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Nonlocal Schrödinger equations in metric measure spaces ☆

Marcelo Actis *, Hugo Aimar, Bruno Bongioanni, Ivana Gómez

Instituto de Matemática Aplicada del Litoral, UNL-CONICET, FIQ, FICH, CCT CONICET Santa Fe, Predio "Dr. Alberto Cassano", Colectora Ruta Nac. 168 km 0, Paraje El Pozo, Santa Fe, Argentina

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A B S T R A C T

In this note we consider the pointwise convergence to the initial data for the solutions of some nonlocal dyadic Schrödinger equations on spaces of homogeneous type. We prove the a.e. convergence when the initial data belongs to a dyadic version of an L^2 based Besov space.

1. Introduction

In quantum mechanics the position of a particle in the *space* is described by the probability density function $|\varphi|^2 = \varphi\overline{\varphi}$ where φ is a solution of the *Schrödinger equation*. In the classical free particle model, the *space* is the Euclidean and the *Schrödinger equation* is associated to the Laplace operator, i.e. $i\frac{\partial \varphi}{\partial t} = \triangle\varphi$. Hence the probability of finding the particle inside the Borel set E of the Euclidean space at time t is given by $\int_E |\varphi(x,t)|^2\, dx$.

The pointwise convergence to the initial data for the classical Schrödinger equation in Euclidean settings is a hard problem. It is well known that some regularity in the initial data is needed [6,9,11,8,19,22,20].

Nonlocal operators instead of the Laplacian in this basic model have been considered previously in the Euclidean space (see for example [15] and references in [21]). The nonlocal fractional derivatives as substitutes of the Laplacian become natural objects when the space itself lacks any differentiable structure and only an analysis of order less than one can be carried out.

We shall be brief in our introduction of the basic setting. For a more detailed approach see [1]. Let (X, d, μ) be a space of homogeneous type (see [17]). Let $\mathscr{D}$ be a dyadic family in X as constructed by

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* Corresponding author.
E-mail addresses: mactis@santafe-conicet.gov.ar (M. Actis), haimar@santafe-conicet.gov.ar (H. Aimar), bbongio@santafe-conicet.gov.ar (B. Bongioanni), ivanagomez@santafe-conicet.gov.ar (I. Gómez).

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M. Christ in [7]. Let $\mathscr{H}$ be a Haar system for $L^2(X,\mu)$ associated to $\mathscr{D}$ as built in [2] (see also [5,3]). Following the basic notation introduced in Section 1, we shall use $\mathscr{H}$ itself as the index set for the analysis and synthesis of signals defined on X. Precisely, by $Q(h)$ we denote the member of $\mathscr{D}$ on which h *is based*. With $j(h)$ we denote the integer scale j for which $Q(h) \in \mathscr{D}^j$.

The system $\mathscr{H}$ is an orthonormal basis for L^2_0, where L^2_0 coincides with $L^2(X,\mu)$ if $\mu(X) = +\infty$ and $L^2_0 = \{f \in L^2 : \int_X f d\mu = 0\}$ if $\mu(X) < \infty$. For a given $Q \in \mathscr{D}$ the number of wavelets h based on Q is $\#\vartheta(Q) - 1$, where $\vartheta(Q)$ is the offspring of Q and $\#\vartheta(Q)$ is its cardinal. The homogeneity property of the space together with the metric control of the dyadic sets guarantees a uniform upper bound for $\#\vartheta(Q)$. On the other hand $\#\vartheta(Q) \geq 1$ for every $Q \in \mathscr{D}$.

Let $(X, d, \mu, \mathscr{D}, \mathscr{H})$ be given as before. For the sake of simplicity we shall assume along this paper that X itself is a quadrant for $\mathscr{D}$. We say that X itself is a quadrant if any two cubes in X have a common ancestor. A distance in X associated to $\mathscr{D}$ can be defined by $\delta(x,y) = \min\{\mu(Q) : Q \in \mathscr{D}$ such that $x, y \in Q\}$ when $x \neq y$ and $\delta(x,x) = 0$. The next lemma, borrowed from [1], reflects the one dimensional character of X equipped with δ and μ.

Lemma 1. *(See Lemma 3.1 in [1].) Let $0 < \varepsilon < 1$, and let Q be a given dyadic cube in X. Then, for $x \in Q$, we have*

$$\int_{X\setminus Q} \frac{d\mu(y)}{\delta(x,y)^{1+\varepsilon}} \simeq \mu(Q)^{-\varepsilon}.$$

Furthermore the integral of $\delta^{-1}(x,\cdot)$ diverges on each dyadic cube containing x and, when the measure of X is not finite, on the complement of each dyadic cube.

For a complex value function f Lipschitz continuous with respect to δ define

$$D^\beta f(x) = \int_X \frac{f(x) - f(y)}{\delta(x,y)^{1+\beta}} d\mu(y).$$

One of the key results relating the operator D^β with the Haar system is provided by the following spectral theorem contained in [1].

Theorem 2. *(See Theorem 3.1 in [1].) Let $0 < \beta < 1$. For each $h \in \mathscr{H}$ we have*

$$D^\beta h(x) = m_h \mu(Q(h))^{-\beta} h(x), \tag{1}$$

where m_h is a constant that may depend on $Q(h)$ but there exist two finite and positive constants M_1 and M_2 such that

$$M_1 < m_h < M_2, \qquad \text{for all } h \in \mathscr{H}. \tag{2}$$

Set $B^\lambda_2(X, \delta, \mu)$ to denote the space of those functions $f \in L^2(X,\mu)$ satisfying

$$\iint_{X\times X} \frac{|f(x) - f(y)|^2}{\delta^{1+2\lambda}(x,y)} d\mu(x)d\mu(y) < \infty.$$

The projection operator defined on L^2 onto V_0 the subspace of functions which are constant on each cube $Q \in \mathscr{D}^0$ is denoted by P_0. We are now in position to state the main results of this paper.

