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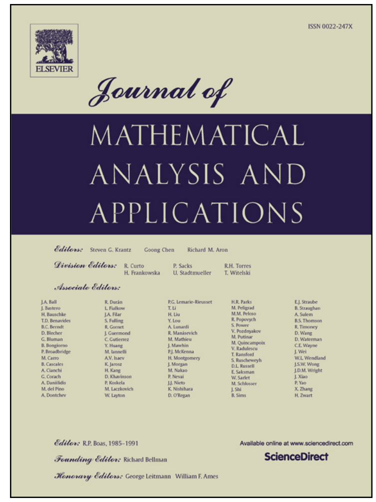

Birkhoff-James orthogonality of linear operators on finite dimensional Banach spaces

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BIRKHOFF-JAMES ORTHOGONALITY OF LINEAR OPERATORS ON FINITE DIMENSIONAL BANACH SPACES

DEBMALYA SAIN

Abstract. In this paper we characterize Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space $\mathbb{X}$. We also explore the left symmetry of Birkhoff-James orthogonality of linear operators defined on $\mathbb{X}$. Using some of the related results proved in this paper, we finally prove that $T \in \mathbb{L}(l_p^2)(p \geq 2, p \neq \infty)$ is left symmetric with respect to Birkhoff-James orthogonality if and only if T is the zero operator.

1. Introduction.

Birkhoff-James orthogonality [2] plays a vital role in the study of geometry of Banach spaces. One of the prominent reasons behind this is the natural connection shared by Birkhoff-James orthogonality with various geometric properties of the space, like smoothness, strict convexity etc. Recently in [5], Sain and Paul have characterized finite dimensional real Hilbert spaces among finite dimensional real Banach spaces in terms of operator norm attainment, using the notion of Birkhoff-James orthogonality. More recently, symmetry of Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Hilbert space $\mathbb{H}$ has been explored by Ghosh et al. in [3]. However, it was remarked in [3] that analogous results corresponding to the far more general setting of Banach spaces remain unknown. The aim of the present paper is twofold: we characterize Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space $\mathbb{X}$ and we also explore the left symmetry of Birkhoff-James orthogonality of linear operators defined on $\mathbb{X}$. Using some of the results proved in this paper, we finally study the left symmetry of Birkhoff-James orthogonality of linear operators defined on $l_p^2(p \geq 2, p \neq \infty)$.

Let $(\mathbb{X}, \|\|)$ be a finite dimensional real Banach space. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ be the unit ball and the unit sphere of the Banach space $\mathbb{X}$ respectively. Let $\mathbb{L}(\mathbb{X})$ denote the Banach space of all linear operators on $\mathbb{X}$, endowed with the usual operator norm.

For any two elements $x, y \in \mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff-James, written as $x \perp_B y$, if

$$\|x\| \leq \|x + \lambda y\| \text{ for all } \lambda \in \mathbb{R}.$$

Likewise, for any two elements $T, A \in \mathbb{L}(\mathbb{X})$, T is said to be orthogonal to A in the sense of Birkhoff-James, written as $T \perp_B A$, if

$$\|T\| \leq \|T + \lambda A\| \text{ for all } \lambda \in \mathbb{R}.$$

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For a linear operator T defined on a Banach space $\mathbb{X}$, let M_T denote the collection of all unit vectors in $\mathbb{X}$ at which T attains norm, i.e.,

$$M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}.$$

In a finite dimensional Hilbert space $\mathbb{H}$, Bhatia and Semrl [1] proved that for any two elements $T, A \in \mathbb{L}(\mathbb{H})$, $T \perp_B A$ if and only if there exists $x \in M_T$ such that $Tx \perp_B Ax$. Sain and Paul [5] generalized the result for linear operators defined on finite dimensional real Banach spaces in Theorem 2.1 of [5] by proving the following result:

Let $\mathbb{X}$ be a finite dimensional real Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that $M_T = \pm D$, where D is a closed, connected subset of $S_{\mathbb{X}}$. Then for $A \in \mathbb{L}(\mathbb{X})$ with $T \perp_B A$, there exists $x \in D$ such that $Tx \perp_B Ax$.

