



Classification and geometrical properties of the $X_{\theta(\cdot)}$ -valued function spaces [☆]



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ABSTRACT

This paper is devoted to investigate the $X_{\theta(\cdot)}$ -valued function spaces. Based on the notions of bounded topological lattice, Banach space net, continuous modular net and the dual space net, we divide $X_{\theta(\cdot)}$ -valued function spaces into two classes: norm-modular spaces and modular-modular spaces. For the first class, we study the separability of $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, give a representation of the dual space $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*$, find sufficient conditions of the equality $L^{p(\cdot)}(I, X_{\theta(\cdot)}) = L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, and prove the reflexivity of $L^{p(\cdot)}(I, X_{\theta(\cdot)})$ under some reasonable conditions. And for the second class, we prove the completeness and uniform convexity of $L^{\rho(\cdot)}(I, X_{\theta(\cdot)})$ in suitable situations. To show the naturality and rationality of these results, some concrete function spaces such as $L_t^{p(\cdot)}(I, L^{p(x,t)}(\Omega))$, $L_t^{p(\cdot)}(I, W^{1,p(x,t)}(\Omega))$ and $L^{p(\cdot)}(I, W_0^{1,p(x,t)}(\Omega))$ together with $L^{Pp(\cdot)}(I, L^{p(x,t)}(\Omega))$ are taken into account.

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1. Introduction and preliminaries

This paper deals with the $X_{\theta(\cdot)}$ -valued function spaces, a new type of abstract-valued function spaces with variable exponents. As useful tools in the study of both theoretical and applied mathematics, variable exponent function spaces have wide applications in many fields of applied mathematics, such as image restoration and electrorheological fluids etc.. Study of these spaces can trace its history to Orlicz [31], 1931, and it was receiving a growing interest of scholars in the latest decades, due to the discovery of boundedness of Hardy–Littlewood maximal operators in $L^{p(\cdot)}$ spaces (see [17,18,32]), and properties of generalized Sobolev spaces $W^{m,p(\cdot)}$ (see [22–25,28] etc.). For the detailed discussions of Lebesgue and Sobolev spaces with variable exponents with the most important progress in the past three decades and related references, we recommend the book [19], and for the open problems in this area, we refer to [21]. As for the applications of variable exponent function spaces, we would mention that, interests of scholars in the study of nonlinear

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parabolic equations turn naturally from the constant type to the nonconstant type, especially those having nonstandard growing nonlinearities, which simulate the realities more appropriately (cf. the survey [27]). Here some works are listed for references, such as [8–12] etc. involving anisotropic spaces, energy estimates and the nonstandard anisotropic growth conditions, [2–5] etc. involving convex functionals, Moreau–Yosida regularization and time discretization, and [15,26,33,35] involving the entropy and renormalization solutions and L^1 data.

It is worth to mention that, in the study of abstract evolution equations with nonstandard growing nonlinearities, a new type of abstract-valued function spaces deserve observation. These spaces have a common property, that is all their elements take values in varying spaces. For example, observe the anisotropic space $L^{p(t,x)}(Q)$, where $Q = I \times \Omega$ is a cylinder in \mathbb{R}^{N+1} and $p : Q \rightarrow [0, 1)$ is a doubly variable exponent measurable in x and continuous in t . It is not difficult to see that, for every $f \in L^{p(t,x)}(Q)$, the corresponding abstract-valued function $Pf(\cdot)$ defined through $Pf(t)(x) = f(t, x)$ takes value in the space $L^{p(t,\cdot)}(\Omega)$, which is varying upon the time. This phenomenon was also observed by Antontsev–Shmarev in [10].

Another example comes from the maximal L^p -regularity theory. Let A be a sectorial operator defined in a Banach space X . Recall that for every $1 < p < \infty$, the maximally regular space $\mathbb{E}_{1,p}(I) := L^p(I, \mathcal{D}(A)) \cap W^{1,p}(I, X)$ can be imbedded continuously into $BUC(I, X_{1-1/p,p})$, where $X_{1-1/p,p} := (X, \mathcal{D}(A))_{1-1/p,p}$ is the real interpolation space between X and $\mathcal{D}(A)$ (refer to [6], p. 180 or [29], Ch. 5). Let $p : I \rightarrow (1, \infty)$ be a continuous exponent, and $\mathbb{E}_{1,p(\cdot)}(I) := L^{p(\cdot)}(I, \mathcal{D}(A)) \cap W^{1,p(\cdot)}(I, X)$, where $L^{p(\cdot)}(I, \mathcal{D}(A))$ and $W^{1,p(\cdot)}(I, X)$ are the Bochner–Lebesgue and Bochner–Sobolev spaces with variable exponents. Then for every $f \in \mathbb{E}_{1,p(\cdot)}(I)$ and every compact subinterval J of I , we have $f|_J \in \mathbb{E}_{1,p_J^-}(J) \hookrightarrow C(J, X_{1-1/p_J^-,p_J^-})$. Let the interval J shrink to a point $t \in I$, then the trace space $X_{1-1/p_J^-,p_J^-}$ of $f|_J$ approaches $X_{1-1/p(t),p(t)}$. A question arises naturally, that is in what condition, does $f(t)$ take value in $X_{1-1/p(t),p(t)}$ together with $\sup_{t \in I} \|f(t)\|_{1-1/p(t),p(t)} < \infty$? In our best knowledge, there is no answer for this question. In [34], the author showed that, if $I = [0, T]$, $p(\cdot)$ satisfies the log-Hölder’s condition, and $-A$ generates a uniformly bounded and analytic semigroup e^{-tA} , then for the function $w(t) = e^{-tA}u_0$ with $u_0 \in X_{1-1/p(0),p(0)}$, both the inclusion $w \in \mathbb{E}_{1,p(\cdot)}(I)$ and the estimate $\sup_{t \in I} \|w(t)\|_{1-1/p(t),p(t)} < \infty$ are satisfied.

The above examples show the real existence of the abstract-valued function spaces with time-varying ranges. So it is necessary to pay attention to these spaces, make some classification and investigation on their geometric properties, so that they can be applied appropriately in concrete problems.

In [34], the authors made some discussions on the $X_{\theta(\cdot)}$ -valued function spaces for the first time. Their arguments depend on the introduction of Banach space net. After establishing suitable “measurability” for the $X_{\theta(\cdot)}$ -valued functions, they gave definitions of two spaces: $L^{p(\cdot)}(I, X_{\theta(\cdot)})$ and $C^-(I, X_{\theta(\cdot)})$, proved their completeness, studied the connections between them and listed some useful examples. As an application, they also investigated an abstract semilinear evolution equation, whose nonlinearity has a time-dependent domain. This model could not be incorporated in the preceding literatures, and has its meaning in the study of evolution equations with nonstandard growth.

This paper focuses on geometric properties of the range-varying function spaces. As preliminaries, in section 2, we give concepts of uniformly bounded topological lattice \mathcal{BTL} , regular Banach space net \mathcal{BSN} , continuous modular net \mathcal{CMN} , together with the dual space net \mathcal{DSN} , and establish the relations between \mathcal{BSN} and \mathcal{CMN} , \mathcal{BSN} and \mathcal{DSN} respectively. After introduction of the function space $L^0(I, X_{\theta(\cdot)})$ with suitable measurability for a \mathcal{DC} exponent θ , we divide the $X_{\theta(\cdot)}$ -valued function spaces into two classes: norm-modular type and modular-modular type. Section 3 is devoted to the first type, which includes integrable ones like $L^{p(\cdot)}(I, X_{\theta_n(\cdot)})$, $L^{p(\cdot)}(I, X_{\theta(\cdot)})$ and $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, and bounded ones like $C^-(I, X_{\theta(\cdot)})$ etc. We investigate the relations among them, seek for sufficient conditions of the equalities $\bigcap_{n \in \mathbb{N}} L^{p(\cdot)}(I, X_{\theta_n(\cdot)}) = L^{p(\cdot)}(I, X_{\theta(\cdot)})$ and $L_+^{p(\cdot)}(I, X_{\theta(\cdot)}) = L^{p(\cdot)}(I, X_{\theta(\cdot)})$. We also pay attention to the separability of $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, and make close observation on its dual space. By laying some reasonable as-

sumptions on both the \mathcal{BSN} and its \mathcal{DSN} , we give an important representation of $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*$, that is,

$$L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^* \cong L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*).$$

These properties exhibit the difference between the $X_{\theta(\cdot)}$ -valued function spaces and the X -valued ones. Additionally as a direct corollary, at the end of Section 2, we derive a sufficient condition for the reflexivity of $L^{p(\cdot)}(I, X_{\theta(\cdot)})$.