Theorem 3. *Let $0 < \beta < \lambda < 1$ and $u_0 \in B_2^\lambda(X, \delta, \mu)$ with $P_0 u_0 = 0$ be given. Then, the function u defined on $\mathbb{R}^+$ by*

$$u(t) = -\sum_{h \in \mathscr{H}} e^{-itm_h \mu(Q(h))^{-\beta}} \langle u_0, h \rangle h, \tag{3}$$

(3.a) belongs to $B_2^\lambda(X, \delta, \mu)$ for every $t > 0$;
(3.b) solves the problem

$$\begin{cases} i\dfrac{du}{dt} = D^\beta u, \ t > 0, \\ u(0) = u_0, \quad \text{on } X. \end{cases} \tag{4}$$

More precisely, $\frac{du}{dt}$ is the Fréchet derivative of $u(t)$ as a function of $t \in (0, \infty)$ with values in $B_2^{\lambda-\beta}(X, \delta, \mu)$ and $\lim_{t \to 0^+} u(t) = u_0$ in $B_2^\lambda(X, \delta, \mu)$.

Theorem 4. *Let $0 < \beta < \lambda < 1$ and $u_0 \in B_2^\lambda(X, \delta, \mu)$ with $P_0 u_0 = 0$ be given. Then,*

(4.a) there exists $Z \subset X$ with $\mu(Z) = 0$ such that the series (3) defining $u(t)$ converges pointwise for every $t \in [0, 1)$ outside Z;
(4.b) $u(t) \to u_0$ pointwise almost everywhere on X with respect to μ when $t \to 0$.

Let us point out that the operator D^β in the framework of p-adic analysis is called the Vladimirov operator and has been extensively studied in recent years, see e.g., [14] and references therein. It has already been proved that the collection of the eigenfunctions of D^β coincides with a wavelet basis. Let us also notice that from a probabilistic approach, in [12,13], the Haar wavelets have been identified as eigenfunctions of differential operators. On the other hand, since the work of S. Petermichl in [18], where a representation for the Hilbert transform is given as an average of dyadic shifts, dyadic techniques has proved to be fundamental and useful in harmonic analysis (see [16] and references therein). From this point of view, our problem can be regarded as a first approximation to the general nonlocal Schrödinger model in Ahlfors metric measure spaces, where instead of δ the metric is the underlying distance d in the space.

The paper is organized in three sections. The second one is devoted to introduce a characterization in terms of Haar coefficients of the dyadic Besov spaces $B_2^\lambda(X, \delta, \mu)$. In Section 3 we prove our main results, which are contained in Theorems 3 and 4. In the last section we also illustrate our results in the Sierpinski gasket.

2. Characterization of the Besov space $B_2^\sigma(X, \delta, \mu)$ in terms of Haar coefficients

The aim of this section is to characterize $B_2^\sigma(X, \delta, \mu)$ in terms of the sequence $\{\langle f, h \rangle : h \in \mathscr{H}\}$ for $0 < \sigma < 1$.

Theorem 5. *Let $0 < \sigma < 1$ be given. The space $B_2^\sigma(X, \delta, \mu)$ coincides with the subspace of $L^2(X, \mu)$ of those functions f for which*

$$\sum_{h \in \mathscr{H}} \frac{|\langle f, h \rangle|^2}{\mu(Q(h))^{2\sigma}} < \infty.$$

Moreover,

$$\|f\|^2_{B^\sigma_2(X,\delta,\mu)} \simeq \|f\|^2_{L^2(X,\mu)} + \sum_{h\in\mathscr{H}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}.$$

Let us state some important lemmas that will be useful for the proof of the above theorem.

Lemma 6. *Let $0<\sigma<1$ and $h,\widetilde{h}\in\mathscr{H}$ be given. Then*

$$\nu(h,\widetilde{h}) := \iint\limits_{X\times X} \frac{[h(x)-h(y)][\widetilde{h}(x)-\widetilde{h}(y)]}{\delta(x,y)^{1+2\sigma}} d\mu(y)d\mu(x) = 0,$$

if $h\neq\widetilde{h}$, and

$$\nu(h,h) = \iint\limits_{X\times X} \frac{[h(x)-h(y)]^2}{\delta(x,y)^{1+2\sigma}} d\mu(y)d\mu(x) \simeq \mu(Q(h))^{-2\sigma},$$

where the equivalence constants depend only on the geometric constants of the space.

Proof. For the first part of the proof, i.e. $\nu(h,\widetilde{h})=0$, when $h\neq\widetilde{h}$, we shall divide our analysis in three cases according to the relative positions of $Q(h):=Q$ and $Q(\widetilde{h}):=\widetilde{Q}$, (i) $Q=\widetilde{Q}$, (ii) $Q\cap\widetilde{Q}=\emptyset$ and (iii) $Q\subsetneqq\widetilde{Q}$.

Let us start by (i). Set $\Pi_{h\widetilde{h}}(x,y):=[h(x)-h(y)][\widetilde{h}(x)-\widetilde{h}(y)]$. Notice that for x and y in $X\setminus Q$ we have $\Pi_{h\widetilde{h}}(x,y)=0$. On the other hand, for $x\in Q$ and $y\in X\setminus Q$ we have $\Pi_{h\widetilde{h}}(x,y)=h(x)\widetilde{h}(x)$. Hence

$$\begin{aligned}\iint\limits_{X\times X} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &= \iint\limits_{Q\times(X\setminus Q)} \frac{h(x)\widetilde{h}(x)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\ &\quad + \iint\limits_{(X\setminus Q)\times Q} \frac{h(y)\widetilde{h}(y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\ &\quad + \iint\limits_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\ &= \iint\limits_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y),\end{aligned}$$

since for the first term $\delta(x,y)$ is constant as a function of $x\in Q$ for y fixed in $X\setminus Q$, for the second $\delta(x,y)$ is constant as a function of $y\in Q$ for x fixed in $X\setminus Q$ and h and $\widetilde{h}$ are orthogonal. Let us prove that

$$\iint\limits_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = 0.$$

Since h and $\widetilde{h}$ are constant on $Q'\in\vartheta(Q)$ we have that $\Pi_{h\widetilde{h}}(x,y)=0$ for $(x,y)\in Q'\times Q'$, hence