It is easy to observe that there exists a bounded linear operator $T \in \mathbb{L}(\mathbb{X})$ such that M_T is not of the form $\pm D$, where D is a closed connected subset of $S_{\mathbb{X}}$. Therefore, Theorem 2.1 of [5] does not completely characterize Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space $\mathbb{X}$. In this paper, one of our main aims would be to obtain a complete characterization of Birkhoff-James orthogonality of linear operators defined on $\mathbb{X}$. The next notion is crucial for our main result:
For any two elements x, y in a real normed linear space $\mathbb{X}$, let us say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Accordingly, we say that $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. Using this notion we completely characterize Birkhoff-James orthogonality of linear operators defined on finite dimensional real Banach spaces.

Next we consider the left symmetry of Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space $\mathbb{X}$. For an element $x \in \mathbb{X}$, let us say that x is left symmetric (with respect to Birkhoff-James orthogonality) if $x \perp_B y$ implies $y \perp_B x$ for any $y \in \mathbb{X}$. It was proved in [3] that if $\mathbb{H}$ is a finite dimensional real Hilbert space then $T \in \mathbb{L}(\mathbb{H})$ is a left symmetric point if and only if T is the zero operator. In this paper we consider the problem in the more general setting of real Banach spaces and prove some related results corresponding to the left symmetry of linear operators defined on a finite dimensional real Banach space. We give example to show that in a finite dimensional real Banach space $\mathbb{X}$, which is not a Hilbert space, there may exist nonzero linear operators $T \in \mathbb{L}(\mathbb{X})$ such that T is a left symmetric point in $\mathbb{L}(\mathbb{X})$. Finally, we prove that $T \in \mathbb{L}(l_p^2)(p \geq 2, p \neq \infty)$ is left symmetric if and only if T is the zero operator.

2. Main results.

In order to characterize Birkhoff-James orthogonality of linear operators defined on finite dimensional real Banach spaces, we have introduced the notions $y \in x^+$ and $y \in x^-$, for any two elements x, y in a real normed linear space $\mathbb{X}$. First we state some obvious but useful properties of this notion which would be used later on in this paper, without giving explicit proofs.

Proposition 2.1. *Let $\mathbb{X}$ be a real normed linear space and $x, y \in \mathbb{X}$. Then the following are true:*

(i) Either $y \in x^+$ *or* $y \in x^-$.
(ii) $x \perp_B y$ *if and only if* $y \in x^+$ *and* $y \in x^-$.
(iii) $y \in x^+$ *implies that* $\eta y \in (\mu x)^+$ *for all* $\eta, \mu > 0$.
(iv) $y \in x^+$ *implies that* $-y \in x^-$ *and* $y \in (-x)^-$.
(v) $y \in x^-$ *implies that* $\eta y \in (\mu x)^-$ *for all* $\eta, \mu > 0$.
(vi) $y \in x^-$ *implies that* $-y \in x^+$ *and* $y \in (-x)^+$.

In the next theorem we use this notion to give a characterization of Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space.

Theorem 2.2. *Let* $\mathbb{X}$ *be a finite dimensional real Banach space. Let* $T, A \in \mathbb{L}(\mathbb{X})$*. Then* $T \perp_B A$ *if and only if there exists* $x, y \in M_T$ *such that* $Ax \in Tx^+$ *and* $Ay \in Ty^-$.