Section 4 is devoted to the modular-modular type spaces. Similar to the generalized scalar Φ function, we introduce a map M taking values in a closed cone V of an ordered topological space, and a series of modular $\{\rho_\alpha\}$ defined on V so that the multiple function series $\rho_\alpha \circ M$ is a \mathcal{CMN} . By putting the proper assumption on V and strong continuity on ρ , we prove that for each $f \in L^0(I, X_{\theta(\cdot)})$, the multiple function $t \mapsto \rho_{\theta(t)}(M(f(t)))$ is measurable, and the integration

$$\Phi_{\theta(\cdot)}(f) = \int_I \rho_{\theta(t)}(M(f(t))) dt$$

derives a modular on $L^0(I, X_{\theta(\cdot)})$. We then define the modular-modular type space $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, and prove its completeness. Under some additional conditions, the uniform convexity of $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}(\Omega))$, and an imbedding relation between the two different types of $X_{\theta(\cdot)}$ -valued spaces are also derived.

In order to show the naturality and rationality of the results obtained here, in Section 3, 4, we list $L^{p_t}(I, L^{p(x,t)}(\Omega))$, $L^{p_t}(I, W^{1,p(x,t)}(\Omega))$, $L^{p_t}(I, W_0^{1,p(x,t)}(\Omega))$ together with $L^{Pp(t)}(I, L^{p(x,t)}(\Omega))$ and $W(Q)$ as examples to study. By careful observations and arguments on the underlying \mathcal{CMN} s, \mathcal{BSN} s and their \mathcal{DSN} s of these spaces, we find that all the assumptions posed in the abstract theorems and propositions are verified by the above spaces exactly. Justification of the truths relies on the achievements we have obtained on the variable exponent Lebesgue and Sobolev spaces, such as the boundedness and convergence of the modified functions $j_\varepsilon * u$ in $L^{p(\cdot)}$ spaces etc. These facts make the applications of the concrete $X_{\theta(\cdot)}$ -valued function spaces much possible. And in the appendix, theories of the abstract Bochner–Lebesgue spaces with variable exponents and fixed valued spaces, including their geometrical properties are gathered for references.

Framework of this paper can be incorporated in linear and nonlinear functional analysis, and the theory of Lebesgue and Sobolev spaces with variable exponents. Results obtained here have their meaning in the study of evolution equations on $L^{p(\cdot)}$ -spaces.

2. Banach space net and continuous modular net

Our arguments begin at the notion of ordered topological spaces. Suppose that \mathcal{A} is a nonempty set, on which there are defined two different structures: a partial order \prec and a topology τ . We say that \mathcal{A} is an ordered topological space, if the two structures are compatible, which means that, for any net $\{\alpha_i\}_{i \in \mathcal{I}}$ convergent in τ with $\alpha_i \prec \beta$ for all $i \in \mathcal{I}$, the limit α satisfies $\alpha \prec \beta$ definitely. Here the index set \mathcal{I} is a direct poset, whose order has the cofinal property (cf. [34]). We say that \mathcal{A} is a topological lattice, we mean that \mathcal{A} is an ordered topological spaces, and for every order-bounded subset of \mathcal{A} , its order supremum and order infimum exist simultaneously. In the following contexts, \mathcal{A} is always assumed to be an totally order-bounded topological lattice, or \mathcal{BTL} in abbreviation. Furthermore, a net $\{\alpha_i\}_{i \in \mathcal{I}}$ is said to be approaching a point β in \mathcal{A} , if the two conditions $\alpha_i \prec \beta$, $\forall i \in \mathcal{I}$ and $\lim_{i \in \mathcal{I}} \alpha_i = \beta$ are both fulfilled.

Definition 2.1. Suppose that $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a collection of Banach spaces. We say $\{X_\alpha\}$ is a Banach space net, or \mathcal{BSN} for short, if

- $\alpha \prec \beta$ implies that $X_\beta \hookrightarrow X_\alpha$.

$\{X_\alpha : \alpha \in \mathcal{A}\}$ is said to be norm-continuous, whenever

- for any net $\{\alpha_i\}_{i \in \mathcal{I}}$ approaching β in \mathcal{A} , the limit $\lim_{i \in \mathcal{I}} \|x\|_{\alpha_i} = \|x\|_\beta$ holds for all $x \in X_\beta$, where $\|\cdot\|_\alpha$ denotes the norm of X_α .

$\{X_\alpha\}$ is called uniformly bounded, if

- there is a constant $C \geq 1$ independent of $\alpha, \beta \in \mathcal{A}$ such that $\|x\|_\alpha \leq C\|x\|_\beta$ for all $\alpha \prec \beta$ and $x \in X_\beta$.

And $\{X_\alpha\}$ is said to be successive, whenever

- conditions $x \in X_{\alpha_i}$ for all $i \in \mathcal{I}$ and $C = \sup_{i \in \mathcal{I}} \|x\|_{\alpha_i} < \infty$ for some net $\{\alpha_i\}_{i \in \mathcal{I}}$ approaching β result in $x \in X_\beta$ and $\|x\|_\beta \leq C$.

Finally, we call $\{X_\alpha\}$ a regular \mathcal{BSN} provided it is norm-continuous, uniformly bounded and successive at the same time.

Remark 2.2. Given a \mathcal{BSN} $\{X_\alpha : \alpha \in \mathcal{A}\}$, the family of dual spaces $\{X_\alpha^* : \alpha \in \mathcal{A}\}$, where \mathcal{A} takes the inverse order \succ instead of \prec , is also a \mathcal{BSN} , called the dual space net, or \mathcal{DSN} in symbol. Here we use the convention: $\langle \xi, x \rangle_\alpha = \langle \xi, x \rangle_\beta$ provided $\xi \in X_\alpha^*$, $x \in X_\beta$ and $\alpha \prec \beta$. It is easy to see that, if $\{X_\alpha\}$ is uniformly bounded, then $\{X_\alpha^*\}$ is also uniformly bounded with the same bounds. However, whether or not $\{X_\alpha^*\}$ inherits the norm-continuity and successive property from $\{X_\alpha\}$ is not clear.

Definition 2.3. Suppose that \mathcal{X} is a linear space, and $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ is a family of semimodulars defined on \mathcal{X} . We say $\{\rho_\alpha\}$ is a continuous modular net, or \mathcal{CMN} in abbreviation, we mean that the following hypotheses are satisfied:

- (1) Every ρ_α generates a Banach space $X_{\rho_\alpha} = X_\alpha$ contained in \mathcal{X} ,
- (2) there exist two positive constants C_1, C_2 such that for all $u \in \mathcal{X}$, inequality

$$\rho_\alpha(u) \leq C_1 \rho_\beta(u) + C_2, \quad (2.1)$$

holds for all $\alpha \prec \beta$, and

- (3) for any sequence α_k approaching β in \mathcal{A} ,

$$\lim_{k \rightarrow \infty} \rho_{\alpha_k}(u) = \rho_\beta(u).$$

Proposition 2.4. Given a \mathcal{CMN} $\{\rho_\alpha : \alpha \in \mathcal{A}\}$, the corresponding space family $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a regular \mathcal{BSN} .

