J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A functional representation of almost isometries [☆]

Javier Cabello [a], Jesús A. Jaramillo [b,*]

[a] *Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas s/n, 06006 Badajoz, Spain*
[b] *IMI and Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain*

A R T I C L E I N F O

Article history:
Received 17 December 2015
Available online xxxx
Submitted by D. Ryabogin

Keywords:
Almost isometries
Quasi-metric spaces
Semi-Lipschitz functions

A B S T R A C T

For each quasi-metric space X we consider the convex lattice $\mathrm{SLip}_1(X)$ of all semi-Lipschitz functions on X with semi-Lipschitz constant not greater than 1. If X and Y are two complete quasi-metric spaces, we prove that every convex lattice isomorphism T from $\mathrm{SLip}_1(Y)$ onto $\mathrm{SLip}_1(X)$ can be written in the form $Tf = c \cdot (f \circ \tau) + \phi$, where τ is an isometry, $c > 0$ and $\phi \in \mathrm{SLip}_1(X)$. As a consequence, we obtain that two complete quasi-metric spaces are almost isometric if, and only if, there exists an almost-unital convex lattice isomorphism between $\mathrm{SLip}_1(X)$ and $\mathrm{SLip}_1(Y)$.

0. Introduction

Suppose you live in a building with no lift. It is clear that your third floor apartment is further from the street than the street is from your apartment, if we think the distance between two places as the time spent for coming from one place to the other one. You may also think about a building with a lift and measure another kind of distance: the time plus the effort – properly weighted. In this case, most people will use the lift when coming home, but not when coming down.

The metrics described above are not symmetric and, moreover, the second one has another strange feature: the ground floor–third floor preferred way (*geodesic*) is different from the third floor–ground floor one.

In this paper we will deal with non-symmetric metrics, called *quasi-metrics*. In the last years there has been an increasing interest about quasi-metric spaces, with applications in a wide variety of topics. We refer to [5,6,8,9] and references therein for further information about the subject. Here we will be interested in some natural transformations of quasi-metric spaces, the so-called *almost isometries* considered in [5]. These are bijections between quasi-metric spaces preserving the triangular function, that is, the quantity

☆ Research partially supported by MINECO (Spain) Projects MTM2012-34341 and MTM2013-45643-C02-01-P and by the Junta de Extremadura project GR10113.

* Corresponding author.
E-mail addresses: coco@unex.es (J. Cabello), jaramil@mat.ucm.es (J.A. Jaramillo).

http://dx.doi.org/10.1016/j.jmaa.2016.04.026
0022-247X/© 2016 Published by Elsevier Inc.

$\mathrm{Tr}(x, y, z) = d(x, y) + d(y, z) - d(x, z)$. This quantity measures how far the points x, y, z are of achieving equality in the triangle inequality. It is easily seen that, in the case that symmetry holds, almost isometries coincide in fact with isometries.

Our aim is to characterize almost isometries by means of a suitable space of real-valued functions, thus obtaining a theorem of Banach–Stone type in this context. In the symmetric case, it has been shown in [1] that a metric on the space X can be recovered, up to Lipschitz homeomorphisms, by using the lattice structure of the space $\mathrm{Lip}(X)$ of real-valued Lipschitz functions on X. Related results have been obtained in [4] in terms of the unital vector lattice structure of $\mathrm{Lip}(X)$. We refer to the survey paper [3] for further developments about Banach–Stone type theorems. In our case, due to the lack of symmetry, it is natural to consider the space of *semi-Lipschitz* functions in the sense of [9]. A function $f : X \to \mathbb{R}$ defined on a quasi-metric space (X, d) is said to be semi-Lipschitz if there exists a constant $L \geq 0$ such that $f(x) - f(y) \leq L \cdot d_X(x, y)$ for every $x, y \in X$. The space of semi-Lipschitz functions on X will be denoted by $\mathrm{SLip}(X)$. Since we are dealing with almost isometries, we will focus on the subspace $\mathrm{SLip}_1(X)$ of semi-Lipschitz functions with constant $L \leq 1$. It is not difficult to see that $\mathrm{SLip}_1(X)$ is a lattice with the usual pointwise operations. Furthermore, it is closed under convex combinations. As a consequence of our main result, we obtain in Corollary 3.8 a characterization of almost isometries of a quasi-metric space X in terms of the *convex lattice* structure of $\mathrm{SLip}_1(X)$.

The contents of the paper are as follows. In section 1 we review some basic facts about quasi-metrics and almost isometries. It is known that a bijection $\tau : (X, d_X) \to (Y, d_Y)$ between quasi-metric spaces is an almost isometry if, and only if, there exists a function $\phi : X \to \mathbb{R}$ (which is unique up to an additive constant) such that

$$d_Y(\tau(x_1), \tau(x_2)) = d_X(x_1, x_2) + \phi(x_2) - \phi(x_1). \tag{1}$$

We will prove that a bijection between quasi-metric spaces is an almost isometry if, and only if, it preserves the length of closed polygonal paths or, equivalently, the length of any rectifiable closed path. In section 2 we obtain the results about semi-Lipschitz functions which are needed along the paper. As we will see, the extension properties of semi-Lipschitz functions will play an important role. Section 3 is devoted to the main result of the paper. It turns out that, given an almost isometry $\tau : (X, d_X) \to (Y, d_Y)$, the function $\phi : X \to \mathbb{R}$ that *governs* τ in the sense of equation (1) allows us to define a bijection $T : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}_1(X, d_X)$ in the following way: $f \mapsto Tf = f \circ \tau + \phi$. It is not difficult to see that this map preserves order and convex combinations, so we say that it is a *convex lattice isomorphism*. In the opposite direction, note that there are three kinds of natural convex lattice isomorphisms to be considered. Namely:

- $T_1 : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}_1(X, d_X), T_1 f = f \circ \tau$, where τ is an isometry.
- $T_2 : \mathrm{SLip}_1(X, d_X) \to \mathrm{SLip}_1(X, c\, d_X), T_2 f = c \cdot f$, where $c \in (0, \infty)$.
- $T_3 : \mathrm{SLip}_1(X, d'_X) \to \mathrm{SLip}_1(X, d_X), T_3 f = f + \phi$, where $d'(x, x') = d(x, x') + \phi(x') - \phi(x)$.

In Theorem 3.1 we show that every convex lattice isomorphism $T : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}_1(X, d_X)$ between complete quasi-metric spaces is, in fact, a composition of one of each kind: T is of the form $Tf = c\cdot(f \circ \tau) + \phi$. Finally, in section 4, we give some further consequences and related results.

1. Quasi-metrics

Let X be a set. We will say $d_X : X \times X \to [0, \infty)$ is a *quasi-metric on* X if the following holds:

- $d_X(x, y) = 0$ if and only if $x = y$.
- $d_X(x, y) \leq d_X(x, z) + d_X(z, y)$ for every $x, y, z \in X$.

We say then that the couple (X, d_X) is a quasi-metric space.

Now, for each $x \in X$ and each $r > 0$, we can consider two kinds of open balls: the *forward* open ball $B_d^+(x,r) = \{y \in X : d(x,y) < r\}$ and the *backward* open ball $B_d^-(x,r) = \{y \in X : d(y,x) < r\}$. We will always consider X endowed with the topology generated by the family $B_d^+(x,r) \cap B_d^-(x,r)$, where $x \in X$ and $r > 0$. It is not difficult to check that this is the topology associated to the metric

$$\widetilde{d_X}(x,y) = \frac{1}{2}(d_X(x,y) + d_X(y,x)),$$

which we will call the *symmetrized metric* associated to d_X.

Following [5], we consider the *triangular function* $\mathrm{Tr}_X : X \times X \times X \to [0,\infty)$, which is given by

$$\mathrm{Tr}_X(x_1,x_2,x_3) = d_X(x_1,x_2) + d_X(x_2,x_3) - d_X(x_1,x_3).$$

Definition 1.1. Let (X,d_X) and (Y,d_Y) be quasi-metric spaces. We will say a bijection $\tau : (X,d_X) \to (Y,d_Y)$ is an *almost isometry* if $\mathrm{Tr}_Y(\tau(x_1),\tau(x_2),\tau(x_3)) = \mathrm{Tr}_X(x_1,x_2,x_3)$ for all $x_1,x_2,x_3 \in X$.