Proof : Let us first prove the easier "if" part.
Suppose there exists $x, y \in M_T$ such that $Ax \in Tx^+$ and $Ay \in Ty^-$. For any $\lambda \geq 0, \|T+\lambda A\| \geq \|Tx+\lambda Ax\| \geq \|Tx\| = \|T\|$. Similarly, for any $\lambda \leq 0, \|T+\lambda A\| \geq \|Ty + \lambda Ay\| \geq \|Ty\| = \|T\|$. This proves that $T \perp_B A$.
Let us now prove the comparatively trickier "only if" part.
Let $T, A \in \mathbb{L}(\mathbb{X})$ be such that $T \perp_B A$. If possible suppose that there does not exist $x, y \in M_T$ such that $Ax \in Tx^+$ and $Ay \in Ty^-$. Using (i) of Proposition 2.1, it is easy to show that either of the following is true:
(i) $Ax \in Tx^+$ for each $x \in M_T$ and $Ax \notin Tx^-$ for any $x \in M_T$
(ii) $Ax \in Tx^-$ for each $x \in M_T$ and $Ax \notin Tx^+$ for any $x \in M_T$.
Without loss of generality, let us assume that $Ax \in Tx^+$ for each $x \in M_T$ and $Ax \notin Tx^-$ for any $x \in M_T$. Consider the function $g : S_{\mathbb{X}} \times [-1, 1] \longrightarrow \mathbb{R}$ defined by

$$g(x, \lambda) = \|Tx + \lambda Ax\|.$$

It is easy to check that g is continuous. Given any $x \in M_T$, since $Ax \notin Tx^-$, there exists $\lambda_x < 0$ such that $g(x, \lambda_x) = \|Tx + \lambda_x Ax\| < \|Tx\| = \|T\|$. By continuity of g, there exists $r_x, \delta_x > 0$ such that

$$g(y, \lambda) < \|T\| \text{ for all } y \in B(x, r_x) \cap S_{\mathbb{X}} \text{ and for all } \lambda \in (\lambda_x - \delta_x, \lambda_x + \delta_x).$$

Using the convexity property of the norm function, it is easy to show that $g(y, \lambda) = \|Ty + \lambda Ay\| < \|T\|$ for all $y \in B(x, r_x) \cap S_{\mathbb{X}}$ and for all $\lambda \in (\lambda_x, 0)$.

For any $z \in S_{\mathbb{X}} \setminus M_T$, we have $g(z, 0) = \|Tz\| < \|T\|$. Thus by continuity of g, there exist $r_z, \delta_z > 0$ such that $g(y, \lambda) = \|Ty + \lambda Ay\| < \|T\|$ for all $y \in B(z, r_z) \cap S_{\mathbb{X}}$ and for all $\lambda \in (-\delta_z, \delta_z)$.

Consider the open cover $\{B(x, r_x) \cap S_{\mathbb{X}} : x \in M_T\} \cup \{B(z, r_z) \cap S_{\mathbb{X}} : z \in S_{\mathbb{X}} \setminus M_T\}$ of $S_{\mathbb{X}}$. Since $\mathbb{X}$ is finite dimensional, $S_{\mathbb{X}}$ is compact. This proves that the considered open cover has a finite subcover and so we get,

$$S_X \subset \cup_{i=1}^{n_1} B(x_i, r_{x_i}) \cup_{k=1}^{n_2} B(z_k, r_{z_k}) \cap S_{\mathbb{X}},$$

for some positive integers n_1, n_2, where each $x_i \in M_T$ and each $z_k \in S_{\mathbb{X}} \setminus M_T$.

Choose $\lambda_0 \in (\cap_{i=1}^{n_1} (\lambda_{x_i}, 0)) \bigcap (\cap_{k=1}^{n_2} (-\delta_{z_k}, \delta_{z_k}))$.
Since $\mathbb{X}$ is finite dimensional, $T + \lambda_0 A$ attains its norm at some $w_0 \in S_{\mathbb{X}}$. Either $w_0 \in B(x_i, r_{x_i})$ for some $x_i \in M_T$ or $w_0 \in B(z_k, r_{z_k})$ for some $z_k \in S_{\mathbb{X}} \setminus M_T$. In

either case, it follows from the choice of λ_0 that $\|T+\lambda_0 A\| = \|(T+\lambda_0 A)w_0\| < \|T\|$, which contradicts our primary assumption that $T \perp_B A$ and thereby completes the proof of the "only if" part.

Theorem 2.1 of Sain and Paul [5] can be deduced as a corollary to the previous theorem.