Proof. Suppose that $\alpha \prec \beta$, then from inequality (2.1) and the definition of Luxemburg norm, we can derive that $X_\beta \hookrightarrow X_\alpha$ and $\|u\|_\alpha \leq C\|u\|_\beta$ with $C = \max\{1, C_1 + C_2\}$ for all $u \in X_\beta$. This shows the directness and uniform boundedness of $\{X_\alpha\}$.

Take a sequence $\{\alpha_k\}$ approaching β in \mathcal{A} and an element $u \in \mathcal{X}$. Suppose firstly $u \in X_\beta$. Without loss of generality, assume that $\|u\|_\beta > 0$, then for all $\varepsilon \in (0, 1)$, by the convexity of ρ_β and the definition of Luxemburg norm, we have

$$\rho_\beta\left(\frac{u}{(1+\varepsilon)\|u\|_\beta}\right) < 1 < \rho_\beta\left(\frac{u}{(1-\varepsilon)\|u\|_\beta}\right).$$

Taking the limits

$$\lim_{k \rightarrow \infty} \rho_{\alpha_k}\left(\frac{u}{(1 \pm \varepsilon)\|u\|_\beta}\right) = \rho_\beta\left(\frac{u}{(1 \pm \varepsilon)\|u\|_\beta}\right)$$

into account, we obtain

$$\rho_{\alpha_k}\left(\frac{u}{(1+\varepsilon)\|u\|_\beta}\right) < 1 < \rho_{\alpha_k}\left(\frac{u}{(1-\varepsilon)\|u\|_\beta}\right),$$

or in other words,

$$(1-\varepsilon)\|u\|_\beta < \|u\|_{\rho_k} < (1+\varepsilon)\|u\|_\beta$$

for sufficiently large k . Thus let $k \rightarrow \infty$, we derive

$$\limsup_{k \rightarrow \infty} \|u\|_{\alpha_k} \leq (1+\varepsilon)\|u\|_\beta \quad \text{and} \quad \liminf_{k \rightarrow \infty} \|u\|_{\alpha_k} \geq (1-\varepsilon)\|u\|_\beta,$$

which yields

$$\lim_{k \rightarrow \infty} \|u\|_{\alpha_k} = \|u\|_\beta$$

by the arbitrariness of ε . This exhibits the continuity of the norm. Finally, suppose that $u \in X_{\alpha_k}$ for all $k \in \mathbb{N}$ and $C = \sup_{k \in \mathbb{N}} \|u\|_{\alpha_k} < \infty$. Assume also $C > 0$, then $\rho_{\alpha_k}(u/C) \leq 1$, $k = 1, 2, \dots$, and consequently

$$\rho_\beta\left(\frac{u}{C}\right) = \lim_{k \rightarrow \infty} \rho_{\alpha_k}\left(\frac{u}{C}\right) \leq 1,$$

which means that $u \in X_\beta$ and $\|u\|_\beta \leq C$. This leads to the successive property of $\{X_\alpha\}$. Thus regularity of $\{X_\alpha\}$ has been completed. \square

Here are several examples of regular \mathcal{BSN} . Among them, some are induced by the \mathcal{CMN} , while some are not.

Example 2.1. Suppose that X is a Banach space. We say a closed linear operator A defined in X is positive, we mean that the residual set $\rho(-A)$ contains a sector $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \theta\}$ with the angle $\theta \in (0, \pi)$ and a disk $B(0, \delta)$ centered at 0 with the radius $\delta > 0$ such that for all $\lambda \in \Sigma_\theta \cup B(0, \delta)$, the estimate

$$\|(\lambda + A)^{-1}\| \leq \frac{M}{1 + |\lambda|}$$

holds for some $M > 0$ independent of λ . A positive operator A is call of \mathcal{BIP} type (cf. [6], §3.4 or [29], Ch. 4), if all its imaginary powers A^{it} s are bounded operators and uniformly bounded on the interval $[-\varepsilon, \varepsilon]$ for any $\varepsilon > 0$. Here the complex power A^z is defined by the Dounford integral

$$A^z = \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^z (\lambda + A)^{-1} d\lambda$$

on a piecewise smooth path Γ located in the interior of $\Sigma_\theta \cup B(0, \delta)$ oriented from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ in the case $\operatorname{Re} z < 0$, and defined by $A^z x = A^n x$ for all $x \in \mathcal{D}(A^z) := \{z \in X : A^{z-n} x \in \mathcal{D}(A^n)\}$ in the case $0 \leq \operatorname{Re} z < n$.

Recall that for a \mathcal{BIP} type operator A , there are constants $C, \theta > 0$ such that $\|A^z\| \leq Ce^{\theta|\operatorname{Im} z|}$, and A^z is strongly continuous in the strip $\{z \in \mathbb{C} : -1 < \operatorname{Re} z \leq 0\}$.

For fixed $\varepsilon > 0$, let \mathcal{A}_ε be the rectangle $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1, |\operatorname{Im} z| \leq \varepsilon\}$ with the order: $\alpha \prec \beta$ whenever $\operatorname{Re} \alpha \leq \operatorname{Re} \beta$ and $\operatorname{Im} \alpha \leq \operatorname{Im} \beta$. It is easy to check that, this order is compatible with the natural topology, and makes \mathcal{A}_ε be a \mathcal{BTL} . For each $\alpha \in \mathcal{A}_\varepsilon$, define $X_\alpha = \mathcal{D}(A^\alpha)$ and equip it with the norm $\|x\|_\alpha = \|A^\alpha x\|$, then we get a uniformly bounded and norm-continuous $\mathcal{BSN} \{X_\alpha : \alpha \in \mathcal{A}_\varepsilon\}$. In addition, if the extra assumption of reflexivity is laid on X , then $\{X_\alpha\}$ is also successive, hence it is regular (cf. [34]).

The first example of \mathcal{BSN} is made by means of complex interpolation, since for each $\alpha = s + ib \in \mathcal{A}_\varepsilon$, $X_\alpha = [X, \mathcal{D}(A^{1+ib})]_s$ in the sense of isomorphism. The second example is made by real interpolation as follows:

Example 2.2. In this example, X is also a Banach space and A is assumed to be a sectorial operator, which means that $\mathcal{D}(A)$ is dense in X , and there are constants $\theta \in (\pi/2, \pi)$, $\omega \in \mathbb{R}$ and $M > 0$ such that $\Sigma_\theta + \omega \subseteq \rho(-A)$ and

$$\|(\lambda + A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}$$

for all $\lambda \in \Sigma_\theta + \omega$. Under this situation, the negative operator $-A$ generates an analytic semigroup e^{-tA} verifying $\|e^{-tA}\| \leq M_0 e^{\omega t}$ for all $t \geq 0$ and some $M_0 \geq 1$. Furthermore, for all $\alpha \in (0, 1)$ and $1 \leq p < \infty$, the real interpolation space $(X, \mathcal{D}(A))_{\alpha, p}$, where $\mathcal{D}(A)$ is endowed with the graph norm, is equal to

$$\{x \in X : \|\tau^{1-\alpha} \|Ae^{-\tau A} x\|\|_{L^p_*(0,1)} := \left\{ \int_0^1 (\tau^{1-\alpha} \|Ae^{-\tau A} x\|)^p \frac{d\tau}{\tau} \right\}^{1/p} < \infty\},$$

with the norms $\|x\|_{(X, \mathcal{D}(A))_{\alpha, p}}$ and $\|x\|_{\alpha, p} = \|x\| + \|\tau^{1-\alpha} \|Ae^{-\tau A} x\|\|_{L^p_*(0,1)}$ equivalent (cf. [6], §3.4 or [29], Ch. 5).

