in ξ_{n-1} into $\mathbb{C}_+$ (in short, are a "plus-symbol" resp. a "minus-symbol", cf. [6,16]).

Carrying the information back to Ω and $\Sigma = \partial\Omega$, we have obtained:

Theorem 3.2. *The principal symbol of the Dirichlet-to-Neumann operator $P_{\gamma,\chi}$ equals $-\kappa_0(x', \xi')$ (expressed in local coordinates in (3.10)–(3.11)), negative and elliptic of order 1. For any smooth subset Σ_+ of Σ, κ_0 is of type $\frac{1}{2}$ and has factorization index $\frac{1}{2}$ relative to Σ_+. An explicit factorization in local coordinates is given in (3.16).*

3.3. Precisions on L and L^{-1}

Define L_1 to be a ψdo on Σ with symbol $\kappa_0(x', \xi')$, and let $L_0 = -P_{\gamma,\chi} - L_1$. Then since L acts like $-P_{\gamma,\chi,+}$, it acts like $L_{1,+} + L_{0,+}$:

$$L\varphi = L_{1,+}\varphi + L_{0,+}\varphi, \quad \text{for } \varphi \in D(L). \tag{3.17}$$

Here L_1, classical of order 1, is principally equal to $-P_{\gamma,\chi}$ and $-P_{\gamma,\nu}$, whereas the operator L_0 is a classical ψdo of order 0, containing both the local term σ and the nonlocal difference between $P_{\gamma,\nu}$ and its principal part.

As shown in Theorem 3.2, L_1 is of type $\frac{1}{2}$ and has factorization index $\frac{1}{2}$ relative to Σ_+. Here $L_{1,+}$, when considered on $\dot H^{1-\varepsilon}(\Sigma_+)$, identifies with the operator r^+L_1 in the homogeneous Dirichlet problem for L_1, going from $\dot H^{1-\varepsilon}(\Sigma_+)$ to $\dot H^{-\varepsilon}(\Sigma_+)$. It has according to [16, Th. 4.4] a parametrix $R: \overline{H}^{s-1}(\Sigma_+^\circ) \to H^{\frac{1}{2}(s)}(\Sigma_+)$ for $s > \frac{1}{2}$; here $H^{\frac{1}{2}(s)}(\Sigma_+) = \dot H^s(\Sigma_+)$ for $\frac{1}{2} < s < 1$, cf. (2.3), and R is of the form

$$R = \Lambda_{+,+}^{(-\frac{1}{2})}(\widetilde{Q}_+ + G)\Lambda_{-,+}^{(-\frac{1}{2})}, \tag{3.18}$$

with a ψdo $\widetilde{Q}$ of order and type 0 and a singular Green operator G of order and class 0. The parametrix property implies that

$$\begin{aligned} L_{1,+}R &= I - S_1, \quad S_1: \overline{H}^t(\Sigma_+) \to C^\infty(\Sigma_+), \quad \text{for } t > -\frac{1}{2}, \\ RL_{1,+} &= I - S_2, \quad S_2: \dot H^{1+t}(\Sigma_+) \to \mathcal{E}_{\frac{1}{2}}(\Sigma_+), \quad \text{for } -\frac{1}{2} < t < 0, \\ &\qquad\qquad\quad S_2: H^{\frac{1}{2}(1+t)}(\Sigma_+) \to \mathcal{E}_{\frac{1}{2}}(\Sigma_+), \quad \text{for } t \ge 0. \end{aligned} \tag{3.19}$$

From (3.17) and the first line in (3.19), we have for the difference S_3 of L^{-1} and R:

$$S_3 = L^{-1} - R = L^{-1}(L_{1,+}R + S_1) - L^{-1}(L_{1,+} + L_{0,+})R = L^{-1}S_1 - L^{-1}L_{0,+}R. \tag{3.20}$$

Some properties of L^{-1} can be obtained by considerations similar to those in [13]:

Proposition 3.3. *The operator $L^{-1}: X^* \to X$ extends to an operator M_0 that maps continuously*

$$M_0: \overline{H}^s(\Sigma_+^\circ) \to \dot{H}^{s+\frac{1}{2}-\varepsilon}(\Sigma_+) \quad \text{for } -1 < s \le \frac{1}{2}, \text{ any } \varepsilon > 0.$$

In particular, the closure of L^{-1} in $L_2(\Sigma_+)$ is a continuous operator from $L_2(\Sigma_+)$ to $\dot{H}^{\frac{1}{2}-\varepsilon}(\Sigma_+)$.

