J. Math. Anal. Appl. 435 (2016) 1573–1590

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the topological entropy of free semigroup actions

Yupan Wang [a], Dongkui Ma [b,*], Xiaogang Lin [c]

[a] *School of Computer Science and Engineering, South China University of Technology, Guangzhou 510641, PR China*
[b] *School of Mathematics, South China University of Technology, Guangzhou 510641, PR China*
[c] *School of Business Administration, South China University of Technology, Guangzhou 510641, PR China*

A R T I C L E I N F O

Article history:
Received 1 September 2015
Available online 14 November 2015
Submitted by B. Bongiorno

Keywords:
Topological entropy
Free semigroup actions
Skew-product transformations
Affine transformations

A B S T R A C T

We extend the notion the topological entropy of a free semigroup action defined by Bufetov [6] to the case of a free semigroup action on a metric space not necessarily compact, provide some properties of this extended topological entropy, extend the topological analogue of the classical Abramov–Rokhlin formula for the entropy of a skew product transformations with respect to a metric space not necessarily compact and give some bounds for the entropy for some particular systems such as a free semigroup with m generators of affine transformations on p-dimensional torus and smooth maps on a Riemannian manifold.

1. Introduction

Topological entropy, which describes the complexity of a system, plays an important role in topological dynamical systems. It was first induced by Adler, Konheim and McAndrew [1] as an invariant of topological conjugacy. Later, Bowen [4] and Dinaburg [7] gave equivalent definitions of topological entropy in a metric space not necessarily compact. Bowen [5] gave a definition of topological entropy for subsets of a compact space in a way which resembles Hausdorff dimension. Later, some researchers tried to find some suitable generalizations of topological entropy for other systems. Friedland [8] gave a survey of entropies for graphs, semigroups and groups, and gave some examples for these entropies. Ghys, Langevin, Walczak and Przytycki [9,12,13] gave the notions of entropy in various abstract topological settings, and studied the relationships between them. Mihailescu and Urbanski [16,17] gave the notion of inverse topological entropy and some applications to dimension estimates. Nitecki [18] introduced definitions of various notions of topological entropy and their relations to preimage sets. Kolyada and Snoha [11] introduced the topological entropy of nonautonomous dynamical systems and Zhu, Liu, Xu and Zhang [23] studied some more properties of

* Corresponding author.
E-mail addresses: wang.yupan@mail.scut.edu.cn (Y. Wang), dkma@scut.edu.cn (D. Ma).

http://dx.doi.org/10.1016/j.jmaa.2015.11.038

this case of topological entropy. Stepin and Tagi-Zade [19] introduced the notion of the topological entropy for amenable group actions and studied some properties of this topological entropy. Biś [2] proposed the notion of topological entropy of a semigroup of finite continuous maps and this case of topological entropy was studied by Biś and Urbański [3], Ma and Wu [15], Ma and Liu [14] and Wang and Ma [21]. Bufetov [6] defined the notion of the topological entropy of free semigroup actions in a different way and identified this notion of topological entropy with fiber entropy of a certain skew-product transformation. Recently, Tang, Li and Cheng [20] gave an equivalent definition of Bufetov's topological entropy [6] by using open cover and some properties.

Wang and Ma [21] extended the notion of the topological entropy defined by Biś [2] to the case of a semigroup of continuous maps on a metric space not necessarily compact. Similar to [21], in the present paper, we extend the notion of the topological entropy defined by Bufetov [6] to the topological entropy of a free semigroup action generated with m generators of uniformly continuous maps on a metric space not necessarily compact, give some properties of this topological entropy, study its relation with the topological entropy of a skew-product transformation, extend the main results of Bufetov [6], and estimate the bounds of the extended topological entropy for some particular systems, such as a free semigroup with m generators of smooth maps on a Riemannian manifold, and affine transformations on p-dimensional torus.

This paper is organized as follows. In section 2, we give some preliminaries. In section 3, we give the definitions of a free semigroup action generated with m generators of uniformly continuous maps on a metric space not necessarily compact and give some fundamental properties of them which are useful to calculate them. In section 4, we extend the topological analogue of the classical Abramov–Rokhlin formula for the entropy of a skew product transformations with respect to a metric space not necessarily compact. In section 5, we estimate the bounds of the topological entropy for some particular systems, such as a free semigroup with m generators of smooth maps on a Riemannian manifold and affine transformations on p-dimensional torus.

2. Preliminaries

Let F_m^+ be the set of all finite words of symbols $0, 1, \cdots, m-1$. For every $w \in F_m^+$, $|w|$ denotes the length of w, i.e., the number of symbols in w. If $w, w' \in F_m^+$, define ww' to be the word obtained by writing w' to the right of w. With respect to this law of composition, F_m^+ is a free semigroup with m generators. We write $w \leq w'$ if there exists a word w'' such that $w' = w''w$.

If $w \in F_m^+$, $w = w_1 w_2 \cdots w_k$ where $w_i \in \{0, 1, \cdots, m-1\}$ for all $i = 1, 2, \cdots, k$, let $f_w = f_{w_1} f_{w_2} \cdots f_{w_k}$. Obviously, $f_{ww'} = f_w f_{w'}$ for any $w' \in F_m^+$.

Denote by Σ_m the set of all two-side infinite sequences of symbols $0, 1, \ldots, m-1$, that is,

$$\Sigma_m = \{\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) | \omega_i = 0, 1, \ldots, m-1 \text{ for all integer } i\}.$$

A metric on Σ_m is introduced by setting

$$d(\omega, \omega') = 1/2^k, \text{ where } k = \inf\{|n| : \omega_n \neq \omega'_n\}.$$

Obviously, Σ_m is compact with respect to this metric. The Bernoulli shift $\sigma_m : \Sigma_m \to \Sigma_m$ is a homeomorphism of Σ_m given by the formula:

$$(\sigma_m \omega)_i = \omega_{i+1}.$$

Assume that $\omega \in \Sigma_m, w \in F_m^+$, a, b are integers, and $a \leq b$. We write $\omega|_{[a,b]} = w$ if $w = \omega_a \omega_{a+1} \ldots \omega_{b-1} \omega_b$.

Let X be a compact metric space with metric d, and $f_0, f_1, \cdots, f_{m-1}$ be continuous maps of X into itself. Then there is a free semigroup with m generators $f_0, f_1, \cdots, f_{m-1}$ acting on X.

For every $w \in F_m^+$, define a metric d_w on X by

$$d_w(x_1, x_2) = \max_{w' \leq w} d(f_{w'}(x_1), f_{w'}(x_2)), \forall x_1, x_2 \in X.$$

Obviously, if $w \leq w'$, then $d_w(x_1, x_2) \leq d_{w'}(x_1, x_2)$.

A subset F of X is said to be a $(w, \epsilon, f_0, \cdots, f_{m-1})$-separated set of X, if $x, y \in F$, $x \neq y$ implies $d_w(x, y) > \epsilon$. Define $N_{\text{sep}}(w, \epsilon, f_0, \cdots, f_{m-1})$ to be the largest cardinality of any $(w, \epsilon, f_0, \cdots, f_{m-1})$-separated set of X. Since X is compact with respect to the metric d_w, $N_{\text{sep}}(w, \epsilon, f_0, \cdots, f_{m-1})$ is a finite number.

A subset E of X is called a $(w, \epsilon, f_0, \cdots, f_{m-1})$-spanning set of X if for every $x \in X$ there exists $y \in E$ such that $d_w(x, y) \leq \epsilon$. Let $N_{\text{span}}(w, \epsilon, f_0, \cdots, f_{m-1})$ denote the smallest cardinality of any $(w, \epsilon, f_0, \cdots, f_{m-1})$-spanning sets of X.

In [6], Bufetov defined

$$N_{\text{span}}(n, \epsilon, f_0, \cdots, f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} N_{\text{span}}(w, \epsilon, f_0, \cdots, f_{m-1}),$$

$$N_{\text{sep}}(n, \epsilon, f_0, \cdots, f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} N_{\text{sep}}(w, \epsilon, f_0, \cdots, f_{m-1}),$$

and also defined the topological entropy of a semigroup action by the formula

$$\begin{aligned} h(f_0, \cdots, f_{m-1}) &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{\text{span}}(n, \epsilon, f_0, \cdots, f_{m-1}) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{\text{sep}}(n, \epsilon, f_0, \cdots, f_{m-1}). \end{aligned}$$

Observe that for $m = 1$, this definition coincides with Bowen's definition [4,22].