Let $p : [0, 1] \rightarrow [1, \infty)$ be a nondecreasing and continuous function fulfilling

$$\left| \frac{1}{p(\beta)} - \frac{1}{p(\alpha)} \right| \leq L|\beta - \alpha|$$

for some $L > 0$. Take $\mathcal{A}_1 = [0, 1)$ as the \mathcal{BTL} with the common order \leq and the natural topology. Let $X_0 = X$ and $X_\alpha = (X, \mathcal{D}(A))_{\alpha, p(\alpha)}$ with the norm $\|\cdot\|_\alpha = \|\cdot\|_{\alpha, p(\alpha)}$ for all $\alpha \in \mathcal{A}_1$, then we obtain another regular $\mathcal{BSN} \{X_\alpha : \alpha \in \mathcal{A}_1\}$ (cf. [34]).

Remark 2.5. In the most applications, we always take the form $p(\alpha) = 1/(1 - \alpha)$.

Suppose that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain on which we define $\mathcal{P}(\Omega)$ as the set of all measurable functions $p : \Omega \rightarrow [1, \infty]$ and

$$\begin{aligned} \mathcal{P}_b(\Omega) &= \{p \in \mathcal{P}(\Omega) : 1 \leq p^- \leq p^+ < \infty\}, \\ \mathcal{P}_0(\Omega) &= \{p \in \mathcal{P}(\Omega) : 1 < p^- \leq p^+ < \infty\}, \end{aligned}$$

where $p^+ = \sup_{x \in \Omega} p(x)$ and $p^- = \inf_{x \in \Omega} p(x)$. Recall that for each $p \in \mathcal{P}_b(\Omega)$,

$$\rho_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx, \quad f \in L^0(\Omega)$$

is a continuous modular, which deduces a Banach space, denoted by $L^{p(\cdot)}(\Omega)$. And $f \in L^{p(\cdot)}(\Omega)$ if and only if $\int_{\Omega} |f(x)|^{p(x)} dx < \infty$, together with the Luxemburg norm.

Example 2.3. Fix two numbers \underline{p} and \bar{p} such that $1 \leq \underline{p} \leq \bar{p} < \infty$, and let

$$\mathcal{A}_b = \{p \in \mathcal{P}_b(\Omega) : p(x) \in [\underline{p}, \bar{p}] \text{ for a.e. } x \in \Omega\}.$$

One can easily check that, with the natural order: $p \prec q$ provided $p(x) \leq q(x)$ a.e. $x \in \Omega$ and the L^∞ -distance, \mathcal{A}_b turns to be a \mathcal{BTL} . Let $\mathcal{X} = L^0(\Omega)$, then the modular family $\{\rho_{p(\cdot)} : p \in \mathcal{A}_b\}$ is a \mathcal{CMN} defined on \mathcal{A}_b . As a matter of fact, for all $p, q \in \mathcal{A}_b$ with $p \prec q$ and all $f \in \mathcal{X}$, by Hölder's inequality ([19], p. 81), we have

$$\rho_{p(\cdot)}(f) \leq \rho_{q(\cdot)}(f) + |\Omega|,$$

which shows the validity of (2.1) with $C_1 = 1$, $C_2 = |\Omega|$.

In addition, if $\{p_k\}$ approaches q in \mathcal{A}_b , then for every $f \in \mathcal{X}$, divide Ω into two disjoint parts: $E_1 = \{x \in \Omega : |f(x)| \leq 1\}$, $E_2 = \{x \in \Omega : |f(x)| > 1\}$, and split the integral $\int_{\Omega} |f(x)|^{p_k(x)} dx$ into two parts correspondingly.

For the first part $\int_{E_1} |f(x)|^{p_k(x)} dx$, using Lebesgue's convergence theorem, we obtain

$$\lim_{k \rightarrow \infty} \int_{E_1} |f(x)|^{p_k(x)} dx = \int_{E_1} |f(x)|^{q(x)} dx.$$

And for the second part $\int_{E_2} |f(x)|^{p_k(x)} dx$, we have

$$\int_{E_2} |f(x)|^{p_k(x)} dx \leq \int_{E_2} |f(x)|^{q(x)} dx,$$

thus

$$\limsup_{k \rightarrow \infty} \int_{E_2} |f(x)|^{p_k(x)} dx \leq \int_{E_2} |f(x)|^{q(x)} dx.$$

Conversely by Fatou's lemma, we also have

$$\liminf_{k \rightarrow \infty} \int_{E_2} |f(x)|^{p_k(x)} dx \geq \int_{E_2} |f(x)|^{q(x)} dx.$$

The above two opposite inequalities jointly yield

$$\lim_{k \rightarrow \infty} \int_{E_2} |f(x)|^{p_k(x)} dx = \int_{E_2} |f(x)|^{q(x)} dx.$$

Consequently $\lim_{k \rightarrow \infty} \rho_{p_k}(f) = \rho_q(f)$. Hence $\{\rho_{p(\cdot)} : p \in \mathcal{A}_b\}$ is a \mathcal{CMN} . Now by invoking Proposition 2.4, we conclude that $\{L^{p(\cdot)}(\Omega) : p \in \mathcal{A}_b\}$ is a regular \mathcal{BSN} .

Remark 2.6. In the above example, the space family $\{L^{p'(\cdot)}(\Omega) : p \in \mathcal{A}_b\}$ is also a regular \mathcal{BSN} , which is the \mathcal{DSN} of $\{L^{p(\cdot)}(\Omega) : p \in \mathcal{A}_b\}$ exactly.

Given $p \in \mathcal{P}_b(\Omega)$, define the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) : u, \partial_i u \in L^{p(\cdot)}(\Omega), i = 1, 2, \dots, N\},$$

where $\partial_i u = \partial u / \partial x_i$ denotes the i -th weak derivative of u . Recall that according to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) + \sum_{i=1}^N \rho_{p(\cdot)}\left(\frac{\partial_i u}{\lambda}\right) \right\},$$

or the equivalent one

$$\|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

$W^{1,p(\cdot)}(\Omega)$ becomes a separable Banach space. If additionally $p^- > 1$, then it is uniformly convex, and of course, it is reflexive.

In the coming contexts, assume $\partial\Omega \in C^1$ and $p \in \mathcal{P}_{\log}^\omega(\Omega)$, which means $p \in C(\overline{\Omega})$ fulfilling the log-Hölder condition:

$$|p(x) - p(y)| \leq \omega(|x - y|), \quad \text{for } |x - y| < 1, \quad (2.2)$$

where $\omega : (0, \infty) \rightarrow [0, \infty)$ is a nonincreasing function satisfying

$$C_\omega = \sup_{r>0} \omega(r) \log \frac{1}{r} < \infty. \quad (2.3)$$

Recall that under this situation, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, so we can define $W_0^{1,p(\cdot)}(\Omega)$ as the complement of $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, or equivalently $W_0^{1,p(\cdot)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$. And for this space, Poincaré's inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega)$$

still holds with the constant $C > 0$ independent of u . Therefore we can define $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ as the equivalent norm of $W_0^{1,p(\cdot)}(\Omega)$. We also know that, $W_0^{1,p(\cdot)}(\Omega)$ is separable, and as a closed subspace of $W^{1,p(\cdot)}(\Omega)$, it is reflexive whenever $p^- > 1$ (refer to [19], Ch. 8, 9).