It is clear that every almost isometry $\tau : (X,d_X) \to (Y,d_Y)$ is an isometry between the corresponding symmetrized spaces $(X,\widetilde{d_X})$ and $(Y,\widetilde{d_Y})$:

$$\begin{aligned}\widetilde{d_Y}(\tau(x_1),\tau(x_2)) &= \frac{1}{2}\left(d_Y(\tau(x_1),\tau(x_2)) + d_Y(\tau(x_2),\tau(x_1))\right) = \\ &= \frac{1}{2}\mathrm{Tr}_Y(\tau(x_1),\tau(x_2),\tau(x_1)) = \frac{1}{2}\mathrm{Tr}_X(x_1,x_2,x_1) = \widetilde{d_X}(x_1,x_2).\end{aligned}$$

In particular, in the case of metric spaces, every almost isometry is in fact an isometry. The following useful characterization of almost-isometries has been obtained in [5].

Lemma 1.2. *(See [5], Proposition 2.8.) Given quasi-metric spaces $(X,d_X),(Y,d_Y)$, a bijection $\tau : (X,d_X) \to (Y,d_Y)$ is an almost isometry if and only if there exists a function $\phi : X \to \mathbb{R}$ such that*

$$d_Y(\tau(x_1),\tau(x_2)) = d(x_1,x_2) + \phi(x_2) - \phi(x_1)$$

for every $x_1,x_2 \in X$.

Remark 1.3. Actually, as seen in [5], ϕ is determined up to an additive constant and given by $\phi(x) = d_X(x,x_0) - d_Y(\tau(x),\tau(x_0))$ for any fixed $x_0 \in X$.

Now, given $x_1,\ldots,x_n$ in a quasi-metric space (X,d_X), we will denote the closed polygonal defined by these points as $[x_1,\ldots,x_n]$. Then the perimeter of this closed polygonal is given by:

$$P_X([x_1,x_2,\ldots,x_n]) = d_X(x_1,x_2) + d_X(x_2,x_3) + \ldots + d_X(x_n,x_1).$$

Lemma 1.4. *Let $\tau : (X,d_X) \to (Y,d_Y)$ be a bijection between quasi-metric spaces. Then, the following conditions are equivalent:*

(1) *τ is an almost isometry.*
(2) $P_Y([\tau(x_1),\tau(x_2),\tau(x_3)]) = P_X([x_1,x_2,x_3])$ for all $x_1,x_2,x_3 \in X$.
(3) $P_Y([\tau(x_1),\tau(x_2),\ldots,\tau(x_n)]) = P_X([x_1,x_2,\ldots,x_n])$ for all $x_1,x_2,\ldots,x_n \in X, n \geq 3$.

Proof. We have seen that any almost isometry τ is an isometry between the corresponding symmetrized spaces $(X,\widetilde{d_X})$ and $(Y,\widetilde{d_Y})$. It is also easy to check that τ is again an isometry whenever 2 or 3 holds. Now,

Please cite this article in press as: J. Cabello, J.A. Jaramillo, A functional representation of almost isometries, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.04.026

as $P_X(x_1, x_2, x_3) = \mathrm{Tr}_X(x_1, x_2, x_3) + 2\widetilde{d_X}(x_1, x_3)$, we obtain that 2 is equivalent to 1. It is obvious that 3 implies 2, so it only that remains to check is that 2 implies 3. Let $x_1, \dots, x_n \in X$. We must show that

$$P_Y([\tau(x_1), \tau(x_2), \dots, \tau(x_n)]) = P_X([x_1, x_2, \dots, x_n]).$$

We will use induction. For $n = 3$, this is our hypothesis, so assume $n \geq 4$. Then,

$$\begin{aligned} P_X([x_1, x_2, \dots, x_n]) &= d_X(x_1, x_2) + \dots + d_X(x_{n-1}, x_n) + d_X(x_n, x_1) = \\ &= d_X(x_1, x_2) + \dots + d_X(x_{n-2}, x_{n-1}) + d_X(x_{n-1}, x_1) + \\ &\quad + d_X(x_1, x_{n-1}) + d_X(x_{n-1}, x_n) + d_X(x_n, x_1) - d_X(x_{n-1}, x_1) - d_X(x_1, x_{n-1}) = \\ &= P_X([x_1, x_2, \dots, x_{n-1}]) + P_X([x_1, x_{n-1}, x_n]) - 2\widetilde{d_X}(x_1, x_{n-1}). \end{aligned}$$

The last term equals

$$P_Y([\tau(x_1), \tau(x_2), \dots, \tau(x_{n-1})]) + P_Y([\tau(x_1), \tau(x_{n-1}), \tau(x_n)]) - 2\widetilde{d_Y}(\tau(x_1), \tau(x_{n-1}))$$

by induction hypothesis. $\square$

Let $\gamma : [a, b] \to X$ be a closed path on a quasi-metric space (X, d_X), that is, a continuous function defined on the interval $[a, b]$ such that $\gamma(a) = \gamma(b)$. As usual, we define the *length* of γ as

$$\ell(\gamma) = \sup P([\gamma(t_1), \gamma(t_2), \dots, \gamma(t_n)]),$$

where the supremum is taken over finite increasing sequences $a = t_0 < t_1 < t_2 < \dots < t_n = b$. We say that γ is *rectifiable* if it has finite length. Then the following is an immediate consequence of Lemma 1.4:

Corollary 1.5. *A bijection $\tau : (X, d_X) \to (Y, d_Y)$ between quasi-metric spaces is an almost isometry if, and only if, $\ell(\tau \circ \gamma) = \ell(\gamma)$ for every rectifiable closed path γ in (X, d_X).*

2. Semi-Lipschitz functions

We recall the following definition from [9]:

Definition 2.1. Let (X, d_X) be a quasi-metric space. A function $f : X \to \mathbb{R}$ is said to be *semi-Lipschitz* if there exists a constant $L \geq 0$ such that

$$f(x) - f(x') \leq L d_X(x, x')$$

for every $x, x' \in X$. The lowest of these constants is called the semi-Lipschitz constant of f, and will be denoted by $\mathrm{SLip}(f)$. The space of all semi-Lipschitz functions $f : X \to \mathbb{R}$ will be denoted by $\mathrm{SLip}(X)$.

Remark 2.2. Note that every d_X-semi-Lipschitz function is continuous for the associated symmetrized distance $\widetilde{d_X}$. In fact, every d_X-semi-Lipschitz function $f : (X, d_X) \to \mathbb{R}$ is $\widetilde{d_X}$-Lipschitz, with Lipschitz constant $\mathrm{Lip}(f) \leq 2\,\mathrm{SLip}(f)$. Indeed, for every $x, x' \in X$ we have:

$$f(x) - f(x') \leq \mathrm{SLip}(f) d(x, x') \leq \mathrm{SLip}(f)\, 2\, \widetilde{d_X}(x, x').$$

Moreover, this bound is sharp, as the following example shows.

Example 2.3. On the real line $\mathbb{R}$ consider the Sorgenfrey quasi-metric d_S, defined by

$$d_S(x,x') = \begin{cases} x - x' & \text{if } x \geq x' \\ 1 & \text{if } x < x' \end{cases} \tag{2}$$

Consider the map $f : (\mathbb{R}, d_S) \to [0,1)$ given by $f(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ stands for the integer part of x. That is, if $x = m + \alpha$, with $m \in \mathbb{Z}$ and $\alpha \in [0,1)$, then $f(x) = \alpha$. Then, f is semi-Lipschitz, $\mathrm{SLip}(f) = 1$ and $\mathrm{Lip}(f) = 2$. Indeed, we have that:

- If $x < x'$, then $d_S(x,x') = 1$ and $f(x) - f(x') \leq 1$.
- If $x \geq x'$, then $f(x) - f(x') \leq x - x' = d_S(x,x')$.