Remark 2.1. It's obvious that $h(f_0, \cdots, f_{m-1})$ is less than or equal to the topological entropy of the semigroup generated by $G_1 = \{\text{id}_X, f_0, \cdots, f_{m-1}\}$ defined by Biś [2].

3. Topological entropy and basic properties in the present paper

Let (X, d) be a metric space which is not necessarily comp act and uniformly continuous maps $f_0, f_1, \cdots, f_{m-1}$ from X into itself.

Let K be a compact subset of X. For any $w \in F_m^+$ and $\epsilon > 0$, a subset $E \subseteq X$ is said to be a $(w, \epsilon, K, f_0, \cdots, f_{m-1})$-spanning set of K, if for any $x \in K$, there exists $y \in E$ such that $d_w(x, y) \leq \epsilon$. Define $N_{\text{span}}(w, \epsilon, K, f_0, \cdots, f_{m-1})$ to be the smallest cardinality of any $(w, \epsilon, K, f_0, \cdots, f_{m-1})$-spanning sets of K. A subset $F \subseteq K$ is said to be a $(w, \epsilon, K, f_0, \cdots, f_{m-1})$-separated set of K, if $x, y \in F$, $x \neq y$ implies $d_w(x, y) > \epsilon$. Let $N_{\text{sep}}(w, \epsilon, K, f_0, \cdots, f_{m-1})$ denote the largest cardinality of any $(w, \epsilon, K, f_0, \cdots, f_{m-1})$-separated sets of K.

For any $n \geq 1$, let

$$N_{\text{span}}(n, \epsilon, K, f_0, \cdots, f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} N_{\text{span}}(w, \epsilon, K, f_0, \cdots, f_{m-1}), \tag{3.1}$$

$$N_{\text{sep}}(n, \epsilon, K, f_0, \cdots, f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} N_{\text{sep}}(w, \epsilon, K, f_0, \cdots, f_{m-1}). \tag{3.2}$$

Obviously,

$$N_{\text{span}}(w, \frac{\epsilon}{2}, K, f_0, \cdots, f_{m-1}) \geq N_{\text{sep}}(w, \epsilon, K, f_0, \cdots, f_{m-1}) \geq N_{\text{span}}(w, \epsilon, K, f_0, \cdots, f_{m-1}),$$

hence,

$$N_{\text{span}}(n, \frac{\epsilon}{2}, K, f_0, \cdots, f_{m-1}) \geq N_{\text{sep}}(n, \epsilon, K, f_0, \cdots, f_{m-1}) \geq N_{\text{span}}(n, \epsilon, K, f_0, \cdots, f_{m-1}). \tag{3.3}$$

Thus we give the following definition.

Definition 3.1. Let (X, d) be a metric space which is not necessarily compact, $f_0, f_1, \cdots, f_{m-1}$ uniformly continuous transformations from X into itself, and K a compact subset of X. Define

$$H(K, f_0, \cdots, f_{m-1}) := \lim_{\epsilon \to 0} N_{\text{span}}(\epsilon, K, f_0, \cdots, f_{m-1}) = \lim_{\epsilon \to 0} N_{\text{sep}}(\epsilon, K, f_0, \cdots, f_{m-1}),$$

where

$$N_{\text{span}}(\epsilon, K, f_0, \cdots, f_{m-1}) = \limsup_{n \to \infty} \frac{1}{n} \log N_{\text{span}}(n, \epsilon, K, f_0, \cdots, f_{m-1}),$$

and

$$N_{\text{sep}}(\epsilon, K, f_0, \cdots, f_{m-1}) = \limsup_{n \to \infty} \frac{1}{n} \log N_{\text{sep}}(n, \epsilon, K, f_0, \cdots, f_{m-1}).$$

Thus, we can define the topological entropy of X with respect to the semigroup with m generators $f_0, f_1, \cdots, f_{m-1}$ by

$$H(f_0, \cdots, f_{m-1}) := \sup\{H(K, f_0, \cdots, f_{m-1}) : K \subseteq X \text{ is compact}\}.$$

Remark 3.2. (1) If $m = 1$ and f is a uniformly continuous map of X, then the new definition of topological entropy generated by f is the same as the topological entropy of f defined by Bowen [4], we denote this topological entropy by $h(f)$.

(2) If X is compact, then $h(f_0, \cdots, f_{m-1}) = H(X, f_0, \cdots, f_{m-1}) = H(f_0, \cdots, f_{m-1})$. Thus the new definition of topological entropy of a free semigroup actions is the same as the topological entropy defined by Bufetov [6]. We write $H_d(f_0, \cdots, f_{m-1})$, $H_d(K, f_0, \cdots, f_{m-1})$ respectively to emphasize d if we need to.

(3) It's obvious that $H(f_0, \cdots, f_{m-1})$ is less than or equal to the topological entropy of the semigroup generated by $G_1 = \{\text{id}_X, f_0, \cdots, f_{m-1}\}$ defined by Wang and Ma [21].

Example 3.3. Let (X, d) be a metric space which is not necessarily compact, $f_0, \cdots, f_{m-1}$ uniformly continuous transformations from X to itself. Denote G the semigroup with m generators $f_0, \cdots, f_{m-1}$ acting on X. If the family G are equicontinuous, then $H_d(f_0, \cdots, f_{m-1}) = 0$.

Proof. For any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$ if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, $\forall f \in G$. For any compact subset K of X, there exists $M > 0$ such that for any $w \in F_m^+$, $N_{\text{span}}(w, \epsilon, K, f_0, \cdots, f_{m-1}) \leq M$ and then $N_{\text{span}}(n, \epsilon, K, f_0, \cdots, f_{m-1}) \leq M$. Thus $N_{\text{span}}(\epsilon, K, f_0, \cdots, f_{m-1}) = 0$. Therefore $H_d(K, f_0, \cdots, f_{m-1}) = 0$. Moreover, we have $H_d(f_0, \cdots, f_{m-1}) = 0$. □

Definition 3.4. Let X be a metric space, d and d' the two metrics on X. We say d and d' are uniformly equivalent if $\text{id}_X : (X, d) \to (X, d')$ and $\text{id}_X : (X, d') \to (X, d)$ are both uniformly continuous.

Theorem 3.5. *Let (X,d) be a metric space, $f_0,\cdots,f_{m-1}$ uniformly continuous transformations from X to itself. If d and d' are uniformly equivalent, then*

$$H_d(f_0,\cdots,f_{m-1}) = H_{d'}(f_0,\cdots,f_{m-1}).$$

Proof. Let $\epsilon_1 > 0$, choose $\epsilon_2 > 0$ such that $d'(x,y) < \epsilon_2$ implies $d(x,y) < \epsilon_1$ for any $x,y \in X$. Choose $\epsilon_3 > 0$ such that $d(x,y) < \epsilon_3$ implies $d'(x,y) < \epsilon_2$ for every $x,y \in X$.

For any compact subset K in X and $w \in F_m^+$, we have

$$N_{\text{span}}(w,\epsilon_1,K,f_0,\cdots,f_{m-1},d) \le N_{\text{span}}(w,\epsilon_2,K,f_0,\cdots,f_{m-1},d')$$

and

$$N_{\text{span}}(w,\epsilon_2,K,f_0,\cdots,f_{m-1},d') \le N_{\text{span}}(w,\epsilon_3,K,f_0,\cdots,f_{m-1},d).$$

Hence

$$\begin{aligned} N_{\text{span}}(n,\epsilon_1,K,f_0,\cdots,f_{m-1},d) &\le N_{\text{span}}(n,\epsilon_2,K,f_0,\cdots,f_{m-1},d') \\ &\le N_{\text{span}}(n,\epsilon_3,K,f_0,\cdots,f_{m-1},d). \end{aligned}$$

Moreover,

$$\begin{aligned} N_{\text{span}}(\epsilon_1,K,f_0,\cdots,f_{m-1},d) &\le N_{\text{span}}(\epsilon_2,K,f_0,\cdots,f_{m-1},d') \\ &\le N_{\text{span}}(\epsilon_3,K,f_0,\cdots,f_{m-1},d). \end{aligned}$$

If $\epsilon_1 \to 0$, then $\epsilon_2 \to 0$ and $\epsilon_3 \to 0$, so we have $H_d(K,f_0,\cdots,f_{-1}) = H_{d'}(K,f_0,\cdots,f_{m-1})$. Therefore we have $H_d(f_0,\cdots,f_{-1}) = H_{d'}(f_0,\cdots,f_{m-1})$. □

Remark 3.6. If X is compact and if d and d' are equivalent metrics, then they are uniformly equivalent. Moreover, each continuous map $f : X \to X$ is uniformly continuous. Therefore, if X is a compact metrizable space, the entropy of $f_0,\cdots,f_{m-1}$ does not rely on the metric chosen on X (provided that metric induces the topology of X).