Corollary 2.2.1. *Let $\mathbb{X}$ be a finite dimensional real Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that $M_T = \pm D$, where D is a closed, connected subset of $S_{\mathbb{X}}$. Then for $A \in \mathbb{L}(\mathbb{X})$ with $T \perp_B A$, there exists $x \in D$ such that $Tx \perp_B Ax$.*

Proof : Since $T \perp_B A$, applying Theorem 2.2, we see that there exists $x, y \in M_T = \pm D$ such that $Ax \in Tx^+$ and $Ay \in Ty^-$. Moreover, it is easy to see that by applying (iv) and (vi) of Proposition 2.1, we may assume without loss of generality that $x, y \in D$. Then following the same line of arguments, as in Theorem 2.1 of [5], it can be proved that there exists $u_0 \in D$ such that $Au_0 \in Tu_0^+$ and $Au_0 \in Tu_0^-$, by using the connectedness of D. However, this is equivalent to $Tu_0 \perp_B Au_0$, completing the proof of Theorem 2.1 of [5].

Remark 2.3. *The previous theorem gives a complete characterization of Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space $\mathbb{X}$. Moreover, as we will see later on in this paper, the theorem is very useful computationally as well as from theoretical point of view. It should be noted that the main idea of the proof of Theorem* 2.2 *was already there in the proof of Theorem* 2.1 *of* [5]*. However, complete characterization of Birkhoff-James orthogonality of linear operators on $\mathbb{X}$ could not be obtained in* [5]*. This reveals the usefulness of the notion introduced by us in this paper to meet this end.*

Next we consider the left symmetry of Birkhoff-James orthogonality of linear operators defined on a finite dimensional real Banach space $\mathbb{X}$. $T \in \mathbb{L}(\mathbb{X})$ is left symmetric if $T \perp_B A$ implies that $A \perp_B T$ for any $A \in \mathbb{L}(\mathbb{X})$. In the following theorem we establish a useful connection between left symmetry of an operator $T \in \mathbb{L}(\mathbb{X})$ and left symmetry of points in the corresponding norm attainment set M_T.

Theorem 2.4. *Let $\mathbb{X}$ be a finite dimensional strictly convex real Banach space. If $T \in \mathbb{L}(\mathbb{X})$ is a left symmetric point then for each $x \in M_T, Tx$ is a left symmetric point.*

Proof : First we observe that the theorem is trivially true if T is the zero operator. Let T be nonzero. Since $\mathbb{X}$ is finite dimensional, M_T is nonempty. If possible suppose that there exists $x_1 \in M_T$ such that Tx_1 is not a left symmetric point. Since T is nonzero, $Tx_1 \neq 0$. Then there exists $y_1 \in S_{\mathbb{X}}$ such that $Tx_1 \perp_B y_1$ but $y_1 \not\perp_B Tx_1$. Since $\mathbb{X}$ is strictly convex, x_1 is an exposed point of the unit ball $B_{\mathbb{X}}$. Let H be the hyperplane of codimension 1 in $\mathbb{X}$ such that $x_1 \perp_B H$. Clearly, any element x of $\mathbb{X}$ can be uniquely written in the form $x = \alpha_1 x_1 + h$, where $\alpha_1 \in \mathbb{R}$ and $h \in H$. Define a linear operator $A \in \mathbb{L}(\mathbb{X})$ as follows:

$$Ax_1 = y_1, Ah = 0 \text{ for all } h \in H.$$

Since $x_1 \in M_T$ and $Tx_1 \perp_B Ax_1$, it follows that $T \perp_B A$. Since T is left symmetric, it follows that $A \perp_B T$.
It is easy to check that $M_A = \{\pm x_1\}$, since $\mathbb{X}$ is strictly convex. Applying Theorem

2.1 of [5] to A, it follows from $A \perp_B T$ that $Ax_1 \perp_B Tx_1$, i.e., $y_1 \perp_B Tx_1$, contrary to our initial assumption that $y_1 \not\perp_B Tx_1$. This contradiction completes the proof of the theorem.

The proof of the following corollary is now obvious.