Thus $\mathrm{SLip}(f) \leq 1$. On the other hand, $f(x) - f(x') = x - x'$ if $\lfloor x \rfloor = \lfloor x' \rfloor$, so that $\mathrm{SLip}(f) = 1$.

As for $\mathrm{Lip}(f)$, note that the symmetrized metric is given by $\widetilde{d_S}(x,x') = \frac{1}{2}(1 + |x - x'|)$ for every $x' \neq x$, and therefore it induces the discrete topology on $\mathbb{R}$. It easy to see that $\mathrm{Lip}(f) = 2$. Take, for instance, the sequences $(\frac{1}{n}), (\frac{-1}{n})$ and observe that $\widetilde{d_S}(\frac{1}{n}, \frac{-1}{n}) = \frac{1}{2} + \frac{1}{n}$ and $f(\frac{-1}{n}) - f(\frac{1}{n}) = 1 - \frac{2}{n}$ for each n. □

For a quasi-metric space (X, d_X), let $\mathrm{SLip}_1(X)$ denote the space of all semi-Lipschitz functions $f : X \to \mathbb{R}$ with constant ≤ 1:

$$\mathrm{SLip}_1(X) = \{f : X \to \mathbb{R} : f(x) - f(x') \leq d_X(x,x'),\ \forall x, x' \in X\}.$$

It is not difficult to check that, given two functions f, g in $\mathrm{SLip}_1(X)$, then their supremum $f \vee g$ and infimum $f \wedge g$ also belong to $\mathrm{SLip}_1(X)$. Thus $\mathrm{SLip}_1(X)$ has a natural lattice structure. Furthermore, it is closed under convex combinations. In this sense we say that $\mathrm{SLip}_1(X)$ has a *convex lattice* structure. If now (Y, d_Y) is another quasi-metric space, we will say that a mapping $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$ is a *convex lattice isomorphism* if the following conditions are satisfied:

- T is a bijection,
- T preserves convex combinations, that is, $T(\lambda f + (1-\lambda)g) = \lambda Tf + (1-\lambda)Tg$ for each $f, g \in \mathrm{SLip}_1(Y)$ and $\lambda \in [0,1]$, and
- T preserves order, that is, $Tf \leq Tg$ if and only if $f \leq g$.

Note that every bijection between lattices that preserves order is automatically a lattice isomorphism. Therefore we have that every convex lattice isomorphism T satisfies that $T(f \vee g) = (Tf) \vee (Tg)$ and $T(f \wedge g) = (Tf) \wedge (Tg)$.

The convex lattice structure of $\mathrm{SLip}_1(X)$ is naturally related with almost isometries, in the following way.

Proposition 2.4. *Let $\tau : (X, d_X) \to (Y, d_Y)$ be an almost isometry between quasi-metric spaces, and let $\phi : X \to \mathbb{R}$ such that $d_Y(\tau(x), \tau(x')) = d_X(x,x') + \phi(x') - \phi(x)$. Then $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$, given by $Tf = f \circ \tau + \phi$, is a convex lattice isomorphism.*

Proof. We need to show that $Tf \in \mathrm{SLip}_1(X)$ for every $f \in \mathrm{SLip}_1(Y)$. Indeed, let $x, x' \in X$. Then

$$Tf(x) - Tf(x') = f(\tau(x)) + \phi(x) - f(\tau(x')) - \phi(x') \leq d_Y(\tau(x), \tau(x')) + \phi(x) - \phi(x') =$$
$$= d_Y(\tau(x), \tau(x')) + d_X(x,x') - d_Y(\tau(x), \tau(x')) = d_X(x,x').$$

As it is clear that T is injective, by symmetry we have that T is a bijection. Given $f, g \in \mathrm{SLip}_1(Y)$, it is easy to see that $Tf \leq Tg$ whenever $f \leq g$. Furthermore, we have that $T(\lambda f + (1-\lambda)g) = \lambda Tf + (1-\lambda)Tg$, for each $\lambda \in [0,1]$. Therefore, T is a convex lattice isomorphism. $\square$

In the next section we will obtain the general form of convex lattice isomorphisms (see Theorem 3.1). In order to achieve this, we need some further developments about semi-Lipschitz functions.

Lemma 2.5. *Let (X, d_X) be a quasi-metric space. For every nonempty $A \subset X$, the functions $f(x) = d_X(x, A)$ and $g(x) = -d_X(A, x)$ belong to $\mathrm{SLip}_1(X)$, where $d_X(A, B)$ is defined as usually as $\inf\{d_X(a, b) : a \in A, b \in B\}$.*

Proof. We will check that $\mathrm{SLip}(f) \leq 1$, for g it is analogous. Let $x, x' \in X$, and consider $\varepsilon > 0$. Choose $a \in A$ such that $d_X(x', a) \leq d_X(x', A) + \varepsilon$. Then

$$f(x) - f(x') = d_X(x, A) - d_X(x', A) \leq d_X(x, A) - d_X(x', a) + \varepsilon \leq$$
$$\leq d_X(x, a) - d_X(x', a) + \varepsilon \leq d_X(x, x') + \varepsilon.$$

As ε is arbitrary, we have $f(x) - f(x') \leq d_X(x, x')$. $\square$

Next we are going to see that semi-Lipschitz functions defined on arbitrary subsets can be extended elsewhere preserving the semi-Lipschitz constant. In fact, the McShane extension of Lipschitz functions (see [7], Theorem 1) can be adapted to the semi-Lipschitz setting.

Lemma 2.6. *Let (X, d_X) be a quasi-metric space, and consider a nonempty subset A of X. If $f : A \to \mathbb{R}$ is a semi-Lipschitz function with constant L, then the function $\tilde{f} : X \to \mathbb{R}$ defined by*

$$\tilde{f}(x) = \inf\{f(a) + L\, d(x, a) \, : \, a \in A\}$$

is a semi-Lipschitz extension of f with constant L.

Proof. It is clear that $\tilde{f}$ is an extension of f. We are going to check that $\mathrm{SLip}(\tilde{f}) \leq L$. Let $x, x' \in X$, and consider $\varepsilon > 0$. Choose $a \in A$ such that $\tilde{f}(x') \geq f(a) + L\, d(x', a) - \varepsilon$. Then

$$\tilde{f}(x) - \tilde{f}(x') \leq f(a) + L\, d(x, a) - f(a) - L\, d(x', a) + \varepsilon = L\,(d(x, a) - d(x', a)) + \varepsilon \leq L\, d(x, x') + \varepsilon.$$

Thus $\tilde{f}(x) - \tilde{f}(x') \leq L\, d(x, x')$. $\square$

In what follows we are going to construct, for every quasi-metric space (X, d_X), a basis of the topology induced by the symmetrized distance $\widetilde{d_X}$, which is defined in terms of the functions in $\mathrm{SLip}_1(X)$, and behaves nicely with respect to its convex lattice structure. This will turn out to be quite useful in the next section. In the symmetric case, this kind of construction is usually done by using cozero-sets. Recall that the cozero-set of a function $f : X \to \mathbb{R}$ is defined as $\{x \in X \, : \, f(x) \neq 0\}$. Nevertheless, as our next example shows, in general the construction cannot be done this way for the non-symmetric case, so we will need to consider a modification of cozero-sets.

Example 2.7. Consider the Sorgenfrey line $(\mathbb{R}, d_S)$, where d_S is given as in Example 2.3. As we have seen, the symmetrized distance $\widetilde{d_S}$ induces the discrete topology on $\mathbb{R}$. Thus for each point $x_0 \in \mathbb{R}$, the singleton $V = \{x_0\}$ is an open set, but it cannot be the cozero-set of any function $f \in \mathrm{SLip}_1(X)$. Indeed, if we have that $f(x) - f(x_0) \leq d_S(x, x_0)$ for all $x \in \mathbb{R}$, we obtain that $f(x') + x_0 - x' \leq f(x_0) \leq f(x) + x_0 - x$ for all $x < x_0 < x'$. So if we assume that $f = 0$ on $\mathbb{R} \setminus \{x_0\}$, we deduce that $f(x_0) = 0$.