Theorem 3.7. *Let (X,d) be a metric space, and $f_0,\cdots,f_{m-1}$ uniformly continuous transformations from X to itself. Let $\delta > 0$. In order to compute $H(f_0,\cdots,f_{m-1})$, it suffices to take the supremum of $H(K,f_0,\cdots,f_{m-1})$ over those compact sets of diameter less than δ.*

Proof. The proof follows [22] and [21] and is omitted. □

Theorem 3.8. *Let $(X_i,d_i)(i=1,2)$ be a metric space. Let $\mathcal{F}^{(1)} = \{f_0^{(1)},\cdots,f_{m-1}^{(1)}\}$ be a set of finite uniformly continuous maps on X_1, and $\mathcal{F}^{(2)} = \{f_0^{(2)},\cdots,f_{k-1}^{(2)}\}$ a set of finite uniformly continuous maps on X_2. Let $\mathcal{F}^{(1)} \times \mathcal{F}^{(2)} = \{(f\times g)_0,\cdots,(f\times g)_{mk-1}\}$, where $(f\times g)_i \in \{f\times g : f \in \mathcal{F}^{(1)}, g \in \mathcal{F}^{(2)}\}$, and $(f\times g)(x_1,x_2) = (f(x_1),g(x_2))$ for any $f\times g \in \mathcal{F}^{(1)}\times\mathcal{F}^{(2)}$ and $x_1 \in X_1, x_2 \in X_2$. A metric d on the product space $X_1\times X_2$ is given by $d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1),d_2(x_2,y_2)\}$. Then*

$$H_d(\mathcal{F}^{(1)}\times\mathcal{F}^{(2)}) \le H_{d_1}(\mathcal{F}^{(1)}) + H_{d_2}(\mathcal{F}^{(2)}).$$

If X_1 is compact and $H_{d_1}(\mathcal{F}^{(1)}) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}N_{sep}(n,\epsilon,X_1,\mathcal{F}^{(1)})$ or X_2 is compact and $H_{d_2}(\mathcal{F}^{(2)}) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}N_{sep}(n,\epsilon,X_2,\mathcal{F}^{(2)})$, then

$$H_d(\mathcal{F}^{(1)}\times\mathcal{F}^{(2)}) = H_{d_1}(\mathcal{F}^{(1)}) + H_{d_2}(\mathcal{F}^{(2)}).$$

Proof. Firstly, $\mathcal{F}^{(1)} \times \mathcal{F}^{(2)}$ is a set of finite uniformly continuous transformations on $X_1 \times X_2$. For any $\nu = \nu_1 \cdots \nu_n \in F_{mk}^+$, there exist unique $w^{(1)} = w_1^{(1)} \cdots w_n^{(1)} \in F_m^+$ and unique $w^{(2)} = w_1^{(2)} \cdots w_n^{(2)} \in F_k^+$ such that $(f \times g)_{\nu_i} = f^{(1)}_{w_i^{(1)}} \times f^{(2)}_{w_i^{(2)}}$ for any $1 \le i \le n$ and thus $(f \times g)_\nu = f^{(1)}_{w^{(1)}} \times f^{(2)}_{w^{(2)}}$. On the other hand, if $w^{(1)} = w_1^{(1)} \cdots w_n^{(1)} \in F_m^+, w^{(2)} = w_1^{(2)} \cdots w_n^{(2)} \in F_k^+$, there exists unique $\nu = \nu_1 \cdots \nu_n \in F_{mk}^+$ such that $f^{(1)}_{w_i^{(1)}} \times f^{(2)}_{w_i^{(2)}} = (f \times g)_{\nu_i}$ for any $1 \le i \le n$ and thus $f^{(1)}_{w^{(1)}} \times f^{(2)}_{w^{(2)}} = (f \times g)_\nu$. Thus for any $n \ge 1$, the map $\nu \mapsto (w^{(1)}, w^{(2)})$ is a one-to-one correspondence.

Let $K_i \subseteq X_i$ be compact, $i = 1, 2$. For any $\epsilon > 0$ and $n \ge 1$ and $w^{(1)} = w_1^{(1)} \cdots w_n^{(1)} \in F_m^+$ and $w^{(2)} = w_1^{(2)} \cdots w_n^{(2)} \in F_k^+$, if F_i is a $(w^{(i)}, \epsilon, K_i, \mathcal{F}^{(i)})$-spanning set of K_i, then for any $(x_1, x_2) \in K_1 \times K_2$ there exist $y_1 \in F_1$ and $y_2 \in F_2$ such that $d_{w^{(1)}}(x_1, y_1) \le \epsilon$ and $d_{w^{(2)}}(x_2, y_2) \le \epsilon$. Moreover, there exists $\nu = \nu_1 \cdots \nu_n \in F_{mk}^+$ such that $(f \times g)_\nu = f^{(1)}_{w^{(1)}} \times f^{(2)}_{w^{(2)}}$. So for any $\nu' = \nu_k \cdots \nu_n$ and $1 \le k \le n$, we have

$$\begin{aligned}
&d((f \times g)_{\nu'}(x_1, x_2), (f \times g)_{\nu'}(y_1, y_2)) \\
&= d((f^{(1)}_{w_k^{(1)} \cdots w_n^{(1)}}(x_1), f^{(2)}_{w_k^{(2)} \cdots w_n^{(2)}}(x_2)), (f^{(1)}_{w_k^{(1)} \cdots w_n^{(1)}}(y_1), f^{(2)}_{w_k^{(2)} \cdots w_n^{(2)}}(y_2))) \\
&= \max(d_1(f^{(1)}_{w_k^{(1)} \cdots w_n^{(1)}}(x_1), f^{(1)}_{w_k^{(1)} \cdots w_n^{(1)}}(y_1)), d_2(f^{(2)}_{w_k^{(2)} \cdots w_n^{(2)}}(x_2), f^{(2)}_{w_k^{(2)} \cdots w_n^{(2)}}(y_2))) \\
&\le \epsilon.
\end{aligned}$$

Therefore $F_1 \times F_2$ is a $(\nu, \epsilon, K_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)})$-spanning set of $K_1 \times K_2$. So

$$N_{\text{span}}(\nu, \epsilon, K_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \le N_{\text{span}}(w^{(1)}, \epsilon, K_1, \mathcal{F}^{(1)}) \cdot N_{\text{span}}(w^{(2)}, \epsilon, K_2, \mathcal{F}^{(2)}),$$

and thus

$$\begin{aligned}
&H_d(K_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \\
&= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \Big[\frac{1}{(mk)^n} \sum_{|\nu|=n} N_{\text{span}}(\nu, \epsilon, K_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \Big] \\
&\le \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \Big[\frac{1}{(mk)^n} \sum_{|\nu|=n} \Big(N_{\text{span}}(w^{(1)}, \epsilon, K_1, \mathcal{F}^{(1)}) \cdot N_{\text{span}}(w^{(2)}, \epsilon, K_2, \mathcal{F}^{(2)}) \Big) \Big] \\
&= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \Big[\Big(\frac{1}{m^n} \sum_{|w^{(1)}|=n} N_{\text{span}}(w^{(1)}, \epsilon, K_1, \mathcal{F}^{(1)}) \Big) \cdot \Big(\frac{1}{k^n} \sum_{|w^{(2)}|=n} N_{\text{span}}(w^{(2)}, \epsilon, K_2, \mathcal{F}^{(2)}) \Big) \Big] \\
&\le H_{d_1}(K_1, \mathcal{F}^{(1)}) + H_{d_2}(K_2, \mathcal{F}^{(2)}).
\end{aligned}$$

Let $\pi_i : X_1 \times X_2 \to X_i$ be the projection map, $i = 1, 2$. If K is a compact subset of $X_1 \times X_2$, then $K_1 = \pi_1(K)$ and $K_2 = \pi_2(K)$ are compact and $K \subseteq K_1 \times K_2$. Hence

$$H_d(K, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \le H_d(K_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}).$$