Corollary 2.4.1. *Let $\mathbb{X}$ be a finite dimensional strictly convex real Banach space such that the unit sphere $S_{\mathbb{X}}$ has no left symmetric point. Then $\mathbb{L}(\mathbb{X})$ can not have any nonzero left symmetric point.*

In the next theorem we prove that if $\mathbb{X}$ is a finite dimensional strictly convex and smooth real Banach space, then a "large" class of operators can not be left symmetric in $\mathbb{L}(\mathbb{X})$.

Theorem 2.5. *Let $\mathbb{X}$ be a finite dimensional strictly convex and smooth real Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that there exists $x, y \in S_{\mathbb{X}}$ satisfying (i) $x \in M_T, (ii)$ $y \perp_B x, (iii)$ $Ty \neq 0$. Then T can not be left symmetric.*

Proof : There exists a hyperplane H of codimension 1 in $\mathbb{X}$ such that $y \perp_B H$. Define a linear operator $A \in \mathbb{L}(\mathbb{X})$ as follows:

$$Ay = Ty, A(H) = 0.$$

Since $y \perp_B x$ and $\mathbb{X}$ is smooth, it follows that $x \in H$, i.e., $Ax = 0$. Since $\mathbb{X}$ is strictly convex, as before it is easy to show that $M_A = \{\pm y\}$. We observe that $T \perp_B A$, since $x \in M_T$ and $Tx \perp_B Ax = 0$. However, $M_A = \{\pm y\}, Ay \not\perp_B Ty$ together implies that $A \not\perp_B T$. This completes the proof of the fact that T can not be left symmetric.

Corollary 2.5.1. *Let $\mathbb{X}$ be a finite dimensional strictly convex and smooth real Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be invertible. Then T can not be left symmetric.*

Proof : Since $\mathbb{X}$ is finite dimensional, there exists $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$. From Theorem 2.3 of James [4], it follows that there exists $y(\neq 0) \in \mathbb{X}$ such that $y \perp_B x$. Using the homogeneity property of Birkhoff-James orthgonality, we may assume without loss of generality that $\|y\| = 1$. Since T is invertible, $Ty \neq 0$. Thus, all the conditions of the previous theorem are satisfied and hence T can not be left symmetric.

Let $\mathbb{H}$ be a finite dimensional real Hilbert space. It was proved in [3] that $T \in \mathbb{L}(\mathbb{H})$ is a left symmetric point in $\mathbb{L}(\mathbb{H})$ if and only if T is the zero operator. In the next example we show that if $\mathbb{X}$ is a finite dimensional real Banach space, which is not a Hilbert space, then there may exist nonzero left symmetric operators in $\mathbb{L}(\mathbb{X})$.

Example 1. Let $\mathbb{X}$ be the 2 dimensional real Banach space l_1^2. Let $T \in \mathbb{L}(\mathbb{X})$ be defined by $T(1,0) = (\frac{1}{2}, \frac{1}{2}), T(0,1) = (0,0)$. We claim that T is left symmetric in $\mathbb{L}(\mathbb{X})$. Indeed, let $A \in \mathbb{L}(\mathbb{X})$ be such that $T \perp_B A$. Since $M_T = \{\pm(1,0)\}$, it follows that $(\frac{1}{2}, \frac{1}{2}) = T(1,0) \perp_B A(1,0)$. Since $(\frac{1}{2}, \frac{1}{2})$ is a left symmetric point in $\mathbb{X}$, it follows that $A(1,0) \perp_B T(1,0)$. We also note that $\{\pm(1,0), \pm(0,1)\}$ are the only extreme points of $S_{\mathbb{X}}$. Since a linear operator defined on a finite dimensional Banach space must attain norm at some extreme point of the unit sphere, either $(1,0) \in M_A$ or $(0,1) \in M_A$. In either case, there exists a unit vector x such that $x \in M_A$ and $Ax \perp_B Tx$. However, this clearly implies that $A \perp_B T$, completing the proof of the fact that T is a nonzero left symmetric point in $\mathbb{L}(\mathbb{X})$.