The operators L^{-1} and M_0 have the same eigenfunctions (for nonzero eigenvalues); they belong to $D(L)$.

Proof. We already know from [13] (cf. (3.9)) that L^{-1} is continuous from $X^* = \overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ to $\dot{H}^{1-\varepsilon}(\Sigma_+)$. Then it has an adjoint M_0 (with respect to dualities consistent with the $L_2(\Sigma_+)$-scalar product) that is continuous from $\overline{H}^{-1+\varepsilon}(\Sigma_+^\circ)$ to $\dot{H}^{-\frac{1}{2}}(\Sigma_+)$. But since L^{-1} is known to be selfadjoint (from X^* to X, consistently with the L_2-scalar product), M_0 must be an extension of L^{-1}. Now the asserted continuity for $-1 < s \le \frac{1}{2}$ follows by interpolation. For $s = 0$ this shows the mapping property of the L_2-closure.

When φ is a distribution in $\overline{H}^{-1+\varepsilon}(\Sigma_+^\circ)$ such that $M_0\varphi = \lambda\varphi$ for some $\lambda \neq 0$, then since $M_0\varphi \in \overline{H}^{-\frac{1}{2}+\varepsilon}(\Sigma_+^\circ) = \dot{H}^{-\frac{1}{2}+\varepsilon}(\Sigma_+)$, φ lies there. Next, it follows that $M_0\varphi \in \overline{H}^{\varepsilon_1}(\Sigma_+^\circ) = \dot{H}^{\varepsilon_1}(\Sigma_+)$, and hence φ also lies there. Finally, we conclude that $M_0\varphi \in \overline{H}^{\frac{1}{2}+\varepsilon_2}(\Sigma_+^\circ)$, so that φ also lies there. Here M_0 coincides with L^{-1}. $\square$

We can now find exact information on the domain of L:

Theorem 3.4. *L^{-1} maps $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ onto $H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$. In other words, the domain of L satisfies*

$$D(L) = H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+) = \Lambda_+^{(-\frac{1}{2})} e^+ \overline{H}^1(\Sigma_+^\circ), \tag{3.21}$$

which is contained in $d^{\frac{1}{2}} e^+ \overline{H}^1(\Sigma_+^\circ) + \dot{H}^{\frac{3}{2}}(\Sigma_+)$.

Proof. It is seen from the second line in (3.19) that $S_3 = L^{-1} - R$ is also described by

$$S_3 = (RL_{1,+} + S_2)L^{-1} - R(L_{1,+} + L_{0,+})L^{-1} = S_2L^{-1} - RL_{0,+}L^{-1}. \tag{3.22}$$

Here S_2L^{-1} maps $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ into $\mathcal{E}_{\frac{1}{2}}(\Sigma_+)$ in view of (3.19). For the other term, we note that $L_{0,+}$ maps $\dot{H}^{1-\varepsilon}(\Sigma_+)$ into $\overline{H}^{1-\varepsilon}(\Sigma_+^\circ)$, since an extension by zero is understood, and R maps the latter space into $H^{\frac{1}{2}(2-\varepsilon)}(\Sigma_+)$. Thus S_3 maps $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ into $H^{\frac{1}{2}(2-\varepsilon)}(\Sigma_+)$. Since R maps $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ into $H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$, it follows that L^{-1} maps $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$ into $H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$. Thus $D(L) \subset H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$.

The opposite inclusion also holds, since r^+L_1 maps $H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$ into $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$, and $H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+) \subset \dot{H}^{\frac{1}{2}}(\overline{\Omega})$ by Lemma 2.2, which r^+L_0 maps into $\overline{H}^{\frac{1}{2}}(\Sigma_+^\circ)$.

This shows the identity. The last statement follows from (2.3). $\square$

Remark 3.5. By this information we can explain more precisely in which way $D(L)$, known to be contained in $\dot{H}^{1-\varepsilon}(\Sigma_+)$, reaches outside of $\dot{H}^1(\Sigma_+)$, namely by certain nontrivial elements of $d^{\frac{1}{2}} e^+ \overline{H}^1(\Sigma_+^\circ)$ (lying in $H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$).