This is related to the question of uniqueness of extension of semi-Lipschitz functions from subspaces. If we consider $X = (\mathbb{R} \setminus \{x_0\}, d_S)$, then the only way to extend the function $g : X \to \mathbb{R}, g(x) = 0$ to a function in $\mathrm{SLip}_1(\mathbb{R}, d_S)$ is by setting $g(x_0) = 0$. Nevertheless, the function $f : X \to \mathbb{R}$ defined by $f(x) = (x - x_0) - \lfloor x - x_0 \rfloor$ can be extended in many ways to a function in $\mathrm{SLip}_1(\mathbb{R}, d_S)$. Actually, it is easy to check that, for any $\delta \in (0, 1)$, if we set $f_\delta(x_0) = \delta$ and $f_\delta(x) = f(x)$ for every $x \neq x_0$, we obtain that f_δ is an extension of f and $f_\delta \in \mathrm{SLip}_1(\mathbb{R}, d_S)$. In this way we can describe the singleton $V = \{x_0\}$ as $V = \{x \in \mathbb{R} : f_{\delta_2}(x) > f_{\delta_1}(x)\}$ for any $0 < \delta_1 < \delta_2 < 1$. This example will be properly generalized in the next result. □

Proposition 2.8. *Let (X, d_X) be a quasi-metric space and X_0 a subset of X. The restriction mapping $f \in \mathrm{SLip}(X) \mapsto f_{|X_0} \in \mathrm{SLip}(X_0)$ is a bijection if, and only if, $\widetilde{d_X}(x, X_0) = 0$ for every $x \in X$, i.e., if and only if X_0 is dense in X for the symmetrized distance $\widetilde{d_X}$.*

Proof. If X_0 is dense in X for $\widetilde{d_X}$ it is clear that, given a semi-Lipschitz function $f : (X_0, \widetilde{d_X}) \to \mathbb{R}$, there is at most one way of extending f continuously to X. As we have seen before (see Lemma 2.6), semi-Lipschitz functions extend from arbitrary subsets, so $f \mapsto f_{|X_0}$ is a bijection in this case.

Conversely, suppose that $x_0 \in X$ satisfies that $\widetilde{d_X}(x_0, X_0) \geq \varepsilon > 0$. We are going to show that there exists a function $f \in \mathrm{SLip}_1(X_0)$ which admits many different extensions in $\mathrm{SLip}_1(X_0 \cup \{x_0\})$, and these can be further extended from $X_0 \cup \{x_0\}$ to X. Consider the sets $A = \{x \in X_0 : 0 < d_X(x_0, x) < \varepsilon/2\}$ and $B = \{x \in X_0 : 0 < d_X(x, x_0) < \varepsilon/2\}$.

If $B = \emptyset$ then $d_X(x_0, x) \geq \varepsilon/2$ for all $x \in X_0$. In this case, we define the function $f = 0$ on X_0 and for each $\delta \in (0, \varepsilon/2)$ we can consider the extension $f : X_0 \cup \{x_0\} \to \mathbb{R}$ given by setting $f(x_0) = \delta$. It is easy to check that $f \in \mathrm{SLip}_1(X_0 \cup \{x_0\})$.

If $B \neq \emptyset$, we define the function $f(x) = d_X(x, B) \wedge (\varepsilon/2)$ on X_0 and for each $\delta \in (0, \varepsilon/2)$ we can consider the extension $f : X_0 \cup \{x_0\} \to \mathbb{R}$ given by setting $f(x_0) = \delta$. We are going to see that $f \in \mathrm{SLip}_1(X_0 \cup \{x_0\})$. We have to show that $f(x) - f(x') \leq d(x, x')$ for every $x, x' \in X_0 \cup \{x_0\}$:

- This is clear if both $x, x' \in X_0$ since, for every nonempty $C \subset X$, the map $x \mapsto d(x, C)$ belongs to $\mathrm{SLip}_1(X)$ (see Lemma 2.5).
- If $x \in B$ then $f(x) = 0$ and therefore

$$f(x) - f(x_0) = -\delta < 0 \leq d_X(x, x_0)$$
$$f(x_0) - f(x) = \delta < \frac{\varepsilon}{2} \leq d_X(x_0, x)$$

- If $x \in A$, taking into account that $d_X(x_0, x) + d_X(x, x_0) = 2\widetilde{d_X}(x, x_0) \geq 2\varepsilon$, we obtain that $d_X(x, x_0) \geq \varepsilon$ and also $d_X(x_0, b) \geq \varepsilon$ for all $b \in B$. Thus $d_X(x, B) \geq \varepsilon/2$ since, for each $b \in B$:

$$\varepsilon \leq d_X(x_0, b) \leq d_X(x_0, x) + d_X(x, b) < \varepsilon/2 + d_X(x, b).$$

Therefore we have that $f(x) = \varepsilon/2$ and

$$f(x) - f(x_0) = \varepsilon/2 - \delta < \varepsilon \leq d_X(x, x_0)$$
$$f(x_0) - f(x) = \delta - \varepsilon/2 < 0 \leq d_X(x_0, x)$$

- Finally, if $x \notin A \cup B$, we have that

$$(f(x) - f(x_0)) \vee (f(x_0) - f(x)) \le \varepsilon/2 \le d_X(x, x_0) \wedge d_X(x_0, x). \quad \square$$

Definition 2.9. Let (X, d_X) be a quasi-metric space. For every pair $f, h \in \mathrm{SLip}_1(X)$ with $f \ge h$, the h-cozero of f will be $\mathrm{coz}_h(f) = \{x \in X : f(x) > h(x)\}$. The h-support of f, denoted $\mathrm{supp}_h(f)$, is the closure of its h-cozero, and we shall write V_h^f for the interior of $\mathrm{supp}_h(f)$. We denote by $RS(X)$ the family of all sets of the form V_h^f, where $f, h \in \mathrm{SLip}_1(X)$ and $f \ge h$.

Recall that an open set in a topological space is said to be *regular* if it agrees with the interior of its closure or, equivalently, if it agrees with the interior of any closed set. From the very definition, we see that each set V_h^f in $RS(X)$ is a regular open set. In the next Lemma we see that the family $RS(X)$ is a basis of the topology for the symmetrized distance $\widetilde{d_X}$ on X.

Lemma 2.10. *Let (X, d_X) be a quasi-metric space, let U be a nonempty open subset of X for the symmetrized distance $\widetilde{d_X}$, and let $x_0 \in U$. Then there exist $f, h \in \mathrm{SLip}_1(X)$ with $f \ge h$, such that $x_0 \in V_h^f \subset U$.*

Proof. We may assume $U \ne X$, otherwise we could take $f = 1, h = 0$.

Since $\widetilde{d_X}(x_0, X \setminus U) > 0$, from the proof of Proposition 2.8 we deduce that there exist two functions $f_1, f_2 \in \mathrm{SLip}_1((X \setminus U) \cup \{x_0\})$ agreeing on $X \setminus U$ and such that $f_1(x_0) > f_2(x_0)$. If we extend these functions to $\tilde{f}_1, \tilde{f}_2 \in \mathrm{SLip}_1(X)$, we may take $f = \tilde{f}_1 \vee \tilde{f}_2$ and $h = \tilde{f}_1 \wedge \tilde{f}_2$ and we are done. $\square$

Remark 2.11. For any $f \ge h \in \mathrm{SLip}_1(X)$, $c \in \mathbb{R}$ and $\lambda \in (0, 1]$, one has

$$V_{h+c}^{f+c} = V_h^{\lambda f + (1-\lambda)h} = V_{\lambda h}^{\lambda f} = V_h^f.$$

With the next two Lemmas, we will show that the inclusion between members of the family $RS(X)$ can be described using the convex lattice structure of $\mathrm{SLip}_1(X)$.