Remark 2.7. As for the dual space $W_0^{1,p(\cdot)}(\Omega)^* =: W^{-1,p'(\cdot)}(\Omega)$, [19], §12.3 reveals that it can be represented in the same way as the constant exponent type. That is each $\xi \in W^{-1,p'(\cdot)}(\Omega)$ equals $f_0 - \sum_{i=1}^N \partial_i f_i$ for some vector function $F = (f_0, f_1, \dots, f_N) \in L^{p'(\cdot)}(\Omega)^{N+1}$ in the generalized function space $\mathcal{D}'(\Omega)$, or equivalently,

$$\langle \xi, u \rangle = \int_{\Omega} f_0(x) u(x) dx + \sum_{i=1}^N \int_{\Omega} f_i(x) \partial_i u(x) dx \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

And the norm $\|\xi\|_{W^{-1,p'(\cdot)}(\Omega)}$ is equivalent to $\sum_{i=0}^N \|f_i\|_{L^{p'(\cdot)}(\Omega)}$. Moreover, $\mathcal{D}(\Omega)$ is dense in $W^{-1,p'(\cdot)}(\Omega)$ provided $p^- > 1$.

Example 2.4. Take \mathcal{A}_b as the underlying \mathcal{BTL} , then following the same lines as in the proof of the preceding example, we can verify that the Sobolev space family $\{W^{1,p(\cdot)}(\Omega) : p \in \mathcal{A}_b\}$ is a regular \mathcal{BSN} . Additionally, if we set $\mathcal{A}_\omega = \mathcal{A}_b \cap \mathcal{P}_{\log}^\omega(\Omega)$ for some ω verifying (2.2) and (2.3) as the ordered and topological subspace of \mathcal{A}_b , then \mathcal{A}_ω is also a \mathcal{BTL} and the homogeneous Sobolev space family $\{W_0^{1,p(\cdot)}(\Omega) : p \in \mathcal{A}_\omega\}$ is a regular \mathcal{BSN} too.

Remark 2.8. In the above example, we should only pay extra extension to the order closedness of \mathcal{A}_ω , i.e. for any order-bounded subset B of \mathcal{A}_ω , both $\sup B = \sup\{p(\cdot) : p \in B\}$ and $\inf B = \inf\{p(\cdot) : p \in B\}$ lie in \mathcal{A}_ω definitely. In fact, for two points $x, y \in \Omega$ satisfying $|x - y| < 1$ and arbitrary $p \in B$,

$$p(x) - \sup B(y) \leq p(x) - p(y) \leq \omega(|x - y|).$$

Take the supremum of $p \in B$, we have

$$\sup B(x) - \sup B(y) \leq \omega(|x - y|).$$

And the opposite inequality is obtained by switching x and y . Thus $\sup B \in \mathcal{P}_{log}(\Omega)$ with the estimate

$$|\sup B(x) - \sup B(y)| \leq \omega(|x - y|).$$

This fact remains true if $\sup B$ is replaced by $\inf B$.

Example 2.5. Denote by $\mathcal{A}_0 = \mathcal{A}_\omega \cap \mathcal{P}_0(\Omega)$ with $\underline{p} > 1$. We known that $W^{-1, \underline{p}'}(\Omega)$ is separable, it can be imbedded densely into $W^{-1, p'(\cdot)}(\Omega)$ for every $p \in \mathcal{A}_0$. Choose $\{\varepsilon_k\}$ as a basis of $W^{-1, \underline{p}'}(\Omega)$, and attached to each ε_k select only one vector function $G_k = (g_{k,0}, g_{k,1}, \dots, g_{k,N}) \in L^{\underline{p}'}(\Omega)^{N+1}$ such that $\varepsilon_k = g_{k,0} - \sum_{i=1}^N \partial_i g_{k,i}$ in $\mathcal{D}'(\Omega)$. Thus for all $\xi \in W^{-1, \underline{p}'}(\Omega)$, by invoking [Remark 2.7](#), we conclude that for any sequence $\{a_k \in \mathbb{R}, k = 1, 2, \dots\}$,

$$\xi = \sum_{k=1}^{\infty} a_k \varepsilon_k \quad \text{in } W^{-1, \underline{p}'}(\Omega)$$

if and only if

$$F = (f_0, f_1, \dots, f_N) = \sum_{k=1}^{\infty} a_k G_k \quad \text{in } L^{\underline{p}'}(\Omega)^{N+1},$$

together with $\xi = f_0 - \sum_{i=1}^N \partial_i f_i$ in $\mathcal{D}'(\Omega)$ holding. Therefore every element of $W^{-1, \underline{p}'}(\Omega)$ has a unique representation depending on the basis $\{\varepsilon_k\}$ and the representation sequence $\{G_k\}$ chosen above.

Take any $\xi \in W^{-1, p'(\cdot)}(\Omega)$, then due to the density of $W^{-1, \underline{p}'}(\Omega)$ in $W^{-1, p'(\cdot)}(\Omega)$, there is a sequence $\{\xi_k\} \subseteq W^{-1, \underline{p}'}(\Omega)$ convergent to ξ in $W^{-1, p'(\cdot)}(\Omega)$. Suppose that $F_k = (f_{k,0}, f_{k,1}, \dots, f_{k,N})$ is the corresponding $L^{\underline{p}'}(\Omega)^{N+1}$ -representation of ξ_k determined by $\{\varepsilon_k\}$ and $\{G_k\}$. Since $\{\xi_k\}$ is a Cauchy sequence of $W^{-1, p'(\cdot)}(\Omega)$, by the equivalence of norms, we can find that $\{F_k\}$ is also a Cauchy sequence of $L^{p'(\cdot)}(\Omega)^{N+1}$. Thus there is a vector function $F = (f_0, f_1, \dots, f_N)$ such that $\{F_k\}$ converges to F in $L^{p'(\cdot)}(\Omega)^{N+1}$. Taking limits in both sides of the following equation as $k \rightarrow \infty$:

$$\langle \xi_k, u \rangle = \int_{\Omega} f_{k,0}(x) u(x) dx + \sum_{i=1}^N \int_{\Omega} f_{k,i}(x) \partial_i u(x) dx, \quad u \in W_0^{1, p(\cdot)}(\Omega),$$

we conclude that $\xi = f_0 - \sum_{i=1}^N \partial_i f_i$ in $W^{-1, p'(\cdot)}(\Omega)$, in other words, F is a representation of ξ in $L^{p'(\cdot)}(\Omega)^{N+1}$. One can easily check that the $L^{p'(\cdot)}(\Omega)^{N+1}$ -representation F of ξ does not depend on the approximation sequence $\{\xi_k\}$ selected in $W^{-1, \underline{p}'}(\Omega)$. Moreover, if $\xi_k \rightarrow \xi$ in $W^{-1, p'(\cdot)}(\Omega)$, then $F_k \rightarrow F$ in $L^{q'(\cdot)}(\Omega)^{N+1}$ provided $q \succ p$, which means that in this setting, every $\xi \in W^{-1, p'(\cdot)}(\Omega)$ has the same representation in $L^{q'(\cdot)}(\Omega)^{N+1}$ for all $q \in \mathcal{A}_0$ with $q \succ p$.

Now, for all $p \in \mathcal{A}_0$ and all $\xi \in W^{-1,p'(\cdot)}(\Omega)$, take the norm $\|\xi\|_{p,*} := \|F\|_{L^{p'(\cdot)}(\Omega)^{N+1}}$, where F is the unique $L^{p'(\cdot)}(\Omega)^{N+1}$ -representation of ξ determined above. Then for every sequence $\{q'_k\} \subseteq \mathcal{A}_0$ approaching p' , following the same process as in Example 2.3, we can prove that,

$$\lim_{k \rightarrow \infty} \|\xi\|_{q_k,*} = \lim_{k \rightarrow \infty} \|F\|_{L^{q'_k(\cdot)}(\Omega)^{N+1}} = \|F\|_{L^{p'(\cdot)}(\Omega)^{N+1}} = \|\xi\|_{p,*}.$$

Conversely, if $\xi \in W^{-1,q'_k(\cdot)}(\Omega)$ for all $k \in \mathbb{N}$ with the bounds $C = \sup_{k \in \mathbb{N}} \|\xi\|_{q_k,*} < \infty$, then $\xi \in W^{-1,p'(\cdot)}(\Omega)$ and $\|\xi\|_{p,*} \leq C$. Hence, $\{W^{-1,p'(\cdot)}(\Omega) : p \in \mathcal{A}_0\}$ is also a regular \mathcal{BSN} , which is the \mathcal{DSN} of $\{W_0^{1,p(\cdot)}(\Omega) : p \in \mathcal{A}_0\}$.