Consider the spaces in local coordinates, where Σ and Σ_+ are replaced by $\mathbb{R}^{n-1}$ and $\overline{\mathbb{R}}^{n-1}_+$. As a typical element of $x_{n-1}^{\frac{1}{2}} e^+ \overline{H}^1(\mathbb{R}^{n-1}_+)$ lying in $H^{\frac{1}{2}(\frac{3}{2})}(\overline{\mathbb{R}}^{n-1}_+)$, we can take

$$\varphi(x') = c x_{n-1}^{\frac{1}{2}} K_0 \psi, \quad c = \Gamma\left(\frac{3}{2}\right)^{-1}, \tag{3.23}$$

where $\psi(x'') \in H^{\frac{1}{2}}(\mathbb{R}^{n-2})$. Here K_0 is the Poisson operator from $H^{\frac{1}{2}}(\mathbb{R}^{n-2})$ to $\overline{H}^1(\mathbb{R}^{n-1}_+)$ solving

$$(1 - \Delta)\zeta(x') = 0 \quad \text{on } \mathbb{R}^{n-1}_+, \qquad \gamma_0\zeta = \psi \quad \text{on } \mathbb{R}^{n-2},$$

namely

$$\zeta = K_0\psi = \mathcal{F}^{-1}_{\xi'\to x'}\big(\big(\langle\xi''\rangle + i\xi_{n-1}\big)^{-1}\hat\psi(\xi'')\big) = \mathcal{F}^{-1}_{\xi''\to x''}\big(e^{-\langle\xi''\rangle x_{n-1}}\hat\psi(\xi'')\big),$$

and $\varphi(x') = cx_{n-1}^{\frac12}\zeta(x')$.

To verify that $\varphi(x') \in H^{\frac12(\frac32)}(\overline{\mathbb{R}}^{n-1}_+)$, we recall from [16, Sect. 5], that the special boundary operator $\gamma_{\frac12,0}\colon H^{\frac12(\frac32)}(\overline{\mathbb{R}}^{n-1}_+) \to H^{\frac12}(\mathbb{R}^{n-2})$ defined there satisfies

$$\gamma_{\frac12,0}\varphi = c^{-1}\gamma_0\big(x_{n-1}^{-\frac12}\varphi(x')\big) = \gamma_0\Xi_+^{\frac12}\varphi, \quad \text{with } \Xi_+^\mu = \mathrm{OP}\big(\big(\langle\xi''\rangle + i\xi_{n-1}\big)^\mu\big),$$

and has the right inverse $K_{\frac12,0}$, where

$$\varphi = K_{\frac12,0}\psi = \Xi_+^{-\frac12}e^+K_0\psi = cx_{n-1}^{\frac12}K_0\psi,$$

cf. [16], Corollary 5.3, and the analysis in the sequel there.

Now φ defined by (3.23) is not in $\dot H^1$ (nor in $\overline{H}^1$) near $x_{n-1} = 0$, since

$$\partial_{x_{n-1}}\varphi(x') = \frac12 x_{n-1}^{-\frac12}\zeta(x') + x_{n-1}^{\frac12}\partial_{x_{n-1}}\zeta(x'),$$

where $x_{n-1}^{\frac12}\partial_{x_{n-1}}\zeta(x')$ is clearly L_2-integrable over $\mathbb{R}^{n-2}\times[0,1]$, but $x_{n-1}^{-\frac12}\zeta(x')$ is not so:

$$\begin{aligned}\int\limits_{\mathbb{R}^{n-2}}\int\limits_{0<x_{n-1}<1}\big|x_{n-1}^{-\frac12}\zeta\big|^2\,dx_{n-1}dx'' &= (2\pi)^{2-n}\lim_{\delta\to0}\int\limits_{\mathbb{R}^{n-2}}\int\limits_{\delta<x_{n-1}<1} x_{n-1}^{-1}e^{-2\langle\xi''\rangle x_{n-1}}\big|\hat\psi(\xi'')\big|^2\,dx_{n-1}d\xi'' \\ &\ge (2\pi)^{2-n}\lim_{\delta\to0}\int\limits_{\mathbb{R}^{n-2}}\int\limits_{\delta<x_{n-1}<1} x_{n-1}^{-1}e^{-2\langle\xi''\rangle}\big|\hat\psi(\xi'')\big|^2\,dx_{n-1}d\xi'' \\ &= (2\pi)^{2-n}\lim_{\delta\to0}|\log\delta|\int\limits_{\mathbb{R}^{n-2}} e^{-2\langle\xi''\rangle}\big|\hat\psi(\xi'')\big|^2\,d\xi'' = +\infty, \end{aligned} \tag{3.24}$$

when $\psi \ne 0$. (It does not help to take ψ very smooth.)