Therefore

$$\begin{aligned}
H_d(\mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) &= \sup_{\substack{K \subset X_1 \times X_2 \\ \text{compact}}} H_d(K, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \\
&\le \sup_{\substack{K_1 \subset X_1 \\ K_2 \subset X_2 \\ \text{compact}}} H_d(K_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)})
\end{aligned}$$

$$\leq \sup_{\substack{K_1\subset X_1\\K_2\subset X_2\\\text{compact}}} (H_{d_1}(K_1,\mathcal{F}^{(1)}) + H_{d_2}(K_2,\mathcal{F}^{(2)}))$$

$$\leq \sup_{\substack{K_1\subset X_1\\\text{compact}}} H_{d_1}(K_1,\mathcal{F}^{(1)}) + \sup_{\substack{K_2\subset X_2\\\text{compact}}} H_{d_2}(K_2,\mathcal{F}^{(2)})$$

$$= H_{d_1}(\mathcal{F}^{(1)}) + H_{d_2}(\mathcal{F}^{(2)}).$$

Now suppose X_1 is compact. (The proof is similar if X_1 is not compact but X_2 is compact.) Since any compact subset of $X_1 \times X_2$ is a subset of $X_1 \times K_2$ for some compact subset K_2 of X_2, we have

$$H_d(\mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) = \sup\{H_d(X_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) :\ K_2 \text{ is a compact subset of } X_2\}.$$

Let K_2 be a compact subset of X_2, $\delta > 0$, and $\nu \in F^+_{mk}$. There exist $w^{(1)} \in F^+_m$ and $w^{(2)} \in F^+_k$ such that $(f\times g)_\nu = f^{(1)}_{w^{(1)}} \times f^{(2)}_{w^{(2)}}$. If E_1 is a $(w^{(1)}, \delta, X_1, \mathcal{F}^{(1)})$-separated set of X_1 and E_2 is a $(w^{(2)}, \delta, K_2, \mathcal{F}^{(2)})$-separated set of K_2, then $E_1 \times E_2$ is a $(\nu, \delta, X_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)})$-separated set of $X_1 \times K_2$. Therefore

$$N_{\rm sep}(\nu, \delta, X_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \geq N_{\rm sep}(w^{(1)}, \delta, X_1, \mathcal{F}^{(1)}) \cdot N_{\rm sep}(w^{(2)}, \delta, K_2, \mathcal{F}^{(2)})$$

and so

$$\begin{aligned}
&N_{\rm sep}(\delta, X_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)})\\
&= \limsup_{n\to\infty} \frac{1}{n} \log\Big[\frac{1}{(mk)^n} \sum_{|\nu|=n} N_{\rm sep}(\nu, \delta, X_1 \times K_2, \mathcal{F}^{(1)} \times \mathcal{F}^{(2)})\Big]\\
&\geq \limsup_{n\to\infty} \frac{1}{n} \log\Big[\frac{1}{(mk)^n} \sum_{|\nu|=n} \Big(N_{\rm sep}(w^{(1)}, \delta, X_1, \mathcal{F}^{(1)}) \cdot N_{\rm sep}(w^{(2)}, \delta, K_2, \mathcal{F}^{(2)})\Big)\Big]\\
&= \limsup_{n\to\infty} \frac{1}{n} \log\Big[\Big(\frac{1}{m^n} \sum_{|w^{(1)}|=n} N_{\rm sep}(w^{(1)}, \delta, X_1, \mathcal{F}^{(1)})\Big) \cdot \Big(\frac{1}{k^n} \sum_{|w^{(2)}|=n} N_{\rm sep}(w^{(2)}, \delta, K_2, \mathcal{F}^{(2)})\Big)\Big]\\
&\geq \liminf_{n\to\infty} \frac{1}{n} \log N_{\rm sep}(n, \delta, X_1, \mathcal{F}^{(1)}) + \limsup_{n\to\infty} \frac{1}{n} \log N_{\rm sep}(n, \delta, K_2, \mathcal{F}^{(2)}).
\end{aligned}$$

According to the condition we give, letting $\delta \to 0$, we have

$$H_d(\mathcal{F}^{(1)} \times \mathcal{F}^{(2)}) \geq H_{d_1}(\mathcal{F}^{(1)}) + H_{d_2}(\mathcal{F}^{(2)}). \qquad \square$$

Remark 3.9. Let $m = 1$ and $f_i : X_i \to X_i$ $(i = 1, 2)$ a uniformly continuous map of a non-compact metric space (X_i, d_i). P. Hulse [10] obtained an example that $H_d(f_1 \times f_2) < H_{d_1}(f_1) + H_{d_2}(f_2)$.

4. Relationship between the topological entropy of a skew-product transformation and the topological entropy of a free semigroup action

Let (X, d) be a metric space which is not necessarily compact, suppose a free semigroup with m generators acts on X, the generators are uniformly continuous transformations $f_0, f_1, \cdots, f_{m-1}$ of X.

To this action, a skew-product transformation $F : \Sigma_m \times X \to \Sigma_m \times X$ is defined by the formula

$$F(\omega, x) = (\sigma_m\omega, f_{\omega_0}(x)),$$

where $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots)$. Here f_{ω_0} stands for f_0 if $\omega_0 = 0$, and for f_1 if $\omega_0 = 1$, and so on. For $w = i_1 \cdots i_k \in F^+_m$, denote $\overline{w} = i_k \cdots i_1$. Let $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \in \Sigma_m$, then

$$\begin{aligned} F^n(\omega, x) &= (\sigma_m^n \omega, f_{\omega_{n-1}} f_{\omega_{n-2}} \cdots f_{\omega_0}(x)) \\ &= (\sigma_m^n \omega, f_{\overline{\omega|_{[0,n-1]}}}(x)). \end{aligned}$$

Our purpose is to find the relationship between the topological entropy $h(F)$ of a skew-product transformation and the topological entropy $H(f_0, \cdots, f_{m-1})$ of a semigroup action, where $h(F)$ denotes the topological entropy defined by Bowen [4,22].

Theorem 4.1. *The topological entropy of the skew product transformation F satisfies*

$$h_D(F) = \log m + H_d(f_0, \cdots, f_{m-1}), \tag{4.1}$$

where the metric D on $\Sigma_m \times X$ is defined as

$$D((\omega, x), (\omega', x)) = \max(d'(\omega, \omega'), d(x, x'))$$

and the metric d' on Σ_m is introduced by setting $d'(\omega, \omega') = 1/2^k$, and $k = \inf\{|n| : \omega_n \neq \omega'_n\}$.

Remark 4.2. Recall that $h(\sigma_m) = \log m$, where $h(\sigma_m)$ denotes the topological entropy of σ_m in symbol space Σ_m.

The proof of this theorem is as analogous as that of Bufetov [6]. We first give the following two lemmas.

Lemma 4.3. *For any compact subset E of X, $n \geq 1$ and $0 \leq \epsilon \leq \frac{1}{2}$, we need to prove*

$$N_{sep}(n, \epsilon, \Sigma_m \times E, F) \geq \sum_{|w|=n} N_{sep}(w, \epsilon, E, f_0, \cdots, f_{m-1}). \tag{4.2}$$

Proof. Let $N = m^n$. There are N distinct words of length n in F_m^+. Denote them as $w_1, w_2, \cdots, w_N$. For each $1 \leq i \leq N$, choose $\omega(i) \in \Sigma_m$ such that $\omega(i)|_{[0,n-1]} = w_i$. It is obvious that for $0 < \epsilon < \frac{1}{2}$ the subset $\{\omega(i) : i = 1, 2, \cdots, N\}$ is a (n, ϵ, σ_m)-separated subset of Σ_m. Let $N_i = N_{\mathrm{sep}}(\overline{w}_i, \epsilon, E, f_0, \cdots, f_{m-1})$ and $\{x_1^i, x_2^i, \cdots, x_{N_i}^i\}$ a $(\overline{w}_i, \epsilon, E, f_0, \cdots, f_{m-1})$-separated set of E. Then the points

$$(\omega(i), x_j^i) \in \Sigma_m \times X, \quad i = 1, \cdots, N, \quad j = 1, \cdots, N_i,$$

form a $(n, \epsilon, \Sigma_m \times E, F)$-separated set of $\Sigma_m \times E$, and the cardinality of this subset is exactly $\sum_{|w|=n} N_{\mathrm{sep}}(w, \epsilon, E, f_0, \cdots, f_{m-1})$. So the inequality (4.2) is proved. □

Lemma 4.4. *For any compact subset E of X, $n \geq 1$ and $\epsilon > 0$, we need to prove*

$$N_{span}(n, \epsilon, \Sigma_m \times E, F) \leq K(\epsilon)\Big(\sum_{|w|=n} N_{span}(w, \epsilon, E, f_0, \cdots, f_{m-1})\Big), \tag{4.3}$$

where $K(\epsilon)$ is a positive constant that depends only on ϵ.