We next wish to prove that in case of the strictly convex and smooth real Banach spaces $l_p^2(p \geq 2, p \neq \infty), T \in \mathbb{L}(l_p^2)(p \geq 2, p \neq \infty)$ is left symmetric if and only if T is the zero operator. Before proving the desired result, we first state two easy propositions. It may be noted that the proofs of both the propositions follow easily from ordinary calculus.

Proposition 2.6. *Let $\mathbb{X}$ be the real Banach space $l_p^2(p \neq 1, \infty)$. $x \in S_{\mathbb{X}}$ is a left symmetric point in $\mathbb{X}$ if and only if $x \in \pm\{(1,0),(0,1),(\frac{1}{2^{1/p}},\frac{1}{2^{1/p}}),(\frac{1}{2^{1/p}},\frac{-1}{2^{1/p}})\}$.*

Proposition 2.7. *Let $\mathbb{X}$ be the real Banach space $l_p^2(p \neq 1, \infty)$. If $x, y \in S_{\mathbb{X}}$ are such that $x \perp_B y$ and $y \perp_B x$ then either of the following is true:*
(i) $x = \pm(1,0)$ and $y = \pm(0,1)$.
(ii) $x = \pm(0,1)$ and $y = \pm(1,0)$.
(iii) $x = \pm(\frac{1}{2^{1/p}},\frac{1}{2^{1/p}}), y = \pm(\frac{1}{2^{1/p}},\frac{-1}{2^{1/p}})$.
(iv) $x = \pm(\frac{1}{2^{1/p}},\frac{-1}{2^{1/p}}), y = \pm(\frac{1}{2^{1/p}},\frac{1}{2^{1/p}})$.

Next we apply these two propositions and some of the results proved in this paper to prove that $T \in \mathbb{L}(l_p^2)(p \geq 2, p \neq \infty)$ is left symmetric if and only if T is the zero operator.

Theorem 2.8. *Let $\mathbb{X}$ be the 2 dimensional real Banach space $l_p^2(p \geq 2, p \neq \infty)$. $T \in \mathbb{L}(\mathbb{X})$ is left symmetric if and only if T is the zero operator.*

Proof : If possible suppose that $T \in \mathbb{L}(\mathbb{X})$ is a nonzero left symmetric point in $\mathbb{L}(\mathbb{X})$. Since Birkhoff-James orthogonality is homogeneous, and T is nonzero, let us assume, without loss of generality, that $\|T\| = 1$. Let T attains norm at $x \in S_{\mathbb{X}}$. From Theorem 2.3 of James [4], it follows that there exists $y \in S_{\mathbb{X}}$ such that $y \perp_B x$. Since $\mathbb{X}$ is strictly convex and smooth, applying Theorem 2.5, we see that $Ty = 0$. Theorem 2.4 ensures that Tx must be a left symmetric point in $\mathbb{X}$. Thus, applying Proposition 2.6, we have that

$$Tx \in \pm\{(1,0),(0,1),(\frac{1}{2^{1/p}},\frac{1}{2^{1/p}}),(\frac{1}{2^{1/p}},\frac{-1}{2^{1/p}})\}.$$