$$\iint\limits_{Q\times Q} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \sum_{Q'\in\vartheta(Q)} \sum_{Q''\in\vartheta(Q)} \int\limits_{Q''}\int\limits_{Q'} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)$$

$$
\begin{aligned}
&= \sum_{Q' \in \vartheta(Q)} \int_{Q'} \int_{Q'} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x) d\mu(y) \\
&\qquad + \sum_{Q' \in \vartheta(Q)} \sum_{\substack{Q'' \in \vartheta(Q) \\ Q' \neq Q''}} \int_{Q''} \int_{Q'} \frac{\Pi_{h\widetilde{h}}(x,y)}{\mu(Q)^{1+2\sigma}} d\mu(x) d\mu(y) \\
&= \mu(Q)^{-1-2\sigma} \sum_{Q'' \in \vartheta(Q)} \int_{Q''} \left(\sum_{Q' \in \vartheta(Q)} \int_{Q'} \Pi_{h\widetilde{h}}(x,y) d\mu(x) \right) d\mu(y) \\
&= \mu(Q)^{-1-2\sigma} \int_Q \int_Q [h(x)\widetilde{h}(x) - h(y)\widetilde{h}(x) \\
&\qquad - h(x)\widetilde{h}(y) + h(y)\widetilde{h}(y)] d\mu(x) d\mu(y) \\
&= 0.
\end{aligned} \tag{5}
$$

Let us now consider the case (ii), that is $Q \cap \widetilde{Q} = \emptyset$. In this case $\Pi_{h\widetilde{h}}(x,y)$ is supported in $(\widetilde{Q} \times Q) \cup (Q \times \widetilde{Q})$. Moreover on $\widetilde{Q} \times Q$ we have $\Pi_{h\widetilde{h}}(x,y) = -h(y)\widetilde{h}(x)$ and on $Q \times \widetilde{Q}$, $\Pi_{h\widetilde{h}}(x,y) = -h(x)\widetilde{h}(y)$. Since, on the other hand $\delta(x,y) = \delta(Q, \widetilde{Q})$ which is a positive constant on the support of $\Pi_{h\widetilde{h}}$, we get

$$
\begin{aligned}
&\iint_{X \times X} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x) d\mu(y) \\
&\quad = -\frac{1}{\delta(Q,\widetilde{Q})^{1+2\sigma}} \left\{ \iint_{\widetilde{Q} \times Q} [h(y)\widetilde{h}(x)] d\mu(x) d\mu(y) + \iint_{Q \times \widetilde{Q}} [h(x)\widetilde{h}(y)] d\mu(x) d\mu(y) \right\} \\
&\quad = 0.
\end{aligned}
$$

Consider now the case (iii). Since $Q \subsetneq \widetilde{Q}$, then $\widetilde{h}$ is constant on Q, hence

$$
\iint_{X \times X} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x) d\mu(y) = \iint_{(X \setminus Q) \times Q} + \iint_{Q \times Q} + \iint_{Q \times (X \setminus Q)} .
$$

Since $\widetilde{h}$ is constant on Q, $\Pi_{h\widetilde{h}}$ is identically zero on $Q \times Q$ and the second term vanishes. For the first term notice that it can be written as

$$
\begin{aligned}
&\int_{X \setminus Q} \left(\int_Q \frac{(-h(y))(\widetilde{h}(x) - \widetilde{h}(y_Q))}{\delta(Q,x)^{1+2\sigma}} d\mu(y) \right) d\mu(x) \\
&\quad = -\int_{X \setminus Q} \frac{\widetilde{h}(x) - \widetilde{h}(y_Q)}{\delta(Q,x)^{1+2\sigma}} d\mu(x) \left(\int_Q h(y) d\mu(y) \right) \\
&\quad = 0,
\end{aligned}
$$

here y_Q denotes any fixed point in Q. For the third term we have similarly

$$\iint\limits_{Q\times(X\setminus Q)} \frac{\Pi_{h\widetilde{h}}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \int\limits_{X\setminus Q} \left(\int\limits_{Q} \frac{(h(x))(\widetilde{h}(x_Q)-\widetilde{h}(y))}{\delta(Q,y)^{1+2\sigma}} d\mu(x) \right) d\mu(y)$$
$$= 0.$$

Finally we have to show that $\nu(h,h) \simeq \mu(Q(h))^{-2\sigma}$. Let $Q = Q(h)$. Notice first for $(x,y) \in (X\setminus Q)\times(X\setminus Q)$ we have $\Pi_{hh}(x,y) = 0$. Hence

$$\begin{aligned}
\iint\limits_{X\times X} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &= \iint\limits_{(X\setminus Q)\times Q} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) + \iint\limits_{Q\times Q} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&\quad + \iint\limits_{Q\times(X\setminus Q)} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= 2\int\limits_{X\setminus Q} \left(\int\limits_{Q} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x) \right) d\mu(y) + \iint\limits_{Q\times Q} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= 2I + II.
\end{aligned}$$

Let us first get an estimate for I. Notice that for any $x, z \in Q$ and $y \in X \setminus Q$, we have that $\delta(x,y) = \delta(z,y)$, hence

$$I = \int\limits_{X\setminus Q} \frac{d\mu(y)}{\delta(z,y)^{1+2\sigma}} \int\limits_{Q} |h(x)|^2 \, d\mu(x),$$

which is equivalent to $\mu(Q)^{-2\sigma}$ by Lemma 1. To get the desired bound for II, we observe that equation (5) holds for $h = \widetilde{h}$ also, then

$$\begin{aligned}
\iint\limits_{Q\times Q} \frac{\Pi_{hh}(x,y)}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &= \mu(Q)^{-1-2\sigma} \iint\limits_{Q\times Q} [h^2(x) + h^2(y) - 2h(x)h(y)] d\mu(x)d\mu(y) \\
&= 2\mu(Q)^{-2\sigma}. \qquad \square
\end{aligned}$$