Proof. Let $C(\epsilon)$ be a positive integer satisfying $2^{-C(\epsilon)} < \frac{\epsilon}{100}$ and $N = m^{n+2C(\epsilon)}$. There are N distinct words of length $n + 2C(\epsilon)$ in F_m^+. Denote them as $w_1, \cdots, w_N$. For each $1 \leq i \leq N$, choose $\omega(i) \in \Sigma_m$ such that $\omega|_{[-C(\epsilon), n+C(\epsilon)-1]} = w_i$. Obviously the subset $\{\omega(i) : i = 1, 2, \cdots, N\}$ is a (n, ϵ, σ_m)-spanning subset of Σ_m. Let $w'_i = \omega(i)|_{[0,n-1]}$, $B_i = N_{\mathrm{span}}(\overline{w}'_i, \epsilon, E, f_0, \cdots, f_{m-1})$ and $\{x_1^i, x_2^i, \cdots, x_{B_i}^i\}$ a $(\overline{w}'_i, \epsilon, E, f_0, \cdots, f_{m-1})$-spanning set of E. Then the points

$$(\omega(i), x_j^i) \in \Sigma_m \times X, \quad i = 1, \cdots, N, \quad j = 1, \cdots, B_i,$$

form a $(n, \epsilon, \Sigma_m \times E, F)$-spanning set of $\Sigma_m \times E$, and the cardinality of this subset is no greater than $K(\epsilon)(\sum_{|w|=n} N_{\text{span}}(w, \epsilon, E, f_0, \cdots, f_{m-1}))$ where $K(\epsilon)$ is a positive constant that depends only on ϵ. So the inequality (4.3) is proved. □

The proof of Theorem 4.1. From Lemma 4.3 we have for any compact subset E of X,

$$N_{\text{sep}}(n, \epsilon, \Sigma_m \times E, F) \geq m^n \cdot N_{\text{sep}}(n, \epsilon, E, f_0, \cdots, f_{m-1}),$$

and then obtain that

$$h_D(F) \geq h_D(\Sigma_m \times E, F) \geq \log m + H_d(E, f_0, \cdots, f_{m-1}).$$

Then

$$h_D(F) \geq \log m + H_d(f_0, \cdots, f_{m-1}).$$

From Lemma 4.4 we have

$$N_{\text{span}}(n, \epsilon, \Sigma_m \times E, F) \leq K(\epsilon) m^n \cdot N_{\text{span}}(n, \epsilon, E, f_0, \cdots, f_{m-1}),$$

and hence

$$h_D(\Sigma_m \times E, F) \leq \log m + H_d(E, f_0, \cdots, f_{m-1}) \leq \log m + H_d(f_0, \cdots, f_{m-1}).$$

Since any compact subset of $\Sigma_m \times X$ is a subset of $\Sigma_m \times E$ for some compact subset of $E \subset X$, we have

$$h_D(F) = \sup\{h_D(\Sigma_m \times E, F) : \; E \text{ is a compact subset of } X\}.$$

Then

$$h_D(F) \leq \log m + H_d(f_0, \cdots, f_{m-1}),$$

and the proof is complete. □

Remark 4.5. If (X, d) is a compact metric space, Bufetov [6] proved that

$$h_D(F) = \log m + h_d(f_0, \cdots, f_{m-1}). \tag{4.4}$$

5. Some estimates of the topological entropies of free semigroup actions

In this section, we will give some estimates of the topological entropy defined by Bufetov [6] and the topological entropy defined in section 3 of some particular systems.

Theorem 5.1. *Let M be a p-dimensional Riemannian manifold and $f_0, f_1, \cdots, f_{m-1}$ the C^1 maps on M, then*

$$H_d(f_0, \cdots, f_{m-1}) \leq \log \Big(\frac{1}{m} \sum_{i=0}^{m-1} \big(\max\{1, \sup_{x \in M} \|d_x f_i\|\} \big)^p \Big),$$

where d denotes the metric on M induced by the Riemannian metric.

Proof. Let $a_i = \max\{1, \sup_{x\in M} \|d_x f_i\|\}$, $0 \le i \le m-1$. For any $w = w_1 w_2 \cdots w_n \in F_m^+$, by the mean-value theorem

$$d(f_{w_i}(x), f_{w_i}(y)) \le (\sup_{x\in M} \|d_x f_{w_i}\|) \cdot d(x,y) \le a_{w_i} d(x,y), \quad \forall x, y \in M, \quad 1 \le i \le n.$$

Suppose K is a compact subset of M. We shall select convenient charts on M that cover K. Let $||| \cdot |||$ denote the norm on $\mathbb{R}^p$ given by

$$||| \, u \, |||= \max \mid u_i \mid \text{ if } u = (u_1, \cdots, u_p) \in \mathbb{R}^p$$

and let $B(0,r)$ denote the open ball in $\mathbb{R}^p$ with center 0 and radius r in this norm. Choose differentiable maps $\varphi_j : B(0,3) \to M, 1 \le j \le q$, such that $K \subset \bigcup_{j=1}^q \varphi_j(B(0,1))$. Let $b > 0$ be so that $d(\varphi_j(u), \varphi_j(v)) \le b \, ||| \, u - v \, |||, \forall u, v \in B(0,2), 1 \le j \le q$. For any $\delta \in (0,1)$, let

$$E(\delta) = \{(t_1\delta, \cdots, t_p\delta) \in \mathbb{R}^p : t_i \in \mathbb{Z}\} \cap B(0,2).$$

The cardinality of $E(\delta)$ is at most $(\frac{4}{\delta})^p$. Each point of $B(0,2)$ is within distance δ of a point of $E(\delta)$. Consider $F(\delta) = \bigcup_{j=1}^q \varphi_j(E(\delta))$. This set is clearly a $(w, (\prod_{i=1}^n a_{w_i})b\delta, f_0, \cdots, f_{m-1})$ spanning set for K. Given $\epsilon > 0$, put $\delta = \frac{\epsilon}{(\prod_{i=1}^n a_{w_i})b}$, then

$$N_{\text{span}}(w, \epsilon, K, f_0, \cdots, f_{m-1}) \le q\Big(\frac{4(\prod_{i=1}^n a_{w_i})b}{\epsilon}\Big)^p = \Big(\prod_{i=1}^n a_{w_i}\Big)^p \Big(\frac{4b}{\epsilon}\Big)^p q.$$

So

$$\begin{aligned} N_{\text{span}}(n, \epsilon, K, f_0, \cdots, f_{m-1}) &\le \frac{1}{m^n} \sum_{|w|=n} \Big(\prod_{i=1}^n a_{w_i}\Big)^p \Big(\frac{4b}{\epsilon}\Big)^p q \\ &= \frac{1}{m^n} \Big(\sum_{i=0}^{m-1} a_i^p\Big)^n \Big(\frac{4b}{\epsilon}\Big)^p q. \end{aligned}$$

Moreover

$$\limsup_{n\to\infty} \frac{1}{n} \log N_{\text{span}}(n, \epsilon, K, f_0, \cdots, f_{m-1}) \le \log\Big(\frac{1}{m} \sum_{i=0}^{m-1} a_i^p\Big).$$

Letting $\epsilon \to 0$ we have

$$H_d(K, f_0, \cdots, f_{m-1}) \le \log\Big(\frac{1}{m} \sum_{i=0}^{m-1} a_i^p\Big).$$

Then

$$H_d(f_0, \cdots, f_{m-1}) \le \log\Big(\frac{1}{m} \sum_{i=0}^{m-1} a_i^p\Big) = \log\Big(\frac{1}{m} \sum_{i=0}^{m-1} \big(\max\{1, \sup_{x\in M} \|d_x f_i\|\}\big)^p\Big). \qquad \square$$

Example 5.2. Let M be a p-dimensional Riemannian manifold, $f_0, \cdots, f_{m-1}$ the Lipschitz maps with a common Lipschitz constant $L \ge 1$. Applying Theorem 5.1,

$$H_d(f_0, \cdots, f_{m-1}) < p \log L.$$

Theorem 5.3. *Let X be a compact metrizable group, $A_0, \cdots, A_{m-1}$ surjective endomorphisms of X and $a_0, \cdots, a_{m-1} \in X$. Let μ denote the Haar measure on X and d a left-invariant metric on X. Then $h(A_0, \cdots, A_{m-1}) = h(a_0 \cdot A_0, \cdots, a_{m-1} \cdot A_{m-1})$ and*

$$h(A_0, \cdots, A_{m-1}) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(e, \epsilon, A_0, \cdots, A_{m-1}))}\big)\Big], \tag{5.1}$$

where

$$D_w(e, \epsilon, A_0, \cdots, A_{m-1}) = \bigcap_{w' \le w} A_{w'}^{-1}(B_d(e, \epsilon))$$

and e is the identity element of X and $B_d(e, \epsilon)$ is the open ball with center e and radius ϵ with respect to the metric d.