Lemma 2.12. *Let (X, d_X) be a quasi-metric space and let $f_1, f_2, h_1, h_2 \in \mathrm{SLip}_1(X)$ with $f_1 \ge h_1$ and $f_2 \ge h_2$. The following conditions are equivalent:*

(i) $V_{h_1}^{f_1} \subset V_{h_2}^{f_2}$

(ii) *For every $\psi, \varphi \in \mathrm{SLip}_1(X)$ with $\psi \wedge \varphi \ge h_1 \vee h_2$, we have that $\psi \ge \varphi$ on $V_{h_2}^{f_2}$ implies that $\psi \ge \varphi$ on $V_{h_1}^{f_1}$.*

Proof. It is clear that (i) implies (ii), so let us prove the converse. First note that, if U and V are regular open sets and $U \setminus V$ is nonempty, then there exists $x_0 \in U$ such that $\widetilde{d_X}(x_0, V) > 0$. Indeed, if $\widetilde{d_X}(x, V) = 0$ for every $x \in U$ we have that $U \subset \overline{V}$. So, $\overline{U} \subset \overline{V}$, and then $U = \mathrm{int}\left(\overline{U}\right) \subset \mathrm{int}\left(\overline{V}\right) = V$. Therefore if $V_{h_1}^{f_1} \setminus V_{h_2}^{f_2}$ is nonempty, there exist a point x_0 and an open ball B for the symmetrized distance $\widetilde{d_X}$, such that

$$x_0 \in B \subset V_{h_1}^{f_1} \setminus \overline{V_{h_2}^{f_2}}.$$

By Lemma 2.10 we can find $f, h \in \mathrm{SLip}_1(X)$ with $f \ge h$ such that $x_0 \in V_h^f \subset B$. Since semi-Lipschitz functions are bounded on $\widetilde{d_X}$-balls, taking into account Remark 2.11 we may also assume that $f \wedge h \ge h_1 \vee h_2$ on B. Now consider $\varphi = f \vee h_1 \vee h_2$ and $\psi = h \vee h_1 \vee h_2$. It is clear that $\psi, \varphi \in \mathrm{SLip}_1(X)$ with $\psi \wedge \varphi \ge h_1 \vee h_2$. Note that $\varphi = f$ and $\psi = h$ on B. On the other hand, on $X \setminus B$ we have that $f = h$ and thus $\varphi = \psi$. In particular, we have that $\psi \ge \varphi$ on $V_{h_2}^{f_2}$. Nevertheless, $\varphi(x_0) > \psi(x_0)$, so it is not true that $\psi \ge \varphi$ on $V_{h_1}^{f_1}$. $\square$

Lemma 2.13. *Let (X, d_X) be a quasi-metric space and let $f, h, \psi, \varphi \in \mathrm{SLip}_1(X)$ with $f \wedge \psi \wedge \varphi \geq h$. The following conditions are equivalent:*

(i) *$\psi \geq \varphi$ on V_h^f.*
(ii) *For every $\lambda \in [0, 1]$, we have that $(\lambda\psi + (1 - \lambda)h) \wedge f \geq (\lambda\varphi + (1 - \lambda)h) \wedge f$.*

Proof. It is easy to check that (i) implies (ii). To prove the converse suppose, on the contrary, that there exists $x_0 \in V_h^f$ such that $\psi(x_0) < \varphi(x_0)$. Now we have two options: either $f(x_0) = h(x_0)$ or $f(x_0) > h(x_0)$.

- Suppose that inequality holds. Then, there exists $\lambda \in (0, 1)$ such that

$$f(x_0) > \lambda\varphi(x_0) + (1 - \lambda)h(x_0) > \lambda\psi(x_0) + (1 - \lambda)h(x_0) \geq h(x_0).$$

  It is obvious that this implies that $(\lambda\psi + (1 - \lambda)h) \wedge f \geq (\lambda\varphi + (1 - \lambda)h) \wedge f$ is not true.
- If equality holds, then $x_0 \in \mathrm{supp}_h(f) = \overline{\mathrm{coz}_h(f)}$, so there exists a sequence (x_n) converging to x_0 in the $\widetilde{d_X}$-distance, such that $f(x_n) > h(x_n)$ for every $n \in \mathbb{N}$. As every semi-Lipschitz function is continuous for the $\widetilde{d_X}$-distance, there exists N such that $\psi(x_n) < \varphi(x_n)$ for every $n \geq N$. Then we may apply the previous case to x_n for some $n \geq N$, and we are done. $\square$

Now, as a direct consequence of the previous characterizations, we obtain the following stability results for convex lattice isomorphisms, which are going to be useful in the next section.

Corollary 2.14. *Let $(X, d_X), (Y, d_Y)$ be quasi-metric spaces, and $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$ a convex lattice isomorphism. Consider $f, h, \psi, \varphi \in \mathrm{SLip}_1(Y)$ with $f \wedge \psi \wedge \varphi \geq h$. Then $\psi \geq \varphi$ on V_h^f if, and only if, $T\psi \geq T\varphi$ on V_{Th}^{Tf}.*

Proof. Suppose that $\psi \geq \varphi$ on V_h^f. From the previous Lemma we have that, for every $\lambda \in [0, 1]$,

$$(\lambda\psi + (1 - \lambda)h) \wedge f \geq (\lambda\varphi + (1 - \lambda)h) \wedge f.$$

Since T is a convex lattice isomorphism, we obtain that $Tf \wedge T\psi \wedge T\varphi \geq Th$ and

$$(\lambda T\psi + (1 - \lambda)Th) \wedge Tf \geq (\lambda T\varphi + (1 - \lambda)Th) \wedge Tf \text{ for every } \lambda \in [0, 1].$$

Using again the previous Lemma we see that $T\psi \geq T\varphi$ on V_{Th}^{Tf}. The converse follows by considering T^{-1}. $\square$

Corollary 2.15. *Let $(X, d_X), (Y, d_Y)$ be quasi-metric spaces, and $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$ a convex lattice isomorphism. Consider $f_1, f_2, h_1, h_2 \in \mathrm{SLip}_1(Y)$ with $f_1 \geq h_1$ and $f_2 \geq h_2$. Then $V_{h_1}^{f_1} \subset V_{h_2}^{f_2}$ if, and only if, $V_{Th_1}^{Tf_1} \subset V_{Th_2}^{Tf_2}$.*

Proof. Assume that $V_{h_1}^{f_1} \subset V_{h_2}^{f_2}$. In order to see that $V_{Th_1}^{Tf_1} \subset V_{Th_2}^{Tf_2}$, we are going to use Lemma 2.12. Suppose that a pair of functions $\xi, \eta \in \mathrm{SLip}_1(X)$ are given, such that $\xi \wedge \eta \geq Th_1 \vee Th_2$ and $\xi \geq \eta$ on $V_{Th_2}^{Tf_2}$. Let $\psi, \varphi \in \mathrm{SLip}_1(X)$ such that $\xi = T\psi$ and $\eta = T\varphi$. Then $\psi \wedge \varphi \geq h_1 \vee h_2$ and, by Corollary 2.14, $\psi \geq \varphi$ on $V_{h_2}^{f_2}$. Thus $\psi \geq \varphi$ on $V_{h_1}^{f_1}$ and, again from Corollary 2.14, we have that $\xi \geq \eta$ on $V_{Th_1}^{Tf_1}$. From Lemma 2.12 we obtain that $V_{Th_1}^{Tf_1} \subset V_{Th_2}^{Tf_2}$. The converse follows by symmetry. $\square$

3. Main result

This section is devoted to obtain the main result in this paper, after all preliminary work in the previous section. First recall that a quasi-metric space (X, d_X) is said to be *complete* if the corresponding symmetrized space $(X, \widetilde{d_X})$ is complete.

Theorem 3.1. *Let (X, d_X) and (Y, d_Y) be complete quasi-metric spaces. If $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$ is a convex lattice isomorphism, then there exist $\alpha > 0$, a homeomorphism $\tau : (X, d_X) \to (Y, d_Y)$ and a quasi-metric d'_X on X, such that*

- *(X, d_X) and (X, d'_X) are almost-isometric, and $d'_X(x, x') = d_X(x, x') + T0(x') - T0(x)$.*
- *$\tau : (X, \alpha \cdot d'_X) \to (Y, d_Y)$ is an isometry.*
- *For every $f \in \mathrm{SLip}_1(Y)$ we have that $Tf = c \cdot (f \circ \tau) + \phi$, where $c = \frac{1}{\alpha}$ and $\phi = T0$.*

Remark 3.2. Observe that, even in the symmetric case, there is no hope of avoiding this α: if we consider $(X, d_X) = (\mathbb{R}, 2|\cdot|)$ and $(Y, d_Y) = (\mathbb{R}, |\cdot|)$, then $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$, given by $f \mapsto 2f$ is a convex lattice isomorphism. Actually, this can be done with any quasi-metric space.