We consequently have for $D(A_{\chi,\Sigma_+})$:

Corollary 3.6. *The domain of A_{χ,Σ_+} satisfies*

$$D(A_{\chi,\Sigma_+}) \subset D(A_\gamma) + K_\gamma H^{\frac12(\frac32)}(\Sigma_+) \subset \overline{H}^2(\Omega) + K_\gamma\big(e^+d(x')^{\frac12}\overline{H}^1(\Sigma_+^\circ)\big) \tag{3.25}$$

(where we recall that e^+ denotes the extension from Σ_+ by zero on Σ_-, and $d(x')$ is a C^∞-function on Σ_+ proportional to $\operatorname{dist}(x', \partial\Sigma_+)$ near $\partial\Sigma_+$).

All elements of $K_\gamma H^{\frac12(\frac32)}(\Sigma_+)$ are reached from $D(A_{\chi,\Sigma_+})$.

Nontrivial elements of $K_\gamma(e^+d(x')^{\frac12}\overline{H}^1(\Sigma_+^\circ))$ are reached, that are not in $K_\gamma\dot H^1(\Sigma_+)$, nor in $K_\gamma(e^+\overline{H}^1(\Sigma_+^\circ))$ (as in Remark 3.5), hence not in $\overline{H}^{\frac32}(\Omega)$.

Proof. It is known from [9, Th. II.1.2] that

$$D(A_{\chi,\Sigma_+}) \subset D(A_\gamma) \dotplus D(T) = D(A_\gamma) \dotplus K_\gamma D(L),$$

when we use that $A_\gamma = A_\beta$ and $K_\gamma D(L) = D(T)$ with the notation used there. Here all elements of $D(T)$ are reached, in the sense that for any $z \in D(T)$ there is a $v \in D(A_\gamma)$ such that $u = v + z \in D(A_{\chi,\Sigma_+})$. Since $D(L) = H^{\frac{1}{2}(\frac{3}{2})}(\Sigma_+)$, this shows the first inclusion in (3.25) and the first statement afterwards.

For the remaining part we use the last information in Theorem 3.4. Since $K_\gamma \dot{H}^{\frac{3}{2}} \subset \overline{H}^2(\Omega)$, this implies the second inclusion in (3.25). Remark 3.5 shows how nontrivial nonsmooth elements occur. □

3.4. The spectrum of the Krein term

The spectral asymptotic behavior of the Krein term

$$M = A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma = \mathrm{i}_V \gamma_V^{-1} L^{-1} \big(\gamma_V^{-1}\big)^* \mathrm{pr}_V \tag{3.26}$$

will now be determined. We assume $n \geq 3$ in this section since applications on Σ of Laptev's result quoted in (2.14) requires the dimension m to be ≥ 2, i.e., $n - 1 \geq 2$. It is used to show that some cut-off terms have a better asymptotic behavior than the one we are aiming for, hence can be disregarded. (We believe that there are ways to handle the case $n - 1 = 1$, either by establishing weaker versions of (2.14), or by using the variable-coefficient factorization of the principal symbol of L, but we refrain from making an effort here. The case $n = 2$ was included in [13] for A principally Laplacian.)

First we study the spectrum of the factor L^{-1}.

Theorem 3.7. *S_3 belongs to $\mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon),\infty}$, and L^{-1} belongs to $\mathfrak{S}_{n-1,\infty}$ (when the operators are extended to $L_2(\Sigma_+)$ by closure).*

The eigenvalues of L^{-1} have the asymptotic behavior:

$$\mu_j\big(L^{-1}\big) j^{1/(n-1)} \to c(L)^{1/(n-1)} \quad \text{for } j \to \infty, \tag{3.27}$$

where

$$c(L) = \frac{1}{(n-1)(2\pi)^{n-1}} \int\limits_{\Sigma_+} \int\limits_{|\xi'|=1} \kappa_0\big(x', \xi'\big)^{-(n-1)}\, d\omega\big(\xi'\big) dx'. \tag{3.28}$$

Proof. Recall that L^{-1} acts as follows:

$$L^{-1} = R + S_3 = \Lambda_{+,+}^{(-\frac{1}{2})}(\widetilde{Q}_+ + G)\Lambda_{-,+}^{(-\frac{1}{2})} + S_3, \tag{3.29}$$

cf. (3.18). By application of Theorem 2.6 to R we find that the singular values $s_j(R)$ behave as in (3.27)–(3.28), where the constant is as in (3.28) since the principal pseudodifferential symbol of R is κ_0^{-1}. In particular, $R \in \mathfrak{S}_{n-1,\infty}$.