Proof. Let $g_i = a_i \cdot A_i$ for any $0 \le i \le m-1$. For any $\epsilon > 0$, $w = i_1 \cdots i_k \in F_m^+$ and $w' = i_l \cdots i_k$ where $1 \le l \le k$, Wang and Ma ([21], in the proof of Theorem 4.2) proved that

$$g_{w'}^{-1} B_d(g_{w'}(x), \epsilon) = x \cdot (A_{i_l} \circ \cdots \circ A_{i_k})^{-1} B_d(e, \epsilon).$$

Putting $D_w(x, \epsilon, g_0, \cdots, g_{m-1}) = \bigcap_{w' \le w} g_{w'}^{-1}(B_d(g_{w'}(x), \epsilon))$, we have

$$\begin{aligned} D_w(x, \epsilon, g_0, \cdots, g_{m-1}) &= \bigcap_{w' \le w} g_{w'}^{-1}(B_d(g_{w'}(x), \epsilon)) \\ &= \bigcap_{w' \le w} x \cdot A_{w'}^{-1} B_d(e, \epsilon) \\ &= x \cdot \bigcap_{w' \le w} A_{w'}^{-1} B_d(e, \epsilon) \\ &= x \cdot D_w(e, \epsilon, A_0, \cdots, A_{m-1}). \end{aligned}$$

Consider a $(w, \epsilon, g_0, \cdots, g_{m-1})$-separated subset E of X with cardinality $N_{\text{sep}}(w, \epsilon, g_0, \cdots, g_{m-1})$, then

$$\bigcup_{x \in E} D_w(x, \frac{\epsilon}{2}, g_0, \cdots, g_{m-1}) = \bigcup_{x \in E} x \cdot D_w(e, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1})$$

is a disjoint union since E is a separated subset of X. Therefore

$$N_{\text{sep}}(w, \epsilon, g_0, \cdots, g_{m-1}) \cdot \mu(D_w(e, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1})) \le 1.$$

Thus

$$N_{\text{sep}}(w, \epsilon, g_0, \cdots, g_{m-1}) \le \frac{1}{\mu(D_w(e, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))},$$

and

$$N_{\text{sep}}(n, \epsilon, g_0, \cdots, g_{m-1}) \le \frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(e, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))}$$

and then

$$N_{\rm sep}(\epsilon, g_0, \cdots, g_{m-1}) \le \limsup_{n\to\infty} \Big[\frac{1}{n}\log\big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(e,\frac{\epsilon}{2}, A_0,\cdots,A_{m-1}))}\big)\Big].$$

Moreover

$$\begin{aligned} h(a_0\cdot A_0,\cdots,a_{m-1}\cdot A_{m-1}) &= h(g_0,\cdots,g_{m-1}) \\ &\le \lim_{\epsilon\to 0}\limsup_{n\to\infty} \Big[\frac{1}{n}\log\big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(e,\frac{\epsilon}{2}, A_0,\cdots,A_{m-1}))}\big)\Big]. \end{aligned}$$

If a set F $(w,\epsilon,g_0,\cdots,g_{m-1})$ spans X with cardinality $N_{\rm span}(w,\epsilon,g_0,\cdots,g_{m-1})$, then

$$X \subset \bigcup_{x\in F} D_w(x, 2\epsilon, g_0,\cdots,g_{m-1}) \subset \bigcup_{x\in F} x\cdot D_w(e,2\epsilon,A_0,\cdots,A_{m-1}).$$

Thus

$$N_{\rm span}(w,\epsilon,g_0,\cdots,g_{m-1})\cdot \mu(D_w(e,2\epsilon,A_0,\cdots,A_{m-1})) \ge 1,$$

and then

$$N_{\rm span}(w,\epsilon,g_0,\cdots,g_{m-1}) \ge \frac{1}{\mu(D_w(e,2\epsilon,A_0,\cdots,A_{m-1}))}.$$

Therefore

$$N_{\rm span}(n,\epsilon,g_0,\cdots,g_{m-1}) \ge \frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(e,2\epsilon,A_0,\cdots,A_{m-1}))},$$

and then

$$N_{\rm span}(\epsilon, g_0,\cdots,g_{m-1}) \ge \limsup_{n\to\infty}\Big[\frac{1}{n}\log\big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(e,2\epsilon,A_0,\cdots,A_{m-1}))}\big)\Big].$$

Moreover,

$$\begin{aligned} h(a_0\cdot A_0,\cdots,a_{m-1}\cdot A_{m-1}) &= h(g_0,\cdots,g_{m-1}) \\ &\ge \lim_{\epsilon\to 0}\limsup_{n\to\infty} \Big[\frac{1}{n}\log\big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(e,2\epsilon, A_0,\cdots,A_{m-1}))}\big)\Big]. \end{aligned}$$

This expression also equals $h(A_0,\cdots,A_{m-1})$ since the right hand is independent of a_i, $0\le i\le m-1$. So the equality (5.1) holds. □

Corollary 1. *Let $\mathbb{T}^p$ be the p-dimensional torus, μ the Haar measure on $\mathbb{T}^p$ and $f_0,\cdots,f_{m-1}$ the surjective endomorphisms on $\mathbb{T}^p$. Then*

$$h(f_0,\cdots,f_{m-1}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\Big[\frac{1}{n}\log\big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(e,\epsilon,f_0,\cdots,f_{m-1}))}\big)\Big],$$

where

$$D_w(e,\epsilon,f_0,\cdots,f_{m-1}) = \bigcap_{w'\le w} f_{w'}^{-1}(B_d(e,\epsilon)).$$

Theorem 5.4. *Let $(\tilde{X}, \tilde{d})$ and (X, d) be metric spaces and $\pi : \tilde{X} \to X$ a continuous surjection such that there exists $\delta > 0$ with*

$$\pi|_{B_{\tilde{d}}(\tilde{x},\delta)} : B_{\tilde{d}}(\tilde{x}, \delta) \to B_d(\pi(\tilde{x}), \delta)$$

an isometric surjection for all $\tilde{x} \in \tilde{X}$. If $\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_{m-1}$ are uniformly continuous transformations on $\tilde{X}$, and $f_0, f_1, \cdots, f_{m-1}$ are uniformly continuous transformations on X satisfying $\pi\tilde{f}_i = f_i\pi$ for any $0 \le i \le m-1$, then

$$H_d(f_0, \cdots, f_{m-1}) = H_{\tilde{d}}(\tilde{f}_0, \cdots, \tilde{f}_{m-1})$$

Proof. The proof follows [21] and is omitted. □

Corollary 2. *Let $\mathbb{T}^p$ be the p-dimensional torus, $A_0, \cdots, A_{m-1}$ endomorphisms of $\mathbb{T}^p$, and $\tilde{A}_0, \cdots, \tilde{A}_{m-1}$ linear transformations on $\mathbb{R}^p$, where $\tilde{A}_i$ is the lift of A_i for $0 \le i \le m-1$, then*

$$H_d(A_0, \cdots, A_{m-1}) = H_{\tilde{d}}(\tilde{A}_0, \cdots, \tilde{A}_{m-1}),$$

where $\tilde{d}$ is the metric on $\mathbb{R}^p$ determined from Euclidean norm.

Lemma 5.5. *Let $A_0, \cdots, A_{m-1}$ be the linear transformations on $\mathbb{R}^p$, μ the Lebesgue measure on $\mathbb{R}^p$ and ρ a metric on $\mathbb{R}^p$ determined by a norm. Then*

$$H_\rho(A_0, \cdots, A_{m-1}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \Big[\frac{1}{n} \log\big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, \epsilon, A_0, \cdots, A_{m-1}))}\big)\Big],$$

where

$$D_w(0, \epsilon, A_0, \cdots, A_{m-1}) = \bigcap_{w' \le w} A_{w'}^{-1}(B_\rho(0, \epsilon))$$

and

$$B_\rho(0, \epsilon) = \{x \in \mathbb{R}^p : \rho(x, 0) < \epsilon\}.$$

Also, $H_\rho(A_0, \cdots, A_{m-1})$ does not depend on the norm chosen.