For $Q \in \mathscr{D}$, set $\mathscr{H}_Q = \{h \in \mathscr{H} : Q(h) \subseteq Q\}$. Let $\mathcal{S}(\mathscr{H}_Q)$ denote the linear span of $\mathscr{H}_Q$. Since $\mathscr{H}_Q$ is countable we write ℓ^2 to denote the space $\ell^2(\mathscr{H}_Q)$ of all square summable complex sequences indexed on $\mathscr{H}_Q$. On the other hand, consider the weighted space $\mathcal{L}^2(Q\times Q) := L^2(Q\times Q, \frac{d\mu(x)d\mu(y)}{\delta(x,y)})$. The next lemma shows that for φ and ψ in $\mathcal{S}(\mathscr{H}_Q)$ the inner product of $\frac{\varphi(x)-\varphi(y)}{\delta(x,y)^\sigma}$ with $\frac{\psi(x)-\psi(y)}{\delta(x,y)^\sigma}$ in $\mathcal{L}^2(Q\times Q)$ is equivalent to the inner product of $\frac{\langle\varphi,h\rangle}{\mu(Q(h))^\sigma}$ with $\frac{\langle\psi,h\rangle}{\mu(Q(h))^\sigma}$ in $\ell^2(\mathscr{H}_Q)$.

Lemma 7. *Let $0 < \sigma < 1$ and $Q \in \mathscr{D}$ be given. For φ, ψ two functions in $\mathcal{S}(\mathscr{H}_Q)$ we have that*

$$\iint\limits_{Q\times Q} \frac{[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)]}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \sum_{h\in\mathscr{H}_Q} \langle\varphi,h\rangle \langle\psi,h\rangle \nu(h,h).$$

In particular,

$$\iint\limits_{Q\times Q} \frac{|\varphi(x)-\varphi(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \simeq \sum_{h\in\mathscr{H}_Q} \frac{|\langle\varphi,h\rangle|^2}{\mu(Q(h))^{2\sigma}}.$$

Proof. Since φ and ψ are in the linear span of $\mathscr{H}_Q$ we easily see that

$$[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)] = \sum_{h\in\mathscr{H}_Q}\sum_{\widetilde{h}\in\mathscr{H}_Q} \langle\varphi,h\rangle \left\langle\psi,\widetilde{h}\right\rangle [h(x)-h(y)][\widetilde{h}(x)-\widetilde{h}(y)]$$

Dividing both members of the above equation by $\delta(x,y)^{1+2\sigma}$, integrating on the product space $Q\times Q$ and then applying Lemma 6 we get

$$\begin{aligned}
&\iint_{Q\times Q} \frac{[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)]}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= \sum_{h\in\mathscr{H}_Q}\sum_{\widetilde{h}\in\mathscr{H}_Q} \langle\varphi,h\rangle \left\langle\psi,\widetilde{h}\right\rangle \iint_{Q\times Q} \frac{(h(x)-h(y))(\widetilde{h}(x)-\widetilde{h}(y))}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= \sum_{h\in\mathscr{H}_Q} \langle\varphi,h\rangle \langle\psi,h\rangle \iint_{Q\times Q} \frac{[h(x)-h(y)]^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&= \sum_{h\in\mathscr{H}_Q} \langle\varphi,h\rangle \langle\psi,h\rangle \nu(h,h). \qquad \square
\end{aligned}$$

Lemma 8. *For $\psi\in\mathcal{S}(\mathscr{H})$ there exists $\varepsilon>0$ such that $\psi(x)-\psi(y)$ vanishes on $\Delta_\epsilon = \{(x,y)\in X\times X : \delta(x,y)<\varepsilon\}$.*

Proof. It is enough to check the result for $\psi=h\in\mathscr{H}$. But since h is constant on each child Q' of $Q(h)$ we have that $h(x)-h(y)=0$ on $\bigcup_{Q' \text{ child of } Q} Q'\times Q'$. On the other hand, $h(x)-h(y)=0$ for x and y both outside $Q(h)$. Hence $h(x)-h(y)$ vanishes on $\{(x,y):\delta(x,y)<\mu(Q(h))\}$. $\square$

The next result shows that Lemma 7 extends to the case of φ in L^2.

Lemma 9. *Let Q be a cube in $\mathscr{D}$, $f\in L^2(Q,\mu)$ and $\psi\in\mathcal{S}(\mathscr{H}_Q)$. Then*

$$\iint_{Q\times Q} \frac{(f(x)-f(y))(\psi(x)-\psi(y))}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \sum_{h\in\mathscr{H}_Q} \langle f,h\rangle \langle\psi,h\rangle \nu(h,h).$$

Proof. Let $\mathscr{F}_n \nearrow \mathscr{H}_Q$ and $f_n = \sum_{h\in\mathscr{F}_n} \langle f,h\rangle h$. Since each $f_n\in\mathcal{S}(\mathscr{H}_Q)$ we have from Lemma 7 that

$$\iint_{Q\times Q} \frac{[f_n(x)-f_n(y)][\psi(x)-\psi(y)]}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) = \sum_{h\in\mathscr{H}_Q} \langle f,h\rangle \langle\psi,h\rangle \nu(h,h)$$

taking n large enough such that $\mathscr{F}_n$ contains all the h's building ψ. On the other hand, since, from Lemma 8, the function $\frac{\psi(x)-\psi(y)}{\delta(x,y)^{1+2\sigma}}\in L^\infty(Q\times Q)$ and $f_n(x)-f_n(y)\to f(x)-f(y)$ in $L^2(Q\times Q)$, hence in $L^1(Q\times Q)$ we get the desired equality. $\square$

The next result allows us to localize to dyadic cubes our characterization of the Besov space $B_2^\sigma(X,\delta,\mu)$ in terms of the Haar coefficients.

Lemma 10. *The set*

$$\triangle_1 = \{(x,y)\in X\times X : \delta(x,y)<1\} = \bigcup_{Q\in\mathscr{M}} Q\times Q,$$

where $\mathscr{M}$ is the family of all maximal dyadic cubes in $\mathscr{D}$ with $\mu(Q)<1$.