Since the closure of L^{-1} maps $L_2(\Sigma_+)$ continuously into $\dot{H}^{\frac{1}{2}-\varepsilon}(\Sigma_+)$ by Proposition 3.3, it belongs to $\mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon),\infty}$. Moreover (cf. (3.19)), $S_1 \in \bigcap_{\tau>0} \mathfrak{S}_{\tau,\infty}$, and $L_{0,+}$ is bounded in $L_2(\Sigma_+)$. Then $L^{-1}S_1$ is in $\bigcap_{\tau>0} \mathfrak{S}_{\tau,\infty}$, and $L^{-1}L_{0,+}R \in \mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon),\infty} \cdot \mathfrak{S}_{n-1,\infty} \subset \mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon),\infty}$ by the rule (2.10), using that S_1 and $L_{0,+}R$ map into spaces where L^{-1} coincides with its L_2-closure. Therefore by (3.20),

$$S_3 \in \mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon),\infty}.$$

Now since L^{-1} acts like $R + S_3$, its closure is in $\mathfrak{S}_{n-1,\infty}$. This shows the first statement in the theorem.

The last statement follows, since S_3 is of a better Schatten class than R, so that (2.13) implies that the L_2-closure of L^{-1} has the same asymptotic behavior of singular values as R. Since L^{-1} is symmetric in L_2, the L_2-closure is selfadjoint, so its singular values are eigenvalues; they are consistent with the eigenvalues of L^{-1} by Proposition 3.3. □

We now turn to the Krein term M recalled in (3.26). Proceeding as in [13, Sect. 5.4], we have for the eigenvalues:

$$\mu_j(M) = \mu_j\big(\mathrm{i}_V \gamma_V^{-1} L^{-1} \big(\gamma_V^{-1}\big)^* \mathrm{pr}_V\big) = \mu_j\big(L^{-1}\big(\gamma_V^{-1}\big)^* \gamma_V^{-1}\big) = \mu_j\big(L^{-1} P_{1,+}\big),$$

where $P_1 = K_\gamma^* K_\gamma$ is a selfadjoint nonnegative invertible elliptic ψdo of order -1; in view of (3.14) it has principal symbol $(\kappa_+ + \bar{\kappa}_+)^{-1} = \underline{a}_{nn}(2\kappa_0)^{-1}$. Let $P_2 = P_1^{\frac{1}{2}}$, then we continue the calculation as follows:

$$\mu_j(M) = \mu_j\big(L^{-1} r^+ P_2 P_2 e^+\big) = \mu_j\big(P_2 e^+ L^{-1} r^+ P_2\big) = \mu_j\big(r^+ P_2 e^+ L^{-1} r^+ P_2 e^+ + S_4\big),$$

where S_4 is a sum of three terms, each one a product of ψdo's and cutoff functions of a total order -2, and each containing a factor either $r^- P_2 e^+$ or $r^+ P_2 e^-$ (or both). To the terms in S_4 we can apply (2.14) together with product rules, concluding that they are in $\mathfrak{S}_{(n-1)/(2+\theta),\infty}$ for some $\theta > 0$.

The operator (cf. (3.25))

$$M_1 = r^+ P_2 e^+ L^{-1} r^+ P_2 e^+ = P_{2,+} \Lambda_{+,+}^{(-\frac{1}{2})} (\widetilde{Q}_+ + G) \Lambda_{-,+}^{(-\frac{1}{2})} P_{2,+} + P_{2,+} S_3 P_{2,+}$$

is selfadjoint nonnegative, so its eigenvalues μ_j coincide with the s-values. We can apply Theorem 2.6 to the first term, obtaining a spectral asymptotic formula (2.16)–(2.17) with t/n replaced by $2/(n-1)$; then the addition of the second term which lies in a better weak Schatten class $\mathfrak{S}_{(n-1)/(2+\theta),\infty}$ preserves the formulas.

Finally M (likewise selfadjoint nonnegative) differs from M_1 by the operator S_4 in a better weak Schatten class, so the spectral asymptotic formula carries over to this operator.