Remark 2.9. Similar to $W_0^{1,p(\cdot)}(\Omega)^*$, for each $p \in \mathcal{A}_0$, every $\xi \in W^{1,p(\cdot)}(\Omega)^*$ has a representation $F \in L^{p'(\cdot)}(\Omega)^{N+1}$, that is

$$\xi = f_0 - \sum_{i=1}^N \partial_{x_i} f_i \quad \text{in } \mathcal{D}'(\overline{\Omega}),$$

or equivalently,

$$\langle \xi, u \rangle = \int_{\Omega} f_0(x) u(x) dx + \sum_{i=1}^N f_i(x) \partial_{x_i} u(x) dx, \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

And the norms $\|\xi\|_{W^{1,p(\cdot)}(\Omega)^*}$ and $\|F\|_{L^{p'(\cdot)}(\Omega)^{N+1}}$ are equivalent (see [1], Theorem 3.9 for an analogous discussion in the constant exponent case). Here, $\mathcal{D}(\overline{\Omega})$ is the restriction of $\mathcal{D}(\mathbb{R}^N)$ on $\overline{\Omega}$, while $\mathcal{D}'(\overline{\Omega})$ is the dual space of $\mathcal{D}(\overline{\Omega})$ in the sense of distribution. Very similar to the above example, we can also derive that $\{W^{1,p(\cdot)}(\Omega)^*, p \in \mathcal{A}_0\}$ is a regular \mathcal{BSN} , it is just the \mathcal{DSN} of $\{W^{1,p(\cdot)}(\Omega) : p \in \mathcal{A}_0\}$.

3. Modular-norm space

Let $I = [0, T]$ for some $0 < T < \infty$ or $I = [0, \infty)$ and $\Lambda(I)$ be the collection of all bounded subintervals of I . Suppose that \mathcal{A} is a \mathcal{BTL} , $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a norm-continuous \mathcal{BSN} , and $\theta : I \rightarrow \mathcal{A}$ is a doubly continuous (\mathcal{DC} in abbreviation) map, which means that θ is continuous according to the topology of \mathcal{A} , and for any nest of intervals $\{J_k \in \Lambda(I) : k = 1, 2, \dots\}$ shrinking to t , limits

$$\lim_{k \rightarrow \infty} \theta_{J_k}^- = \lim_{k \rightarrow \infty} \theta_{J_k}^+ = \theta(t)$$

always hold simultaneously, where $\theta_J^- = \inf_{t \in J} \theta(t)$ and $\theta_J^+ = \sup_{t \in J} \theta(t)$ in the order sense.

Denote by $\theta_I^\pm = \theta_I^\pm$, and define

$$L_-^0(I, X_{\theta(\cdot)}) = \{f \in L^0(I, X_{\theta^-}) : f|_J \in L^0(J, X_{\theta_J^-}) \text{ for all } J \in \Lambda(I)\},$$

and

$$L^0(I, X_{\theta(\cdot)}) = \{f \in L_-^0(I, X_{\theta(\cdot)}) : f(t) \in X_{\theta(t)} \text{ for a.e. } t \in I\}.$$

Obviously, both of them are linear spaces according to the sum and scalar multiplication of abstract-valued functions. The following proposition gives a description of the suitable measurability of the elements in $L^0(I, X_{\theta(\cdot)})$. Its proof has been completed in [34].

Proposition 3.1. For each $f \in L^0(I, X_{\theta(\cdot)})$, the norm-valued function $t \mapsto \|f(t)\|_{\theta(t)}$ is measurable.

Besides the norm-continuity, assume that $\{X_\alpha\}$ is a dense \mathcal{BSN} , i.e. X_β is a densely injected in X_α whenever $\alpha \prec \beta$. Denote by $\mathcal{S}(I, X_{\theta+})$ the set all $X_{\theta+}$ -valued simple functions with compact supporting sets. It is easy to see that $\mathcal{S}(I, X_{\theta+})$ is contained in $L^0(I, X_{\theta(\cdot)})$, consequently for every $\varphi \in \mathcal{S}(I, X_{\theta+})$, the norm function $t \mapsto \|\varphi(t)\|_{\theta(t)}$ is measurable.

Define

$$L_+^0(I, X_{\theta(\cdot)}) = \{f \in L^0(I, X) : f(t) \in X_{\theta(t)} \text{ and there exists a sequence } \{\varphi_n\} \\ \text{in } \mathcal{S}(I, X_{\theta+}) \text{ s.t. } \|\varphi_n(t) - f(t)\|_{\theta(t)} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for a.e. } t \in I\}$$

Remark 3.2. Here, the set $\mathcal{S}(I, X_{\theta+})$ can be replaced by $L^0(I, X_{\theta+})$. And by this definition, one can easily check that $L_+^0(I, X_{\theta(\cdot)})$ is a subspace of $L^0(I, X_{\theta(\cdot)})$.

For each $p \in \mathcal{P}(I)$, define a semimodular on $L^0(I, X_{\theta(\cdot)})$ as follows:

$$\rho_{p(\cdot)}(\|f(\cdot)\|_{\theta(\cdot)}) = \int_I \varphi_{p(t)}(\|f(t)\|_{\theta(t)}) dt,$$

where

$$\varphi_q(s) = \begin{cases} s^q, & \text{if } 1 \leq q < \infty, \\ \infty \cdot \chi_{(1, \infty)}(s) & \text{if } q = \infty \end{cases}$$

for $s \geq 0$. This semimodular generates an abstract Bochner–Lebesgue space, denoted by $L^{p(\cdot)}(I, X_{\theta(\cdot)})$, i.e.

$$L^{p(\cdot)}(I, X_{\theta(\cdot)}) = \{f \in L^0(I, X_{\theta(\cdot)}) : \|f(\cdot)\|_{\theta(\cdot)} \in L^{p(\cdot)}(I)\},$$

and the norm is defined by

$$\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} = \|\|f(\cdot)\|_{\theta(\cdot)}\|_{L^{p(\cdot)}(I)}.$$

This space has similar properties as the scalar ones, such as the unit ball property, that is $\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq 1$ whenever $\rho_{p(\cdot)}(\|f(\cdot)\|_{\theta(\cdot)}) \leq 1$, and vice versa. If $p \in \mathcal{P}_b(I)$, then $\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} = 1$ if and only if $\rho_{p(\cdot)}(\|f(\cdot)\|_{\theta(\cdot)}) = 1$. Furthermore,

Theorem 3.3. For every $p \in \mathcal{P}(I)$, $L^{p(\cdot)}(I, X_{\theta(\cdot)})$ is a Banach space.

For a proof of this theorem, please refer to [34].

Given a positive integer n , let $t_{n,k} = kT/2^n$, $J_{n,k} = (t_{n,k-1}, t_{n,k}]$ and $\theta_{n,k}^- = \theta_{J_{n,k}}^-$, where $k = 1, 2, \dots, 2^n$ if $I = [0, T]$ or $k = 1, 2, \dots$ if $I = [0, \infty)$. Define $\theta_n(t) = \theta_{n,k}^-$ if $t \in J_{n,k}$, then we obtain a series of variable exponents $\{\theta_n\}$ which is increasing in n and tends to $\theta(t)$ as $n \rightarrow \infty$ for all $t \in I$. Since every exponent θ_n is a step function, function space $L^{p(\cdot)}(I, X_{\theta_n(\cdot)})$ is well defined with the norm $\|\|f(\cdot)\|_{\theta_n(\cdot)}\|_{L^{p(\cdot)}(I)}$.