The next result is a statement of the intuitive fact that the regularity requirement additional to the L^2 integrability involved in the definition of $B^\sigma_2(X,\delta,\mu)$ is only relevant around the diagonal of $X \times X$.

Lemma 11. *For $0 < \sigma < 1$ an L^2 function f belongs to $B^\sigma_2(X,\delta,\mu)$ if and only if*

$$\mathscr{E}(f) = \iint\limits_{\delta(x,y)<1} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) < \infty.$$

Moreover, the $B^\sigma_2(X,\delta,\mu)$ norm of f is equivalent to $\|f\|_2 + [\mathscr{E}(f)]^{\frac{1}{2}}$.

Proof. The "only if" is obvious. Since

$$\iint\limits_{\delta(x,y)\geq 1} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \leq C \int\limits_{x\in X} |f(x)|^2 \left(\int\limits_{\delta(x,y)\geq 1} \frac{d\mu(y)}{\delta(x,y)^{1+2\sigma}} \right) d\mu(x).$$

From Lemma 1 we have that the first term in the above inequality is bounded by the L^2 norm of f, as desired. □

The following result is elementary but useful.

Lemma 12. *For an $L^2(X,\mu)$ function f the quantities $\|f\|_2^2 + \sum_{h\in\mathscr{H}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}$ and $\|f\|_2^2 + \sum_{\substack{h\in\mathscr{H}\\ \mu(Q(h))<1}} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}}$ are equivalent.*

The characterization of $B^\sigma_2(X,\delta,\mu)$ in terms of the Haar system is just a completion argument built on the result in Lemma 6.

Proof of Theorem 5. From Lemmas 8, 9 and 10 it is enough to prove that quantities

$$\iint\limits_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \tag{6}$$

and

$$\sum_{h\in\mathscr{H}_Q} \frac{|\langle f,h\rangle|^2}{\mu(Q(h))^{2\sigma}} \tag{7}$$

are equivalent for each cube Q in $\mathscr{M}$ with constants independent of Q.

Let us start by showing that the double integral in (6) is bounded by the sum (7). Let $(\mathscr{F}_n)$ be an increasing sequence of finite subfamilies of $\mathscr{H}_Q$ that covers $\mathscr{H}_Q$. In other words $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ and $\mathscr{H}_Q = \bigcup_n \mathscr{F}_n$. Let $f_n = \sum_{h\in\mathscr{F}_n} \langle f\chi_Q, h\rangle h$. Since f_n converge pointwise almost everywhere for x in Q to $f(x)$, then we have that $|f_n(x) - f_n(y)| \to |f(x) - f(y)|$ as $n \to \infty$ for $y \in Q$ also. Hence from Fatou's lemma and Lemma 7

$$\begin{aligned}\iint\limits_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &= \iint\limits_{Q\times Q} \lim_{n\to\infty} \frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\ &\leq \liminf_{n\to\infty} \iint\limits_{Q\times Q} \frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\end{aligned}$$

$$\leq C \liminf_{n\to\infty} \sum_{h\in\mathscr{H}_Q} \frac{|\langle f_n, h\rangle|^2}{\mu(Q(h))^{2\sigma}}$$

$$\leq C \sum_{h\in\mathscr{H}_Q} \frac{|\langle f, h\rangle|^2}{\mu(Q(h))^{2\sigma}}.$$

To prove the converse we shall argue by duality. Take $\boldsymbol{b} = \{b_h : h \in \mathscr{H}_Q\}$ a sequence of scalars indexed on $\mathscr{H}_Q$ with $b_h = 0$ except for a finite number of $h \in \mathscr{H}_Q$. Assume that $\sum_{h\in\mathscr{H}_Q} |b_h|^2 \leq 1$. Set $\psi = \sum_{h\in\mathscr{H}_Q} b_h \nu(h,h)^{-1/2} h \in \mathcal{S}(\mathscr{H}_Q)$. So that from Lemma 9,

$$\begin{aligned}
\sum_{h\in\mathscr{H}_Q} \langle f, h\rangle \nu(h,h)^{\frac{1}{2}} b_h &= \sum_{h\in\mathscr{H}_Q} \langle f, h\rangle \langle \psi, h\rangle \nu(h,h) \\
&= \iint_{Q\times Q} \frac{(f(x)-f(y))(\psi(x)-\psi(y))}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) \\
&\leq \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac{1}{2}} \\
&\quad \times \left(\iint_{Q\times Q} \frac{|\psi(x)-\psi(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac{1}{2}}.
\end{aligned}$$

Notice that from Lemma 7 and Lemma 6 we have,

$$\begin{aligned}
\iint_{Q\times Q} \frac{|\psi(x)-\psi(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y) &\leq C \sum_{h\in\mathscr{H}_Q} \frac{|\langle \psi, h\rangle|^2}{\mu(Q(h))^{2\sigma}} \\
&\leq C \sum_{h\in\mathscr{H}_Q} |b_h|^2 \left[\frac{\nu(h,h)^{-\frac{1}{2}}}{\mu(Q(h))^{\sigma}}\right]^2 \\
&\leq C.
\end{aligned}$$

Hence

$$\left|\sum_{h\in\mathscr{H}_Q} \langle f, h\rangle \nu(h,h)^{\frac{1}{2}} b_h\right| \leq C \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac{1}{2}},$$

so that

$$\left(\sum_{h\in\mathscr{H}_Q} |\langle f, h\rangle|^2 \nu(h,h)\right)^{\frac{1}{2}} \leq C \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac{1}{2}}$$

or

$$\left(\sum_{h\in\mathscr{H}_Q} \frac{|\langle f, h\rangle|^2}{\mu(Q(h))^{2\sigma}}\right)^{\frac{1}{2}} \leq C \left(\iint_{Q\times Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\sigma}} d\mu(x)d\mu(y)\right)^{\frac{1}{2}}$$

as desired.