We next claim that $x \perp_B y$.
From Theorem 2.3 of James [4], it follows that there exists a real number a such that $ay + x \perp_B y$. Since $y \perp_B x$ and $x, y \neq 0$, $\{x, y\}$ is linearly independent and hence $ay + x \neq 0$. Let $z = \frac{ay+x}{\|ay+x\|}$. We note that if $Tz = 0$ then T is the zero operator. Let $Tz \neq 0$. Clearly, $\{y, z\}$ is a basis of $\mathbb{X}$.
Let $\|c_1 z + c_2 y\| = 1$, for some scalars c_1, c_2. Then we have, $1 = \|c_1 z + c_2 y\| \geq \mid c_1 \mid$. Since $\mathbb{X}$ is strictly convex, $1 > \mid c_1 \mid$, if $c_2 \neq 0$. We also have, $\|T(c_1 z + c_2 y)\| = \|c_1 Tz\| = \mid c_1 \mid \|Tz\| \leq \|Tz\|$ and $\|T(c_1 z + c_2 y)\| = \|Tz\|$ if and only if $c_1 = \pm 1$ and $c_2 = 0$. This proves that $M_T = \{\pm z\}$. However, we have already assumed that $x \in M_T$. Thus, we must have $x = \pm z$. Since $z \perp_B y$, our claim is proved.
Thus, $x, y \in S_{\mathbb{X}}$ are such that $x \perp_B y$ and $y \perp_B x$. Therefore, by applying Proposition 2.7, we see that we have the following information about T :
(i) T attains norm at x, $x \perp_B y, y \perp_B x, Ty = 0$.
(ii) $x, y, Tx \in \pm\{(1,0),(0,1),(\frac{1}{2^{1/p}},\frac{1}{2^{1/p}}),(\frac{1}{2^{1/p}},\frac{-1}{2^{1/p}})\}$.

This effectively ensures that in order to prove that $T \in \mathbb{L}(\mathbb{X})$ is left symmetric if and only if T is the zero operator, we only need to consider 32 different operators that satisfy (i) and (ii) and show that none among them is left symmetric.
Let us first consider one such typical linear operator and prove that it is not left symmetric.
Let $T \in \mathbb{L}(\mathbb{X})$ be defined by: $T(1,0) = (1,0), T(0,1) = (0,0)$. Define $A \in \mathbb{L}(\mathbb{X})$ by $A(1,0) = (0,1), A(0,1) = (1,1)$. Clearly, T attains norm only at $\pm(1,0)$. Since $T(1,0) = (1,0) \perp_B (0,1) = A(1,0)$, it follows that $T \perp_B A$. We claim that $A \not\perp_B T$. Now, $\|A(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}})\|^p = \frac{1}{2} + 2^{p-1} > 2 = \|A(0,1)\|^p$, since $p \geq 2$. This proves that $\pm(1,0),(0,1) \notin M_A$. It is also easy to observe that if $(\alpha,\beta) \in M_A$ then either $\alpha,\beta \geq 0$ or $\alpha,\beta \leq 0$.
For any $\alpha,\beta > 0$ and for sufficiently small negative λ,
$\|A(\alpha,\beta)+\lambda T(\alpha,\beta)\|^p = \|(\beta+\lambda\alpha, \alpha+\beta)\|^p = |\beta+\lambda\alpha|^p+|\alpha+\beta|^p < |\beta|^p+|\alpha+\beta|^p = \|A(\alpha,\beta)\|^p$.
Similarly, for any $\alpha,\beta < 0$, and for sufficiently small negative λ, $\|A(\alpha,\beta)+\lambda T(\alpha,\beta)\|^p < \|A(\alpha,\beta)\|^p$.
This proves that for any $w \in M_A, Tw \notin Aw^-$. Applying Theorem 2.2, it now follows that $A \not\perp_B T$. Thus, T is not left symmetric in $\mathbb{L}(\mathbb{X})$, contradicting our initial assumption.
Next we describe a general method to prove that none among these 32 operators are left symmetric.
Let T attains norm at $x, x \perp_B y, y \perp_B x, Ty = 0$ and $x, y, Tx \in \pm\{(1,0),(0,1),(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}),(\frac{1}{2^{1/p}}, \frac{-1}{2^{1/p}})\}$.
Define a linear operator $A \in \mathbb{L}(\mathbb{X})$ by $Ax = y, Ay = (1,0)$ *or* $(1,1)$ such that
(i) A does not attain its norm at $\pm x, \pm y$.
(ii) $Tw \notin Aw^-$ for any $w \in M_A$.
Then, as before, it is easy to see that $T \perp_B A$ but $A \not\perp_B T$. Thus, T is not left symmetric.
This completes the proof of the theorem.

We would like to end the present paper with the concluding remark that it would be indeed interesting to extend the above theorem to higher dimensional l_p spaces, and more generally, to finite-dimensional strictly convex and smooth real Banach spaces, if possible.

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