Remark 3.4. A fact easily ignored is that continuity and order-boundedness of θ could not guarantee the realization of its infimum on a bounded and closed interval. Hence for each $J_{n,k}$, we do not expect the existence of a point, say $s_{n,k}$, lying in $\bar{J}_{n,k}$ such that the value $\theta(s_{n,k})$ equals the order infimum $\theta_{n,k}^-$.

The following proposition can be verified straightly.

Proposition 3.5. Suppose that $\{X_\alpha\}$ is norm-continuous and uniformly bounded with the uniform bounds C , then

$$L^{p(\cdot)}(I, X_{\theta(\cdot)}) \hookrightarrow L^{p(\cdot)}(I, X_{\theta_n(\cdot)})$$

and $\|f\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})} \leq C\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}$ for all $n \in \mathbb{N}$ and all $f \in L^{p(\cdot)}(I, X_{\theta(\cdot)})$.

Theorem 3.6. If $\{X_\alpha\}$ is regular, then $L^{p(\cdot)}(I, X_{\theta(\cdot)})$ is equivalent to

$$\{f \in L^0_-(I, X_{\theta(\cdot)}) : \sup_{n \geq 1} \|f\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})} < \infty\}$$

with the estimates

$$\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq \sup_{n \geq 1} \|f\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})} \leq C\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}$$

holding.

Proof. Thanks to the preceding proposition, it suffices to show that if $f \in L^0_-(I, X_{\theta(\cdot)})$ and $K = \sup_{n \geq 1} \|f\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})} < \infty$, then $f \in L^{p(\cdot)}(I, X_{\theta(\cdot)})$ and $\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq K$. Without loss of generality, assume that $K > 0$, then by the unit ball property, we have

$$\int_I \varphi_{p(t)}\left(\left\|\frac{f(t)}{K}\right\|_{\theta_n(t)}\right) dt \leq 1,$$

which yields

$$\int_I \varphi_{p(t)}\left(\liminf_{n \rightarrow \infty} \left\|\frac{f(t)}{K}\right\|_{\theta_n(t)}\right) dt \leq \int_I \liminf_{n \rightarrow \infty} \varphi_{p(t)}\left(\left\|\frac{f(t)}{K}\right\|_{\theta_n(t)}\right) dt \leq 1$$

by the lower semi-continuity of $\varphi_{p(t)}$ and Fatou's lemma. Thus the integrand $\varphi_{p(t)}(\liminf_{n \rightarrow \infty} \|f(t)/K\|_{\theta_n(t)})$ and consequently $\liminf_{n \rightarrow \infty} \|f(t)/K\|_{\theta_n(t)}$ is finite for a.e. $t \in I$. Denote by E the set of all points t at which

$$\liminf_{n \rightarrow \infty} \left\|\frac{f(t)}{K}\right\|_{\theta_n(t)} < \infty,$$

then E is measurable with zero measure complement. And for all $t \in E$ and arbitrary $\varepsilon > 0$, there exists a subsequence, say $\{\theta_{n_i}\}$, such that

$$\sup_{i \geq 1} \left\|\frac{f(t)}{K}\right\|_{\theta_{n_i}(t)} \leq (1 + \varepsilon) \liminf_{n \rightarrow \infty} \left\|\frac{f(t)}{K}\right\|_{\theta_n(t)} < \infty.$$

Notice that $\lim_{i \rightarrow \infty} \theta_{n_i}(t) = \theta(t)$, by the successive property of $\{X_\alpha\}$, we can derive that $f(t) \in X_{\theta(t)}$ and

$$\left\|\frac{f(t)}{(1 + \varepsilon)K}\right\|_{\theta(t)} \leq \liminf_{n \rightarrow \infty} \left\|\frac{f(t)}{K}\right\|_{\theta_n(t)}.$$

Thus $f \in L^0(I, X_{\theta(\cdot)})$ and

$$\int_I \varphi_{p(t)}\left(\left\|\frac{f(t)}{(1 + \varepsilon)K}\right\|_{\theta(t)}\right) dt \leq \int_I \liminf_{n \rightarrow \infty} \varphi_{p(t)}\left(\left\|\frac{f(t)}{K}\right\|_{\theta_n(t)}\right) dt \leq 1.$$

This tells us that $f \in L^{p(\cdot)}(I, X_{\theta(\cdot)})$, and

$$\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq (1 + \varepsilon) \sup_{n \geq 1} \|f\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})},$$

which leads to the desired estimate by the arbitrariness of ε . \square

Suppose that $\{X_\alpha\}$ is a norm-continuous and dense \mathcal{BSN} and $p \in \mathcal{P}_b(I)$, then similar to $L^{p(\cdot)}(I, X_{\theta(\cdot)})$, we can define another space $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ through

$$L_+^{p(\cdot)}(I, X_{\theta(\cdot)}) = \{f \in L_+^0(I, X_{\theta(\cdot)}) : \|f(\cdot)\|_{\theta(\cdot)} \in L^{p(\cdot)}(I)\}$$

with the same norm as in $L^{p(\cdot)}(I, X_{\theta(\cdot)})$. Note that in such situation, $\mathcal{S}(I, X_{\theta+})$ is a linear subspace of $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$. Moreover, we can derive that

Proposition 3.7. *For each $f \in L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, there is a sequence $\{\varphi_n\} \subseteq \mathcal{S}(I, X_{\theta+})$ convergent to f according to the norm $\|\cdot\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}$.*

Proof. Take a sequence $\{\psi_n\}$ in $\mathcal{S}(I, X_{\theta+})$ such that $\psi_n(t) \rightarrow f(t)$ in $X_{\theta(t)}$ as $n \rightarrow \infty$ for a.e. $t \in I$. Since the norm function $t \rightarrow \|f(t)\|_{\theta(t)}$ is measurable, $E_0 = \{t \in I : \|f(t)\|_{\theta(t)} = 0\}$ is a measurable subset of I . Evidently, for every $n \in \mathbb{N}$, $\psi_n = \psi_n \chi_{I \setminus E_0}$ is also a $X_{\theta+}$ -valued simple function, and $\lim_{n \rightarrow \infty} \|\psi_n(t) - f(t)\|_{\theta(t)} = 0$ for a.e. $t \in I$. Let $J_n = \{t \in I : \|\bar{\psi}_n(t)\|_{\theta(t)} \leq 2\|f(t)\|_{\theta(t)}\}$ and $\varphi_n = \bar{\psi}_n \chi_{J_n}$, then we have $\{\varphi_n\} \subseteq \mathcal{S}(I, X_{\theta+})$, $\|\varphi_n\|_{\theta+} \leq 2\|f(t)\|_{\theta(t)}$ for a.e. $t \in I$ and

$$\begin{aligned} \{t \in I : \|\bar{\psi}_n(t) - f(t)\|_{\theta(t)} \rightarrow 0\} &= \bigcap_{\varepsilon > 0} \bigcup_{N \geq 1} \bigcap_{n \geq N} \{t \in J_n : \|\bar{\psi}_n(t) - f(t)\|_{\theta(t)} \leq \varepsilon\} \\ &= \bigcap_{\varepsilon > 0} \bigcup_{N \geq 1} \bigcap_{n \geq N} \{t \in I : \|\varphi_n(t) - f(t)\|_{\theta(t)} \leq \varepsilon\} \\ &= \{t \in I : \|\varphi_n(t) - f(t)\|_{\theta(t)} \rightarrow 0\}. \end{aligned}$$