3. Proofs of Theorems 3 and 4

Before starting to prove the main results, let us mention that from the result of the previous section the Sobolev type condition required in [1, Theorem 4.1] is the same as our current hypothesis $f \in B_2^\lambda(X,\delta,\mu)$ when $p = 2$. Then for any function $f \in B_2^\sigma(X,\delta,\mu)$ we can write

$$D^\sigma f(x) = \sum_{h\in\mathscr{H}} m_h \mu(Q(h))^{-\sigma} \langle f,h\rangle\, h(x),$$

where the series converges in $L^2(X,\mu)$.

Proof of Theorem 3. From Theorem 5 we see that for each $t > 0$, $u(t)$ defined in (3) belongs to $B_2^\lambda(X,\delta,\mu)$, since $u_0 \in B_2^\lambda(X,\delta,\mu)$. Moreover, for $t,s \geq 0$,

$$\begin{aligned}\|u(t)-u(s)\|^2_{B_2^\lambda(X,\delta,\mu)} &= \left\| \sum_{h\in\mathscr{H}} \left(e^{itm_h\mu(Q(h))^{-\beta}} - e^{ism_h\mu(Q(h))^{-\beta}} \right) \langle u_0,h\rangle\, h \right\|^2_{B_2^\lambda(X,\delta,\mu)} \\ &= \sum_{h\in\mathscr{H}} \left| e^{itm_h\mu(Q(h))^{-\beta}} - e^{ism_h\mu(Q(h))^{-\beta}} \right|^2 |\langle u_0,h\rangle|^2 \\ &\qquad + \sum_{h\in\mathscr{H}} \left| e^{itm_h\mu(Q(h))^{-\beta}} - e^{ism_h\mu(Q(h))^{-\beta}} \right|^2 \frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2\lambda}}.\end{aligned}$$

In order to see that both series above tend to zero as $s \to t$, for each positive ε we can take a finite subfamily $\mathscr{F}$ of $\mathscr{H}$ such that

$$\sum_{h\in\mathscr{H}\setminus\mathscr{F}} |\langle u_0,h\rangle|^2 \left(1+\mu(Q(h))^{-2\lambda}\right) < \varepsilon.$$

For the sums on $\mathscr{F}$ we can argue with the continuity of the complex exponential.

Let us prove that the formal derivative of $u(t)$ is actually the derivative in the sense of $B_2^{\lambda-\beta}(X,\delta,\mu)$. In fact, for $t > 0$ and τ small enough,

$$\begin{aligned}&\left\| \frac{u(t+\tau)-u(t)}{\tau} - i \sum_{h\in\mathscr{H}} e^{itm_h\mu(Q(h))^{-\beta}} \mu(Q(h))^{-\beta} \langle u_0,h\rangle\, h \right\|^2_{B_2^{\lambda-\beta}(X,\delta,\mu)} \\ &= \left\| \sum_{h\in\mathscr{H}} e^{itm_h\mu(Q(h))^{-\beta}} \left[\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau} - i\mu(Q(h))^{-\beta} \right] \langle u_0,h\rangle\, h \right\|^2_{B_2^{\lambda-\beta}(X,\delta,\mu)} \\ &\leq C \left\{ \left\| \sum_{h\in\mathscr{H}} e^{itm_h\mu(Q(h))^{-\beta}} \left[\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau} - i\mu(Q(h))^{-\beta} \right] \langle u_0,h\rangle\, h \right\|^2_{L^2} \right. \\ &\qquad \left. + \sum_{h\in\mathscr{H}} \left| \frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau} - i\mu(Q(h))^{-\beta} \right|^2 \frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2(\lambda-\beta)}} \right\} \\ &\leq C \sum_{h\in\mathscr{H}} \mu(Q(h))^{2\beta} \left| \frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau} - i\mu(Q(h))^{-\beta} \right|^2 \frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2\lambda}}.\end{aligned}$$

Since, from Theorem 5, $\sum_{h\in\mathscr{H}} \frac{|\langle u_0,h\rangle|^2}{\mu(Q(h))^{2\lambda}} < \infty$ and

$$\mu(Q(h))^{2\beta}\left|\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\tau} - i\mu(Q(h))^{-\beta}\right|^2 = \left|\frac{e^{i\tau m_h\mu(Q(h))^{-\beta}}-1}{\mu(Q(h))^{-\beta}\tau} - i\right|^2$$

tends to zero as $\tau \to 0$ for each $h \in \mathscr{H}$, arguing as before we obtain the result.

On the other hand since $u(t) \in B_2^\lambda(X,\delta,\mu)$ and since $\lambda > \beta$, $D^\beta u(t)$ is well defined and it is given by

$$\begin{aligned} D^\beta u(t) &= D^\beta\left(\sum_{h\in\mathscr{H}} e^{itm_h\mu(Q(h))^{-\beta}}\langle u_0,h\rangle\, h\right) \\ &= \sum_{h\in\mathscr{H}} e^{itm_h\mu(Q(h))^{-\beta}}\mu(Q(h))^{-\beta}\langle u_0,h\rangle\, h = -i\frac{du}{dt}. \end{aligned}$$

Hence $u(t)$ is a solution of the nonlocal equation and *(3.b)* is proved. □

Before proving Theorem 4 we shall obtain some basic maximal estimates involved in the proofs of *(4.a)* and *(4.b)*. With M_{dy} we denote the Hardy–Littlewood dyadic maximal operator given by

$$M_{dy}f(x) = \sup \frac{1}{\mu(Q)}\int_Q |f(y)|\, d\mu(y)$$

where the supremum is taken on the family of all dyadic cubes $Q \in \mathscr{D}$ for which $x \in Q$. Calderón's dyadic maximal operator of order λ is defined by

$$M^{\#}_{\lambda,dy}f(x) = \sup \frac{1}{\mu(Q)^{1+\lambda}}\int_Q |f(y)-f(x)|\, d\mu(y),$$

where the supremum is taken on the family of all dyadic cubes Q of X such that $x \in Q$. The following lemma can be seen as an extension of a result by DeVore and Sharpley in [10, Corollary 11.6].