In the proof of the Theorem, we will use the following result from [1]:

Lemma 3.3. *(See [1].) Let X and Y be complete metric spaces. Suppose that there exist bases of their topologies $\mathcal{B}_X, \mathcal{B}_Y$ and a bijection $\mathfrak{T} : \mathcal{B}_Y \to \mathcal{B}_X$ that preserves the inclusion. Then there exist dense subsets $X_0 \subset X$, $Y_0 \subset Y$ and a homeomorphism $\tau : X_0 \to Y_0$ such that for every $x \in X_0$ and every $V \in \mathcal{B}_Y$, we have that $x \in \mathfrak{T}(V)$ if and only if $\tau(x) \in V$.*

Proof of Theorem 3.1. For the symmetrized spaces $(X, \widetilde{d_X})$ and $(Y, \widetilde{d_Y})$ we consider the respective bases of their topologies $RS(X)$ and $RS(Y)$ (see Lemma 2.10). We now define $\mathfrak{T} : RS(Y) \to RS(X)$ by setting

$$\mathfrak{T}(V_h^f) = V_{Th}^{Tf}$$

for every $f, h \in \mathrm{SLip}_1(Y)$ with $f \geq h$. From Corollary 2.15 we have that $\mathfrak{T}$ is a well-defined bijection that preserves the inclusion. Thus from Lemma 3.3 we obtain that there exist dense subsets $Y_0 \subset Y$, $X_0 \subset X$ and a homeomorphism $\tau : X_0 \to Y_0$ such that, for every set $V_h^f \in RS(Y)$ and every $x \in X_0$, we have that $x \in V_{Th}^{Tf}$ if and only if $\tau(x) \in V_h^f$. We will need the following claims:

Claim 3.4. *Let $g_1, g_2 \in \mathrm{SLip}_1(Y)$ and $x_0 \in X_0$. Then, $g_1(\tau(x_0)) = g_2(\tau(x_0))$ if and only if $Tg_1(x_0) = Tg_2(x_0)$.*

Proof. Let $y_0 = \tau(x_0)$ and suppose that $g_1(y_0) = g_2(y_0)$. If y_0 is isolated, then we may take $h, f \in \mathrm{SLip}_1(Y)$ such that $g_1(y_0) \geq f(y_0) > h(y_0)$ and $f(y) = h(y)$ whenever $y \neq y_0$. Furthermore, we may suppose $g_1 \wedge g_2 \geq f \geq h$ as in the proof of Lemma 2.12. Applying Corollary 2.14 with $\varphi = g_1, \psi = g_2$, we have $Tg_1(x_0) = Tg_2(x_0)$.

If y_0 is non-isolated, then we may find a sequence $(y_n) \subset Y_0$ such that, for every n,

$$\max\{d_Y(y_0, y_{n+1}), d_Y(y_{n+1}, y_0)\} < \frac{1}{4} \min\{d_Y(y_0, y_n), d_Y(y_n, y_0)\} \leq 1.$$

We may suppose $g_1 \geq g_2$. We are looking for $g \in \mathrm{SLip}_1(Y)$ such that $g(y_{2n}) > g_1(y_{2n})$, $g(y_{2n-1}) < g_2(y_{2n-1})$ for every $n \in \mathbb{N}$ (as in [2], Lemma 3). This cannot be done for every couple of functions, so we must reduce the difference between g_1 and g_2. As $g_1(y_0) = g_2(y_0)$, we have $\frac{1}{4}g_1(y_0) = \frac{1}{4}g_2(y_0)$. Now we consider the function g_0 defined on the sequence (y_n) by setting, for $n \in \mathbb{N}$,

$$g_0(y_{2n}) = \frac{1}{4}g_1(y_{2n}) + \frac{1}{8}|g_1(y_{2n}) - g_1(y_0)|, g_0(y_{2n-1}) = \frac{1}{4}g_2(y_{2n-1}) - \frac{1}{8}|g_2(y_{2n-1}) - g_2(y_0)|.$$

Now we show that $g_0(y_m) - g_0(y_k) \leq d_Y(y_m, y_k)$ for any $m, k \in \mathbb{N}$. Indeed, let $m < k$ (the other case is similar). Then,

$$\begin{aligned} g_0(y_m) - g_0(y_k) &\leq \frac{1}{4}g_1(y_m) + \frac{1}{8}|g_1(y_m) - g_1(y_0)| - \frac{1}{4}g_2(y_k) + \frac{1}{8}|g_2(y_k) - g_2(y_0)| = \\ &= \frac{1}{4}\big(g_1(y_m) - g_1(y_0) - g_2(y_k) + g_2(y_0)\big) + \frac{1}{8}\big(|g_1(y_m) - g_1(y_0)| + |g_2(y_k) - g_2(y_0)|\big) \leq \\ &\leq \frac{1}{4}(d_Y(y_m, y_0) + d_Y(y_0, y_k)) + \frac{1}{8}(d_Y(y_m, y_0) + d_Y(y_0, y_k)) < \frac{3}{4}d_Y(y_m, y_0) \leq \\ &\leq d_Y(y_m, y_0) - d_Y(y_0, y_k) \leq d_Y(y_m, y_k), \end{aligned}$$

and this g_0 extends to $g \in \mathrm{SLip}_1(Y)$. Now, applying Corollary 2.14 to $\varphi = h = g \wedge \frac{1}{4}g_1, \psi = f = \frac{1}{4}g_1$, one has $Tg \geq T\left(\frac{1}{4}g_1\right)$ in a neighborhood of each $x_{2n} = \tau^{-1}(y_{2n})$. Analogously, $Tg \leq T\left(\frac{1}{4}g_2\right)$ in a neighborhood of each $x_{2n-1} = \tau^{-1}(y_{2n-1})$. As τ is a homeomorphism, x_n tends to x_0, and so, $T\left(\frac{1}{4}g_1\right)(x_0) \leq Tg(x_0) \leq T\left(\frac{1}{4}g_2\right)(x_0)$. As T preserves order, this implies $T\left(\frac{1}{4}g_1\right)(x_0) = Tg(x_0) = T\left(\frac{1}{4}g_2\right)(x_0)$. To finish we just need to check that $Tg_1(x_0) = Tg_2(x_0)$, but this is obvious since $T\left(\frac{1}{4}g_1\right) = \frac{1}{4}Tg_1 + \frac{3}{4}T0$ and $T\left(\frac{1}{4}g_2\right) = \frac{1}{4}Tg_2 + \frac{3}{4}T0$.