Hereby we obtain the theorem:

Theorem 3.8. *The eigenvalues of* $M = A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1}$ *have the asymptotic behavior:*

$$\mu_j(M) j^{2/(n-1)} \to c(M)^{2/(n-1)} \quad \textit{for } j \to \infty, \tag{3.30}$$

where

$$c(M) = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+} \int_{|\xi'|=1} \left(\frac{\underline{a}_{nn}(x')}{2\kappa_0(x',\xi')^2}\right)^{(n-1)/2} d\omega(\xi') dx'. \tag{3.31}$$

Proof. It remains to account for the value of the constant $c(M)$. It follows, since $P_2^2 = P_1$ has principal symbol $\underline{a}_{nn}(2\kappa_0)^{-1}$ and the ψdo part of L^{-1} has principal symbol κ_0^{-1}. $\square$

Remark 3.9. We take the opportunity to recall two corrections to [13] (already mentioned in [14]): Page 351, line 4 from below, delete "$H^{\frac{1}{2}}(\Sigma_+^\circ) \subset$", replace "$H^1(\Sigma)$" by "$L_2(\Sigma)$". Page 361, line 4, replace "(Th. 3.3)" by "(Th. 4.3)".

References

[1] M.S. Birman, Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions, Vestnik Leningrad. Univ. 17 (1962) 22–55; English translation in: Spectral Theory of Differential Operators, in: Amer. Math. Soc. Transl. Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 19–53.

[2] M.S. Birman, M.Z. Solomyak, Asymptotic behavior of the spectrum of pseudodifferential operators with anisotropically homogeneous symbols, Vestnik Leningrad. Univ. 13 (1977) 13–21; English translation in: Vestnik Leningrad. Univ. Math. 10 (1982) 237–247.

[3] B.M. Blumenthal, R.K. Getoor, The asymptotic distribution of the eigenvalues for a class of Markov operators, Pacific J. Math. 9 (1959) 399–408.

[4] M. Bonforte, Y. Sire, J.L. Vazquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, arXiv:1404.6195.
[5] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math. 126 (1971) 11–51.
[6] G. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, Amer. Math. Soc., Providence, RI, 1981.
[7] R.L. Frank, L. Geisinger, Refined semiclassical asymptotics for fractional powers of the Laplace operator, J. Reine Angew. Math. (2014), in press, arXiv:1105.5181.
[8] L. Geisinger, A short proof of Weyl's law for fractional differential operators, J. Math. Phys. 55 (2014) 011504.
[9] G. Grubb, A characterization of the non-local boundary value problems associated with an elliptic operator, Ann. Sc. Norm. Super. 22 (1968) 425–513.
[10] G. Grubb, Singular Green operators and their spectral asymptotics, Duke Math. J. 51 (1984) 477–528.
[11] G. Grubb, Functional Calculus of Pseudodifferential Boundary Problems, second ed., Progr. Math., vol. 65, Birkhäuser, Boston, 1996, first edition issued 1986.
[12] G. Grubb, Distributions and Operators, Grad. Texts in Math., vol. 252, Springer, New York, 2009.
[13] G. Grubb, The mixed boundary value problem, Krein resolvent formulas and spectral asymptotic estimates, J. Math. Anal. Appl. 382 (2011) 339–363.
[14] G. Grubb, Spectral asymptotics for Robin problems with a discontinuous coefficient, J. Spectr. Theory 1 (2011) 155–177.
[15] G. Grubb, Perturbation of essential spectra of exterior elliptic problems, Appl. Anal. 90 (2011) 103–123.
[16] G. Grubb, Fractional Laplacians on domains, a development of Hörmander's theory of mu-transmission pseudodifferential operators, arXiv:1310.0951, in press.
[17] G. Grubb, Spectral asymptotics for nonsmooth singular Green operators, Comm. Partial Differential Equations 39 (2014) 530–573.
[18] G. Grubb, Local and nonlocal boundary conditions for mu-transmission and fractional order elliptic pseudodifferential operators, Anal. PDE (2014), in press, arXiv:1403.7140.
[19] L. Hörmander, The Analysis of Linear Partial Differential Operators, III, Springer-Verlag, Berlin, New York, 1985.
[20] A. Laptev, Spectral asymptotics of a class of Fourier integral operators, Tr. Mosk. Mat. Obs. 43 (1981) 92–115; English translation in: Trans. Moscow Math. Soc. (1983) 101–127.
[21] X. Ros-Oton, J. Serra, Local integration by parts and Pohozaev identities for higher order fractional Laplacians, arXiv:1406.1107.
[22] R. Servadei, E. Valdinoci, A Brezis–Nirenberg result for non-local critical equations in low dimension, Commun. Pure Appl. Anal. 12 (2013).
[23] E. Shamir, Regularization of mixed second-order elliptic problems, Israel J. Math. 6 (1968) 150–168.
[24] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 2nd edition, J.A. Barth, Leipzig, 1995.