Lemma 13. *If $f \in B_2^\lambda(X,\delta,\mu)$, then* $\left\|M^{\#}_{\lambda,dy}f\right\|_{L^2} \le \|f\|_{B_2^\lambda}$.

Proof. Let $Q \in \mathscr{D}$ and $x \in Q$ be given. Applying Schwarz's inequality, since $\delta(x,y) \le \mu(Q)$ for $y \in Q$, we have

$$\begin{aligned} \frac{1}{\mu(Q)^{1+\lambda}}\int_Q |f(y)-f(x)|\, d\mu(y) &\le \frac{1}{\mu(Q)^{1+\lambda}}\left(\int_Q |f(y)-f(x)|^2\, d\mu(y)\right)^{\frac12}\mu(Q)^{\frac12} \\ &\le \left(\frac{1}{\mu(Q)^{1+2\lambda}}\int_Q |f(y)-f(x)|^2\, d\mu(y)\right)^{\frac12} \\ &\le \left(\int_X \frac{|f(y)-f(x)|^2}{\delta(x,y)^{1+2\lambda}}\, d\mu(y)\right)^{\frac12}. \end{aligned}$$

In this section we shall need a better description of the structure of $\mathscr{H}$ in terms of scales. As we said before $\mathscr{D}^j$ denotes the dyadic sets at the scale $j \in \mathbb{Z}$. With $\mathscr{H}^j$ we denote the Haar wavelets which are based in the cubes of $\mathscr{D}^j$ for j a fixed integer. Since from our construction of the Haar system we have that

$$V_{j+1} = V_j \oplus W_j,$$

then $\mathscr{H}^j$ is an orthonormal basis for W_j. Hence, for $N > 0$, $V_{N+1} = V_0 \oplus W_0 \oplus \ldots \oplus W_N$. So that, for $P_0 f = 0$, the projection of f onto V_{N+1} is given by

$$P_{N+1} f = \sum_{j=0}^{N} \mathcal{Q}_j f,$$

where $\mathcal{Q}_j$ is the projector onto W_j, precisely

$$\mathcal{Q}_j f = \sum_{h \in \mathscr{H}^j} \langle f, h \rangle\, h.$$

As it is easy to see, since V_{N+1} is the space of those L^2 functions on X which are constant on each cube of $\mathscr{D}^{N+1}$, $|P_{N+1} f(x)| \leq M_{dy} f(x)$, pointwise.

Let us next introduce two maximal operators related to the series (3). For fixed $t > 0$ set

$$S_t^* f(x) = \sup_{N \in \mathbb{N}} \left| S_t^N f(x) \right|$$

with

$$S_t^N f(x) = \sum_{j=0}^{N} \sum_{h \in \mathscr{H}^j} e^{-itm_h \mu(Q(h))^{-\beta}} \langle f, h \rangle\, h(x).$$

Set

$$S^* f(x) = \sup_{0<t<1} S_t^* f(x).$$

Lemma 14. *Let $f \in B_2^\lambda(X, \delta, \mu)$ with $0 < \beta < \lambda < 1$ and $P_0 f = 0$. Then, the inequalities*

(14.a) $S_t^* f(x) \leq C t M_{\lambda,dy}^{\#} f(x) + 2 M_{dy} f(x)$ *for $t \geq 0$ and $x \in X$;*
(14.b) $S^* f(x) \leq C M_{\lambda,dy}^{\#} f(x) + 2 M_{dy} f(x)$ *for $x \in X$;*
(14.c) $\left\| S^* f \right\|_{L^2} \leq C \left\| f \right\|_{B_2^\lambda(X,\delta,\mu)}$,

hold for some constant C which does not depend on f.

Proof. For $f \in B_2^\lambda(X, \delta, \mu)$, $t \geq 0$ and $N \in \mathbb{N}$, we have

$$\left| S_t^N f(x) \right| \leq \left| S_t^N f(x) - S_0^N f(x) \right| + \left| S_0^N f(x) \right|. \tag{8}$$

Since $S_0^N f(x) = P_N f(x)$, we have $\sup_N \left| S_0^N f(x) \right| \leq M_{dy} f(x)$. Let us now estimate the first term on the right hand side of (8). Let $Q(j, x)$ be the only cube Q in $\mathscr{D}^j$ for which $x \in Q$. Let $\mathscr{H}(j, x)$ be the set of all the wavelets based on $Q(j, x)$. Recall that $\#(\mathscr{H}(j, x))$ is bounded by a purely geometric constant. Then

$$
\begin{aligned}
\left|S_t^N f(x) - S_0^N f(x)\right| &= \left|\sum_{j=0}^{N} \sum_{h\in\mathscr{H}(j,x)} \left[e^{-itm_h\mu(Q(j,x))^{-\beta}} - 1\right] \langle f, h\rangle\, h(x)\right| \\
&= \left|\sum_{j=0}^{N} \sum_{h\in\mathscr{H}(j,x)} \left[e^{-itm_h\mu(Q(j,x))^{-\beta}} - 1\right] \int_{Q(j,x)} (f(y) - f(x))h(y)d\mu(y)\, h(x)\right| \\
&= C\,t \sum_{j=0}^{N} \sum_{h\in\mathscr{H}(j,x)} \frac{\left|e^{-itm_h\mu(Q(j,x))^{-\beta}} - 1\right|}{m_h t\mu(Q(j,x))^{-\lambda}} \frac{1}{\mu(Q(j,x))^{1+\lambda}} \int_{Q(j,x)} |f(y) - f(x)|\, d\mu(y) \\
&\le C\,t \sum_{j=0}^{N} \mu(Q(j,x))^{\lambda-\beta} M^{\#}_{\lambda,dy} f(x) \\
&\le \widetilde{C}\, t M^{\#}_{\lambda,dy} f(x)
\end{aligned}
$$

and *(14.a)* is proved. Since $t < 1$, *(14.b)* follows. And applying Lemma 13 we get *(14.c)*. □