The "if" part comes by symmetry. □

Claim 3.5. *$Tg - T0$ is constant if and only if g is. Moreover, there exists $\alpha > 0$ such that $T\lambda = T0 + \frac{\lambda}{\alpha}$ for every $\lambda \in \mathbb{R}$.*

Proof. For each $\lambda \in \mathbb{R}$ let $g_\lambda \in \mathrm{SLip}_1(Y)$ be such that $Tg_\lambda = T0 + \lambda$. Note that, for $\lambda \geq 1$,

$$T\left(\frac{1}{\lambda}g_\lambda\right) = T\left(\frac{1}{\lambda}g_\lambda + \frac{\lambda - 1}{\lambda}0\right) = \frac{1}{\lambda}Tg_\lambda + \frac{\lambda - 1}{\lambda}T0 = \frac{1}{\lambda}T0 + 1 + \frac{\lambda - 1}{\lambda}T0 = T0 + 1 = T(g_1).$$

This implies that $g_1 = \frac{1}{\lambda}g_\lambda$, and so $\mathrm{SLip}(g_1) \leq \frac{1}{\lambda}$ for every $\lambda \geq 1$. Thus we obtain that g_1 is a constant function. From now on, α will be the constant value g_1 takes. Since $T(g_1) = T0 + 1 > T0$ we have that $\alpha > 0$. Now, it is not hard to check that g_λ must be constant and $g_\lambda = \alpha\lambda$ for every $\lambda \geq 0$. In particular, $T\lambda = T0 + \frac{\lambda}{\alpha}$. Furthermore, for every $\lambda \geq 0$

$$T0 = T\left(\frac{1}{2}\cdot\lambda + \frac{1}{2}\cdot(-\lambda)\right) = \frac{1}{2}T\lambda + \frac{1}{2}T(-\lambda) = \frac{1}{2}\left(T0 + \frac{\lambda}{\alpha}\right) + \frac{1}{2}T(-\lambda)$$

and we obtain $T(-\lambda) = T0 - \frac{\lambda}{\alpha}$. □

Claim 3.6. *Taking $c = \frac{1}{\alpha} = T1 - T0$ and $\phi = T0$ we have that, for every $f \in \mathrm{SLip}_1(Y)$ and $x \in X_0$,*

$$Tf(x) = c \cdot f(\tau(x)) + \phi(x).$$

Proof. Fix $f \in \mathrm{SLip}_1(Y)$ and $x_0 \in X_0$. Consider the functions $g_1 = f$ and g_2 the constant function on Y with value $f(\tau(x_0))$. By applying the two preceding Claims we obtain that $Tf(x_0) = Tg_2(x_0) = T0(x_0) + \frac{1}{\alpha}f(\tau(x_0))$. □

Claim 3.7. *The function $\phi = T0$ satisfies that $T0(x_1) - T0(x_2) < d_X(x_1, x_2)$ for every $x_1 \neq x_2 \in X_0$.*

Proof. Suppose that there exist $x_1 \neq x_2 \in X_0$ such that $T0(x_1) - T0(x_2) = d_X(x_1, x_2)$ and let $y_1 = \tau(x_1)$, $y_2 = \tau(x_2)$, $\beta = d_Y(y_1, y_2) > 0$. Consider $f(y) = d_Y(y, y_2)$. This function is 1-semi-Lipschitz, takes value $\beta > 0$ on y_1 and vanishes on y_2. Then,

$$Tf(x_1) - Tf(x_2) = T0(x_1) + \frac{\beta}{\alpha} - T0(x_2) = d_X(x_1, x_2) + \frac{\beta}{\alpha} > d_X(x_1, x_2),$$

a contradiction. □

To continue with the proof, we define on X the quasi-metric $d'_X(x, x') = d_X(x, x') + \phi(x') - \phi(x)$, where $\phi = T0$. Then d'_X is almost-isometric to d_X. Consider now the convex lattice isomorphisms

$$R : \mathrm{SLip}_1(X, d_X) \to \mathrm{SLip}(X, d'_X),$$

defined by $Rg = g - \phi$,

$$S : \mathrm{SLip}_1(X, d'_X) \to \mathrm{SLip}(X, \alpha d'_X),$$

defined by $Sh = \alpha \cdot h$, and finally the composition

$$\widehat{T} : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}(X, \alpha d'_X),$$

given by $\widehat{T} = S \circ R \circ T$. Then we have that $\widehat{T}1 = 1$ and, moreover, $\widehat{T}f(x) = f(\tau(x))$ for every $f \in \mathrm{SLip}_1(Y)$ and every $x \in X_0$.

We are going to see that (Y, d_Y) and $(X, \alpha d'_X)$ are isometric. First choose x_1 and $x_2 \in X_0$ and let $y_1 = \tau(x_1)$, $y_2 = \tau(x_2)$. Consider the function $f \in \mathrm{SLip}_1(Y)$ given by $f(y) = d_Y(y, y_2)$. Then $\widehat{T}f(x_1) = d_Y(y_1, y_2)$ and $\widehat{T}f(x_2) = 0$, so we obtain that

$$d_Y(y_1, y_2) = \widehat{T}f(x_1) - \widehat{T}f(x_2) \leq \alpha d'_X(x_1, x_2).$$

By symmetry, if we consider $\widehat{T}^{-1}$, we obtain that $\alpha d'_X(x_1, x_2) \leq d_Y(\tau(x_1), \tau(x_2))$ for every $x_1, x_2 \in X_0$. Thus $\tau : (X_0, \alpha d'_X) \to (Y_0, d_Y)$ is in fact an isometry. Taking into account that $(X, \widetilde{d_X})$ and $(Y, \widetilde{d_Y})$ are complete, and that every quasi-metric is continuous for the corresponding symmetrized distance, we obtain that this isometry τ extends to the whole X, and we have finished the proof of the theorem. □

Let (X, d_X) and (Y, d_Y) be complete quasi-metric spaces. A convex lattice isomorphism $T : \mathrm{SLip}_1(Y) \to \mathrm{SLip}_1(X)$ is said to be *almost-unital* if $T1 - T0 = 1$. According to the representation obtained in the previous theorem, T is of the form $Tf(x) = c \cdot (f \circ \tau) + \phi$, where $c = T1 - T0$ and $\phi = T0$. Then T is almost-unital if, and only if, $c = 1$, and this is equivalent to the fact that (X, d_X) and (Y, d_Y) are almost isometric. So we obtain the following

Corollary 3.8. *Let (X, d_X) and (Y, d_Y) be complete quasi-metric spaces. Then, the following are equivalent:*

- *(X, d_X) and (Y, d_Y) are almost isometric.*
- *There exists an almost unital convex lattice isomorphism between* $\mathrm{SLip}_1(X)$ *and* $\mathrm{SLip}_1(Y)$.

4. Consequences and related questions

Since every almost isometry between metric spaces is an isometry (see [5], Corollary 2.9), we readily obtain the following

Corollary 4.1. *Let X and Y be complete metric spaces and $T : \mathrm{Lip}_1(Y) \to \mathrm{Lip}_1(X)$ a convex lattice isomorphism. Then, there exist $\tau : (X, d_X) \to (Y, d_Y)$, $c \in (0, \infty)$ and $\beta \in \mathbb{R}$ such that*

- $Tf = \frac{1}{c} \cdot (f \circ \tau) + \beta$ *for every* $f \in \mathrm{Lip}(Y)$.
- $d_X(x, x') = c \cdot d_Y(\tau(x), \tau(x'))$ *for every* $x, x' \in X$.

We next consider the case of two quasi-metrics on the same space inducing the same symmetrized distance.

Corollary 4.2. *Let (X, d_X) be a complete quasi-metric space and d'_X another quasi-metric on X. If $\widetilde{d'_X}(x, x') = \widetilde{d_X}(x, x')$ for all $x, x' \in X$, then the following are equivalent:*

(i) $\mathrm{Id} : (X, d_X) \to (X, d'_X)$ *is an almost isometry.*
(ii) *For every $x_0 \in X$ and every $f \in \mathrm{SLip}_1(X, d'_X)$, the function defined as $f(x) + d_X(x, x_0) - d'_X(x, x_0)$ is 1-semi-Lipschitz on (X, d_X).*
(iii) *For some $x_0 \in X$ and every $f \in \mathrm{SLip}_1(X, d'_X)$, the function defined as $f(x) + d_X(x, x_0) - d'_X(x, x_0)$ is 1-semi-Lipschitz on (X, d_X).*

Proof. We will show that (i) implies (ii) and that (iii) implies (i).

As shown before (see Proposition 2.4), if $\mathrm{Id} : (X, d_X) \to (X, d'_X)$ is an almost isometry, with $d'_X(x, x') = d_X(x, x') + \phi(x') - \phi(x)$, then $T : \mathrm{SLip}_1(X, d'_X) \to \mathrm{SLip}_1(X, d_X)$ given by $Tf = f + \phi + c$ is a bijection for any $c \in \mathbb{R}$. As seen in Remark 1.3, for every $x_0 \in X$ there exists $c \in \mathbb{R}$ such that $\phi(x) = d_X(x, x_0) - d'_X(x, x_0) - c$ for every x, so we obtain the first implication.