Proof. Since all norms on $\mathbb{R}^p$ are equivalent, they induce uniformly equivalent metrics on $\mathbb{R}^p$. So by Theorem 3.5 we have $H_\rho(A_0, \cdots, A_{m-1}) = H_d(A_0, \cdots, A_{m-1})$ where d is the Euclidean distance on $\mathbb{R}^p$. Also it is clear that the expression given in the theorem is also independent of the norm. Hence we can suppose ρ is the Euclidean distance as well.

Let K be a compact subset of $\mathbb{R}^p$ with $\mu(K) > 0$, $\epsilon > 0$ and $w \in F_m^+$. Similar as the proof in [21] (Lemma 4.5), we have

$$N_{\text{span}}(w, \epsilon, K, A_0, \cdots, A_{m-1}) \ge \frac{\mu(K)}{\mu(D_w(0, 2\epsilon, A_0, \cdots, A_{m-1}))}.$$

Hence

$$N_{\text{span}}(n, \epsilon, K, A_0, \cdots, A_{m-1}) \ge \frac{1}{m^n} \sum_{|w|=n} \frac{\mu(K)}{\mu(D_w(0, 2\epsilon, A_0, \cdots, A_{m-1}))}.$$

Then

$$N_{\text{span}}(\epsilon, K, A_0, \cdots, A_{m-1}) \geq \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{\mu(K)}{\mu(D_w(0, 2\epsilon, A_0, \cdots, A_{m-1}))}\big)\Big]$$
$$= \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, 2\epsilon, A_0, \cdots, A_{m-1}))}\big)\Big].$$

Therefore

$$H_\rho(A_0, \cdots, A_{m-1}) \geq \lim_{\epsilon\to 0} \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, \epsilon, A_0, \cdots, A_{m-1}))}\big)\Big].$$

On the other hand, let $w \in F_m^+$. Let K_q be the closed p-cube with center $0 \in \mathbb{R}^p$ and side length $2q$. Similar as the proof in [21] (Lemma 4.5), we have

$$N_{\text{sep}}(w, \epsilon, K_q, A_0, \cdots, A_{m-1}) \leq \frac{2^p(q+\epsilon)^p}{\mu(D_w(0, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))},$$

and then

$$N_{\text{sep}}(n, \epsilon, K_q, A_0, \cdots, A_{m-1}) \leq \frac{1}{m^n} \sum_{|w|=n} \frac{2^p(q+\epsilon)^p}{\mu(D_w(0, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))}.$$

Moreover,

$$N_{\text{sep}}(\epsilon, K_q, A_0, \cdots, A_{m-1}) \leq \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))}\big)\Big].$$

If K is any compact subset of $\mathbb{R}^p$ then there exists $q > 0$ such that $K \subset K_q$. Thus

$$N_{\text{sep}}(\epsilon, K, A_0, \cdots, A_{m-1}) \leq N_{\text{sep}}(\epsilon, K_q, A_0, \cdots, A_{m-1})$$
$$\leq \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))}\big)\Big],$$

and then

$$H_\rho(K, A_0, \cdots, A_{m-1}) = \lim_{\epsilon\to 0} N_{\text{sep}}(\epsilon, K, A_0, \cdots, A_{m-1})$$
$$\leq \lim_{\epsilon\to 0} \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, \frac{\epsilon}{2}, A_0, \cdots, A_{m-1}))}\big)\Big].$$

Therefore

$$H_\rho(A_0, \cdots, A_{m-1}) = \sup\{H_\rho(K, A_0, \cdots, A_{m-1}) : K \subset \mathbb{R}^p \text{ is compact}\}$$
$$\leq \lim_{\epsilon\to 0} \limsup_{n\to\infty} \Big[\frac{1}{n} \log \big(\frac{1}{m^n} \sum_{|w|=n} \frac{1}{\mu(D_w(0, \epsilon, A_0, \cdots, A_{m-1}))}\big)\Big]. \qquad \square$$

Theorem 5.6. *Let V be a p-dimensional vector space, ρ a metric on V induced by a norm on V, $A_0, \cdots, A_{m-1}$ the linear transformations on V. If for each $0 \leq i \leq m-1$ all eigenvalues of A_i is of modulus greater than or equal to 1, then*

$$\log \frac{1}{m}\Big(\sum_{i=0}^{m-1}\prod_{j=1}^{p}|\lambda_j^{(i)}|\Big) \leq H_\rho(A_0,\cdots,A_{m-1}) \leq \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\Lambda_i^p\Big)$$

where $\lambda_1^{(i)},\lambda_2^{(i)},\ldots,\lambda_p^{(i)}$ *are the eigenvalues of* A_i, $0\leq i\leq m-1$, *counted with their multiplicities, and* Λ_i *is the biggest eigenvalue of* $\sqrt{A_iA_i^T}$, $0\leq i\leq m-1$.

Particularly in the case $p=1$ *and* $V=\mathbb{R}^1$, *we have*

$$H_\rho(A_0,\cdots,A_{m-1}) = \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}|\lambda_1^{(i)}|\Big),$$

where $\lambda_1^{(i)}$ *is the proportionality constant of* $A_i:\mathbb{R}\to\mathbb{R}, x\mapsto\lambda_1^{(i)}x, 0\leq i\leq m-1$.

Proof. By choosing a basis in V, we can suppose $V=\mathbb{R}^p$. Let μ be the Lebesgue measure on $\mathbb{R}^p$. Since all norms on $\mathbb{R}^p$ are equivalent they induce uniformly equivalent metrics on $\mathbb{R}^p$ and by Theorem 3.5 $H_\rho(A_0,\cdots,A_{m-1}) = H_d(A_0,\cdots,A_{m-1})$ where d is the Euclidean distance. Hence we may suppose that ρ is the Euclidean distance. By Lemma 5.5 we have

$$H_\rho(A_0,\cdots,A_{m-1}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\Big[\frac{1}{n}\log\Big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(0,\epsilon,A_0,\cdots,A_{m-1}))}\Big)\Big].$$

Let $w=w_1\cdots w_n\in F_m^+$ and $n\geq 1$. Similar as the proof in [21] (Thm. 4.6), we have

$$\begin{aligned}\mu(D_w(0,\epsilon,A_0,\cdots,A_{m-1})) &\leq \mu(A_w^{-1}B_\rho(0,\epsilon))\\ &= |\det(A_w)^{-1}|\cdot\mu(B_\rho(0,\epsilon))\\ &= \frac{1}{\prod_{i=1}^n\prod_{j=1}^p|\lambda_j^{(w_i)}|}\cdot\mu(B_\rho(0,\epsilon)).\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{n}\log\Big(\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(0,\epsilon,A_0,\cdots,A_{m-1}))}\Big)\\ &\geq \frac{1}{n}\log\Big(\frac{1}{m^n}\sum_{|w|=n}\prod_{i=1}^{n}\prod_{j=1}^{p}|\lambda_j^{(w_i)}|\Big) - \frac{1}{n}\log\mu(B_\rho(0,\epsilon))\\ &= \frac{1}{n}\log\Big[\frac{1}{m^n}\Big(\sum_{i=0}^{m-1}\prod_{j=1}^{p}|\lambda_j^{(i)}|\Big)^n\Big] - \frac{1}{n}\log\mu(B_\rho(0,\epsilon))\\ &= \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\prod_{j=1}^{p}|\lambda_j^{(i)}|\Big) - \frac{1}{n}\log\mu(B_\rho(0,\epsilon))\end{aligned}$$

and

$$H_d(A_0,\cdots,A_{m-1}) \geq \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\prod_{j=1}^{p}|\lambda_j^{(i)}|\Big).$$

On the other hand, for any $n\geq 1$ and $w=w_1\cdots w_n\in F_m^+$ and $w'=w_i\cdots w_n\leq w$, we can omit from [21] (Thm. 4.6) that

$$(A_{w'})^{-1}B_\rho(0,\epsilon) \supset B_\rho(0, \frac{1}{\prod_{i=1}^n \Lambda_{w_i}}\epsilon).$$

Thus

$$D_w(0,\epsilon, A_0, \cdots, A_{m-1}) = \bigcap_{w' \leq w} (A_w)^{-1}B_\rho(0,\epsilon) \supset B_\rho(0, \frac{1}{\prod_{i=1}^n \Lambda_{w_i}}\epsilon).$$