Proof of Theorem 4. Since for each $Q \in \mathscr{D}$ we have that $\mathcal{X}_Q$ belongs to $Lip(X, \delta)$, in fact $|\mathcal{X}_Q(x) - \mathcal{X}_Q(y)| \le \frac{\delta(x,y)}{\mu(Q)}$, then $\mathcal{S}(\mathscr{H})$ is a dense subspace of $L^2(X, \mu)$ such that $S_t^N g(x)$ converges as $N \to \infty$ for every $x \in X$ and every $t \ge 0$. From Lemmas 13, 14, and the above remark the result of Theorem 4 follows the standard argument of pointwise convergence (see [4]). □

Let us finally illustrate the above result in the Sierpinski triangle. Set S to denote the Sierpinski triangle in $\mathbb{R}^2$ equipped with the normalized Hausdorff measure $\mathcal{H}_s$ of order $s = \frac{\log 3}{\log 2}$. For each positive integer j, the set S can be written as the union of 3^j translations of the contraction of S by a factor 3^{-j}, with respect to any one of the three vertices of the convex hull of S. Except for sets of $\mathcal{H}_s$ measure zero this covering of S is disjoint. Each one of these disjoint pieces is denoted by T_k^j, $k = 1, \dots, 3^j$. We shall also write T_1^0 to denote S. Set $\mathscr{D}^j$ to denote the family $\{T_k^j : k = 1, \dots, 3^j\}$ and $\mathscr{D} = \cup_{j\ge 0}\mathscr{D}^j$. Notice that $\mathcal{H}_s(T_k^j) = 3^{-j}$.

For each j and each $k = 1, \dots, 3^j$ we have that T_k^j contains and is covered by exactly three pieces $T_{k_1}^{j+1}$, $T_{k_2}^{j+1}$, $T_{k_3}^{j+1}$ of the generation $j + 1$. In order to abbreviate the notation, given $T \in \mathscr{D}$ we write $T(1)$, $T(2)$ and $T(3)$ to denote these pieces of T. The three dimensional space of all functions defined in T which are constant on $T(1)$, $T(2)$ and $T(3)$ has as a basis $\left\{\mathcal{X}_T, \mathcal{X}_{T(1)}, \mathcal{X}_{T(2)}\right\}$. By orthonormalization of this basis with the inner product of $L^2(S, d\mathcal{H}_s)$ keeping $3^{j/2}\mathcal{X}_T$, $T \in \mathscr{D}^j$, we get two other functions h_T^1 and h_T^2 such that $\left\{3^{j/2}\mathcal{X}_T, h_T^1, h_T^2\right\}$ is an orthonormal basis for that 3-dimensional space. From the self similarity of our setting, we can take h_T^1 and h_T^2 as the corresponding scalings and translations of those associated to T_1^0. If

$$
\begin{aligned}
h^1_{T_1^0}(x) &= \sqrt{\tfrac{3}{2}}\left(\mathcal{X}_{T(1)} - \mathcal{X}_{T(2)}\right), \\
h^2_{T_1^0}(x) &= \tfrac{1}{\sqrt{2}}\left(\mathcal{X}_{T(1)} + \mathcal{X}_{T(2)} - 2\mathcal{X}_{T(3)}\right),
\end{aligned}
$$

then

$$
h^1_{T_k^j}(x) = 3^{\frac{j}{2}}\sqrt{\tfrac{3}{2}}\left(\mathcal{X}_{T_k^j(1)} - \mathcal{X}_{T_k^j(2)}\right)
$$

and

$$
h^2_{T_k^j}(x) = 3^{\frac{j}{2}}\tfrac{1}{\sqrt{2}}\left(\mathcal{X}_{T_k^j(1)} + \mathcal{X}_{T_k^j(2)} - 2\mathcal{X}_{T_k^j(3)}\right).
$$

The system $\mathscr{H}$ of all these functions h is an orthonormal basis for the subspace of $L^2(S, d\mathcal{H}_s)$ of those functions with vanishing mean. Or, equivalently, the system $\mathscr{H} \cup \{1\}$ is an orthonormal basis for $L^2(S, d\mathcal{H}_s)$.

For a given h in $\mathscr{H}$ we shall use $T(h)$ to denote the triangle in which h is based, i.e. $h = h^i_T$ for $i = 1$ or $i = 2$. For each $T \in \mathscr{D}$ there are exactly two wavelets $h \in \mathscr{H}$ based on T.

The function $\delta(x, y) = \min\{\mathcal{H}_s(T) : T \in \mathscr{D}$ such that $x, y \in T\}$ is a distance on S provided we agree at defining $\delta(x, x) = 0$.

Set $j(h)$ to denote the integer j such that $T(h) \in \mathscr{D}^j$. From Lemma 2 for $0 < \beta < 1$ we have that each $h \in \mathscr{H}$ is an eigenfunction with eigenvalue $m_\beta 3^{j(h)\beta}$ of the dyadic fractional differential operator

$$D^\beta g(x) = \int_S \frac{g(x) - g(y)}{\delta(x, y)^{1+\beta}} d\mathcal{H}_s(y),$$

where the constant $m_\beta = 1 + \frac{1}{2}\frac{1}{3^\beta - 1}$ is independent of h.

Our main results throughout this paper applied to the particular setting S read as follows.

Corollary 15. *Let f be an $L^2(S, d\mathcal{H}_s)$ function with vanishing mean such that*

$$\iint_{S\times S} \frac{|f(x) - f(y)|^2}{\delta(x, y)^{1+2\lambda}} d\mathcal{H}_s(x) d\mathcal{H}_s(y) < \infty$$

for some $\lambda > \beta > 0$. Then, the wave function

$$u(x, t) = \sum_{h\in\mathscr{H}} e^{-itm_\beta 3^{j(h)\beta}} \langle f, h\rangle\, h(x),$$

solves the Schrödinger type problem

$$\begin{cases} i\dfrac{\partial u}{\partial t} = D^\beta u, & x \in S,\ t > 0, \\ \lim\limits_{t\to 0^+} u(x, t) = f(x), & \text{for } \mathcal{H}_s\text{-almost every } x \in S. \end{cases}$$

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