For the second one, we will check $P_X([x_1, x_2, x_3]) = P'_X([x_1, x_2, x_3])$ for every $x_1, x_2, x_3 \in X$. Let us denote by T the application that maps $f \in \mathrm{SLip}_1(X, d'_X)$ to $f + d_X(x, x_0) - d'_X(x, x_0) \in \mathrm{SLip}_1(X, d_X)$. For some suitable $f_1, f_2, f_3 \in \mathrm{SLip}_1(X, d'_X)$ we have:

$$\begin{aligned}
P'_X([x_1, x_2, x_3]) &= d'_X(x_1, x_2) + d'_X(x_2, x_3) + d'_X(x_3, x_1) = \\
&= f_1(x_2) - f_1(x_1) + f_2(x_3) - f_2(x_2) + f_3(x_1) - f_3(x_3) = \\
&= f_1(x_2) + d_X(x_2, x_0) - d'_X(x_2, x_0) - f_1(x_1) - d_X(x_1, x_0) + d'_X(x_1, x_0) + \\
&\quad + f_2(x_3) + d_X(x_3, x_0) - d'_X(x_3, x_0) - f_2(x_2) - d_X(x_2, x_0) + d'_X(x_2, x_0) + \\
&\quad + f_3(x_1) + d_X(x_1, x_0) - d'_X(x_1, x_0) - f_3(x_3) - d_X(x_3, x_0) + d'_X(x_3, x_0) = \\
&= Tf_1(x_2) - Tf_1(x_1) + Tf_2(x_3) - Tf_2(x_2) + Tf_3(x_1) - Tf_3(x_3) \leq \\
&\leq d_X(x_1, x_2) + d_X(x_2, x_3) + d_X(x_3, x_1) = P_X([x_1, x_2, x_3]).
\end{aligned}$$

As both symmetrized metrics agree,

$$P'_X([x_1, x_2, x_3]) + P'_X([x_3, x_2, x_1]) = P_X([x_1, x_2, x_3]) + P_X([x_3, x_2, x_1]),$$

and we obtain $P'_X([x_1, x_2, x_3]) = P_X([x_1, x_2, x_3])$. □

Finally, we characterize when a quasi-metric space is almost isometric to a metric space.

Corollary 4.3. *Let (X, d_X) be a quasi-metric space. Then, the following are equivalent:*

(i) (X, d_X) *is almost isometric to* $(X, \widetilde{d_X})$.
(ii) (X, d_X) *is almost isometric to (X, d'_X), where d'_X is the reverse quasi-metric of d_X: $d'_X(x, x') = d_X(x', x)$.*

(iii) (X, d_X) *is almost isometric to a metric space.*
(iv) *There exists* $\phi \in \mathrm{SLip}_1(X, d_X)$ *such that* $\phi + f \in \mathrm{SLip}_1(X, d_X)$ *if and only if* $\phi - f \in \mathrm{SLip}_1(X, d_X)$.

Proof. We will show some implications:

- (iii) implies (i): If (X, d_X) is almost isometric to a metric space (Y, d_Y), then $(X, \widetilde{d_X})$ is isometric to $(Y, \widetilde{d_Y}) = (Y, d_Y)$, so (X, d_X) is almost isometric to $(X, \widetilde{d_X})$.
- (i) implies (iii): It is trivial.
- (i) if and only if (ii): Since $\widetilde{P_X}([x_1, x_2, x_3]) = \frac{1}{2}(P'_X([x_1, x_2, x_3]) + P_X([x_1, x_2, x_3]))$, it is clear that $P'_X([x_1, x_2, x_3]) = P_X([x_1, x_2, x_3])$ if and only if $\widetilde{P_X}([x_1, x_2, x_3]) = P_X([x_1, x_2, x_3])$.
- (i) implies (iv): Suppose (X, d_X) is almost isometric to $(X, \widetilde{d_X})$. Then, $\widetilde{d_X}(x, x') = d_X(x, x') + \phi(x') - \phi(x)$ for some $\phi \in \mathrm{SLip}_1(X, d_X)$ and $f \in \mathrm{SLip}_1(X, \widetilde{d_X}) \mapsto f + \phi \in \mathrm{SLip}_1(X, d_X)$ is an isomorphism. As $-f \in \mathrm{SLip}_1(X, \widetilde{d_X})$ if and only if $f \in \mathrm{SLip}_1(X, \widetilde{d_X})$, we obtain (iv).
- (iv) implies (ii): The only we must show is that $P_X([x_1, x_2, x_3]) \leq P_X([x_3, x_2, x_1])$ whenever $x_1, x_2, x_3 \in X$. Let $f_1, f_2, f_3 \in \mathrm{SLip}_1(X)$ such that

$$\begin{aligned} P_X([x_1, x_2, x_3]) &= d_X(x_1, x_2) + d_X(x_2, x_3) + d_X(x_3, x_1) = \\ &= f_1(x_2) - f_1(x_1) + f_2(x_3) - f_2(x_2) + f_3(x_1) - f_3(x_3). \end{aligned}$$

As (iv) implies that $2\phi - f_i \in \mathrm{SLip}_1(X)$ for $i = 1, 2, 3$, we get

$$\begin{aligned} P_X([x_3, x_2, x_1]) &= d_X(x_3, x_2) + d_X(x_2, x_1) + d_X(x_1, x_3) \geq \\ &\geq (2\phi - f_2)(x_2) - (2\phi - f_2)(x_3) + (2\phi - f_1)(x_1) - (2\phi - f_1)(x_2) + \\ &\quad + (2\phi - f_3)(x_3) - (2\phi - f_3)(x_1) = \\ &= P_X([x_1, x_2, x_3]). \end{aligned}$$

By symmetry, $P_X([x_1, x_2, x_3]) = P_X([x_3, x_2, x_1])$, which equals $P'_X([x_1, x_2, x_3])$, so we are done. □

Taking a look at [5], we see that another natural environment for almost isometries is that of Finsler manifolds. It is shown there that, given an almost isometry τ between Finsler manifolds, the function ϕ that governs τ is smooth and has differential with norm lower than 1 at every point. It would be interesting to know whether almost isometries of Finsler manifolds can be functionally characterized in terms of a suitable space of *smooth* functions.

Acknowledgments

This paper was finished while Jesús A. Jaramillo was visiting Kent State University. It is a pleasure to thank the Mathematics Department of this University, and very specially Professor Richard M. Aron, for their kind hospitality and support.

References

[1] F. Cabello Sánchez, J. Cabello Sánchez, Nonlinear isomorphisms of lattices of Lipschitz functions, Houston J. Math. 37 (2011) 181–202.
[2] F. Cabello Sánchez, J. Cabello Sánchez, Lattices of uniformly continuous functions, Topology Appl. 160 (2013) 50–55.
[3] M.I. Garrido, J.A. Jaramillo, Variations on the Banach–Stone theorem, Extracta Math. 17 (2002) 351–385.
[4] M.I. Garrido, J.A. Jaramillo, Homomorphisms on function lattices, Monatsh. Math. 141 (2004) 127–146.
[5] M.A. Javaloyes, L. Lichtenfelz, P. Piccione, Almost isometries of non-reversible metrics with applications to stationary spacetimes, J. Geom. Phys. 89 (2015) 38–49.

Please cite this article in press as: J. Cabello, J.A. Jaramillo, A functional representation of almost isometries, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.04.026

[6] G. Mayor, O. Valero, Aggregation of asymmetric distances in computer science, Inform. Sci. 180 (2010) 803–812.
[7] E.J. McShane, Extension of range of functions, Bull. Amer. Math. Soc. 40 (1934) 837–842.
[8] A.C.G. Mennucci, On asymmetric distances, Anal. Geom. Metr. Spaces 1 (2013) 200–231.
[9] S. Romaguera, M. Sanchís, Semi-Lipschitz functions and best approximation in quasi-metric spaces, J. Approx. Theory 103 (2000) 292–301.

Please cite this article in press as: J. Cabello, J.A. Jaramillo, A functional representation of almost isometries, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.04.026