Then

$$\begin{aligned}
&\frac{1}{n}\log\Big[\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(D_w(0,\epsilon,A_0,\cdots,A_{m-1}))}\Big]\\
&\leq \frac{1}{n}\log\Big[\frac{1}{m^n}\sum_{|w|=n}\frac{1}{\mu(B_\rho(0,\frac{1}{\prod_{i=1}^n \Lambda_{w_i}}\epsilon))}\Big]\\
&= \frac{1}{n}\log\Big[\frac{1}{m^n}\sum_{|w|=n}\Big(\prod_{i=1}^n \Lambda_{w_i}\Big)^p\Big] - \frac{1}{n}\log\mu(B_\rho(0,\epsilon))\\
&= \frac{1}{n}\log\Big[\frac{1}{m^n}\Big(\sum_{i=0}^{m-1}\Lambda_i^p\Big)^n\Big] - \frac{1}{n}\log\mu(B_\rho(0,\epsilon))\\
&= \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\Lambda_i^p\Big) - \frac{1}{n}\log\mu(B_\rho(0,\epsilon)).
\end{aligned}$$

Therefore

$$H_\rho(A_0,\cdots,A_{m-1}) \leq \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\Lambda_i^p\Big). \qquad \square$$

Example 5.7. Let $A_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto 3x$, and $A_2 : \mathbb{R} \to \mathbb{R}$, $x \mapsto 5x$. Applying Theorem 5.6, we cam get $H(A_1, A_2) = \log\frac{1}{2}(3 + 5) = 2\log 2$. Denote $G_1 = \{\mathrm{id}_\mathbb{R}, A_1, A_2\}$. For the topological entropy $H(G_1)$ defined in [21], we have $H(G_1) = \log 5$. Obviously, $H(A_1, A_2) < H(G_1)$.

Remark 5.8. Example 5.7 shows that $H(A_1, A_2)$ is strictly less than the topological entropy $H(G_1)$ defined in [21].

Theorem 5.9. *Let $A_0, \cdots, A_{m-1}$ be endomorphisms of $\mathbb{T}^p$. If for each $0 \leq i \leq m-1$ all eigenvalues of the matrix $[A_i]$ which represents A_i are of modulus greater than or equal to 1, then*

$$\log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\prod_{j=1}^{p}|\lambda_j^{(i)}|\Big) \leq H_\rho(A_0,\cdots,A_{m-1}) \leq \log\frac{1}{m}\Big(\sum_{i=0}^{m-1}\Lambda_i^p\Big)$$

where $\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_p^{(i)}$ are the eigenvalues of $[A_i]$, $0 \leq i \leq m-1$, counted with their multiplicities, and Λ_i is the biggest eigenvalue of $\sqrt{[A_i][A_i]^\top}$, $0 \leq i \leq m-1$.

Particularly in the case $p = 1$, we have

$$H_\rho(A_0,\cdots,A_{m-1}) = \log\frac{1}{m}\big(\sum_{i=1}^{m}|\lambda_1^{(i)}|\big),$$

where $\lambda_1^{(i)}$ is the degree of the automorphism A_i of S^1, for every $0 \leq i \leq m-1$, where S^1 denotes the unite circle.

Remark 5.10. (1) For the system generated by the iteration of a single linear map A of $\mathbb{R}^p$, we already have

$$h(A) = \sum_{\{i:|\lambda_i|>1\}} \log|\lambda_i|$$

where $h(A)$ is the topological entropy of the single map A and $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of the linear map A [22].

(2) if G is a semigroup generated by $G_1 = \{\mathrm{id}_V, f_0, \cdots, f_{m-1}\}$, Wang and Ma [21] gave

$$\max_{0\le i\le m-1} \sum_{j=1}^{p} \log|\lambda_j^{(i)}| \le H(G_1) \le p \max_{0\le i\le m-1} \log \Lambda_i,$$

where $H(G_1)$ is the topological entropy of the semigroup defined in [21].

Example 5.11. Let A and B be the hyperbolic automorphisms on $\mathbb{T}^2$ induced by the matrices

$$\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \text{and} \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}$$

respectively. Then we can easily get the bounds of the entropy $H(A, B)$ as

$$\log\frac{9}{2} \le H(A,B) \le \log(\frac{19+3\sqrt{29}}{4} + 3 + \sqrt{5}).$$

Remark 5.12. This example is appeared in [21]. If $G_1 = \{\mathrm{id}_{\mathbb{T}^2}, A, B\}$, then the topological entropy $H(G_1)$ defined in [21] follows

$$\log 5 \le H(G_1) \le 2\log\frac{3+\sqrt{29}}{2}.$$

Problem 5.13. It is well known that for a homeomorphism $f : X \to X$ of a compact metric space X the equality $h(f) = h(f^{-1})$ holds where $h(f)$ denotes the topological entropy of f. If $f_0, \cdots, f_{m-1}$ are homeomorphisms of X, is it true that $h(f_0, \cdots, f_{m-1}) = h(f_0^{-1}, \cdots, f_{m-1}^{-1})$?

Acknowledgments

The authors really appreciate the referees' valuable remarks and suggestions that helped a lot. The work was supported by NSFC 11401220, Guangdong Natural Science Foundation 2014A030313230 and Fundamental Research Funds for the Central Universities SCUT (2015ZZ055, 2015ZZ127).

References

[1] R. Adler, A. Konheim, J. McAndrew, Topological entropy, Trans. Amer. Math. Soc. 114 (1965) 309–319.
[2] A. Biś, Entropies of a semigroup of maps, Discrete Contin. Dyn. Syst. Ser. A 11 (2004) 639–648.
[3] A. Biś, M. Urbański, Some remarks on topological entropy of a semigroup of continuous maps, Cubo 8 (2006) 63–71.
[4] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Amer. Math. Soc. 153 (1971) 401–414, Erratum: Trans. Amer. Math. Soc. 181 (1971) 509–510.
[5] R. Bowen, Topological entropy for noncompact sets, Trans. Amer. Math. Soc. 184 (1973) 125–136.
[6] A. Bufetov, Topological entropy of free semigroup actions and skew-product transformations, J. Dyn. Control Syst. 5 (1999) 137–143.
[7] E.I. Dinaburg, The relation between topological entropy and metric entropy, Sov. Math., Dokl. 11 (1970) 13–16.
[8] S. Friedland, Entropy of graphs, semigroups and groups, in: Ergodic Theory of $\mathbb{Z}^d$ Actions, Warwick, 1993–1994, in: London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 319–343.

[9] E. Ghys, R. Langevin, P. Walczak, Entropie geometrique des feuilletages, Acta Math. 160 (1988) 105–142.
[10] P. Hulse, Counterexamples to the product rule for entropy, Dyn. Syst. 24 (2009) 81–95.
[11] S. Kolyada, L. Snoha, Topological entropy of nonautonomous dynamical systems, Random Comput. Dyn. 4 (1996) 205–233.
[12] R. Langevin, F. Przytycki, Entropie de l'image inverse d'une application, Bull. Soc. Math. France 120 (1992) 237–250.
[13] R. Langevin, P. Walczak, Entropie d'une dynamique, Math. Acad. Sci. Paris, Sér. I 312 (1991) 141–144.
[14] D. Ma, S. Liu, Some properties of topological pressure of a semigroup of continuous maps, Dyn. Syst. 29 (2014) 1–17.
[15] D. Ma, M. Wu, Topological pressure and topological entropy of a semigroup of maps, Discrete Contin. Dyn. Syst. 31 (2011) 545–557.
[16] E. Mihailescu, M. Urbański, Inverse topological pressure with applications to holomorphic dynamics of several complex variables, Commun. Contemp. Math. 6 (2004) 653–679.
[17] E. Mihailescu, M. Urbański, Inverse pressure estimates and the independence of stable dimension for non-invertible maps, Canad. J. Math. 60 (2008) 658–684.
[18] Z. Nitecki, Topological entropy and the preimage structure of maps, Real Anal. Exchange 29 (2003–2004) 9–41.
[19] A.M. Stepin, A.T. Tagri-Zade, Variational characterization of topological pressure for amenable groups of transformations, Dokl. Akad. Nauk SSSR 254 (1980) 545–549.
[20] J. Tang, B. Li, W.C. Cheng, Some properties on topological entropy of free semigroup action, preprint.
[21] Y. Wang, D. Ma, On the topological entropy of a semigroup of continuous maps, J. Math. Anal. Appl. 427 (2015) 1084–1100.
[22] P. Waters, An Introduction to Ergodic Theory, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
[23] Y. Zhu, Z. Liu, X. Xu, W. Zhang, Entropy of nonautonomous dynamical systems, J. Korean Math. Soc. 49 (2012) 165–185.