Consequently,

$$m(\{t \in I : \varphi_n(t) \rightarrow f(t) \text{ in } X_{\theta(t)}\}) = m(\{t \in I : \bar{\psi}_n(t) \rightarrow f(t) \text{ in } X_{\theta(t)}\}),$$

which means $\varphi_n(t) \rightarrow f(t)$ in $X_{\theta(t)}$ as $n \rightarrow \infty$ for a.e. $t \in I$. Notice that, for all $\lambda > 0$, $\int_I \|\lambda f(t)\|_{\theta(t)}^{p(t)} dt < \infty$, thus using Lebesgue's convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \rho_{p(\cdot)}(\lambda \|\varphi_n - f\|_{\theta(\cdot)}) = \lim_{n \rightarrow \infty} \int_I (\lambda \|\varphi_n(t) - f(t)\|_{\theta(t)})^{p(t)} dt = 0,$$

which results in $\varphi_n \rightarrow f$ in $\|\cdot\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}$ by the arbitrariness of λ . \square

Theorem 3.8. *Under the norm-continuity and dense assumptions upon $\{X_\alpha\}$, $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ is a Banach space.*

Proof. Taken a Cauchy sequence $\{f_n\}$ in $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, by virtue of [Theorem 3.3](#), there is a function $f \in L^{p(\cdot)}(I, X_{\theta(\cdot)})$ for which $\|f_n - f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \rightarrow 0$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, on account of [Proposition 3.7](#), there is a $\varphi_n \in \mathcal{S}(I, X_{\theta+})$ such that $\|\varphi_n - f_n\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} < 1/n$, which in turn yields $\|\varphi_n - f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \rightarrow 0$ as $n \rightarrow \infty$. Hence there is a subsequence, say $\{\varphi_n\}$ itself, satisfying $\varphi_n(t) \rightarrow f(t)$ in $X_{\theta(t)}$ as $n \rightarrow \infty$ for a.e. $t \in I$. Therefore $f \in L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ and the proof is completed. \square

Remark 3.9. By reviewing the above proof, one can easily find that, under present situations, condition $L^0(I, X_{\theta(\cdot)}) = L_+^0(I, X_{\theta(\cdot)})$ is sufficient for $L^{p(\cdot)}(I, X_{\theta(\cdot)}) = L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$.

Theorem 3.10. Besides the norm-continuity and dense conditions, assume that X_{θ^+} is separable, then $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ is also a separable space.

This theorem is a straight derivation of Proposition 3.7.

Denote by $L^{p(\cdot)}(I, X_{\theta(\cdot)})^*$ and $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*$ the dual space of $L^{p(\cdot)}(I, X_{\theta(\cdot)})$ and $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ respectively. Here and later, the exponent $p(\cdot)$ is required to belong to $\mathcal{P}_0(I)$. Define four functions on $[0, \infty)$ as follows:

$$\sigma_{p(\cdot)}^+(s) = \max\{s^{p^+}, s^{p^-}\}, \quad \sigma_{p(\cdot)}^-(s) = \min\{s^{p^+}, s^{p^-}\}$$

and

$$\varsigma_{p(\cdot)}^+(s) = \max\{s^{1/p^+}, s^{1/p^-}\}, \quad \varsigma_{p(\cdot)}^-(s) = \min\{s^{1/p^+}, s^{1/p^-}\}.$$

There are some interesting properties of these functions listed below.

Lemma 3.11.

(1) All of $\sigma_{p(\cdot)}^\pm, \varsigma_{p(\cdot)}^\pm$ are increasing strictly, and their inverse functions satisfy

$$(\sigma_{p(\cdot)}^+)^{-1} = \varsigma_{p(\cdot)}^-, \quad (\sigma_{p(\cdot)}^-)^{-1} = \varsigma_{p(\cdot)}^+,$$

(2) if $s \leq C\varsigma_{p(\cdot)}^+(s)$ for some $C > 0$, then $s \leq \sigma_{p'(\cdot)}^+(C)$, where $p'(\cdot)$ is the point-wise conjugate exponent of $p(\cdot)$, i.e. $1/p(t) + 1/p'(t) = 1$ for all $t \in I$, and

(3) for all $s > 0$,

$$\sigma_{p(\cdot)}^+(1/\varsigma_{p(\cdot)}^+(s)) = \sigma_{p(\cdot)}^-(1/\varsigma_{p(\cdot)}^-(s)) = \varsigma_{p(\cdot)}^+(1/\sigma_{p(\cdot)}^+(s)) = \varsigma_{p(\cdot)}^-(1/\sigma_{p(\cdot)}^-(s)) = 1/s.$$

Proof. The first assertion can be checked by direct calculations, and the last one has been verified in [34]. As for the second assertion, notice that, if $C \leq 1$, then from $s \leq C\varsigma_{p(\cdot)}^+(s)$, we obtain $s \leq 1$ and consequently $s \leq Cs^{1/p^+}$. Solving it we get $s \leq C^{(p')^-}$. If $C > 1$, then from $s \leq C\varsigma_{p(\cdot)}^+(s)$, we can derive that,

$$s \leq \begin{cases} C^{(p')^-}, & \text{if } s \leq 1 \\ C^{(p')^+}, & \text{if } s > 1 \end{cases} \leq C^{(p')^+}.$$

The above two inequalities jointly produce $s \leq \sigma_{p'(\cdot)}^+(C)$. \square

Similar to scalar functions with variable exponents, for all $f \in L^{p(\cdot)}(I, X_{\theta(\cdot)})$, we have

$$\sigma_{p(\cdot)}^-(\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}) \leq \rho_{p(\cdot)}(\|(f(\cdot))\|_{\theta(\cdot)}) \leq \sigma_{p(\cdot)}^+(\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}),$$

and

$$\varsigma_{p(\cdot)}^-(\rho_{p(\cdot)}(\|(f(\cdot))\|_{\theta(\cdot)})) \leq \|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq \varsigma_{p(\cdot)}^+(\rho_{p(\cdot)}(\|(f(\cdot))\|_{\theta(\cdot)})).$$

Theorem 3.12. Suppose that $\{X_\alpha : \alpha \in \mathcal{A}\}$ is a dense and regular BSN, whose DSN $\{X_\alpha^* : \alpha \in \mathcal{A}\}$ is regular. Suppose also for every $\alpha \in \mathcal{A}$, X_α^* is norm attainable, and has the Radon–Nikodym’s property w.r.t. the Lebesgue measure space (I, \mathcal{L}, m) . Then,

$$L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^* = L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*)$$

in the sense of isomorphism.

Proof. Firstly for each $g \in L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*)$, define Ψ_g by

$$\langle \langle \Psi_g, f \rangle \rangle_{\theta(\cdot)} = \int_I \langle g(t), f(t) \rangle_{\theta(t)} dt, \quad \forall f \in L_+^{p(\cdot)}(I, X_{\theta(\cdot)}).$$

It is easy to check that, $\Psi_g \in L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*$ and $\|\Psi_g\|_{L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*} \leq 2\|g\|_{L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*)}$ by Hölder’s inequality of variable exponent type.

Conversely, take any $\xi \in L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*$. If $\xi = 0$, then we take $g = 0$ and there is nothing to do. If $\xi \neq 0$, assume firstly $\|\xi\|_{L_+^{p(\cdot)}(I, X_{\theta(\cdot)})^*} = 1$. Notice that the restriction of ξ to $L^{p(\cdot)}(I, X_{\theta+})$, denoted by $\xi|_{\theta+}$, is also a continuous functional, i.e. $\xi|_{\theta+} \in L^{p(\cdot)}(I, X_{\theta+})^*$. Thus according to Proposition A.7, there is a unique function $g \in L^{p'(\cdot)}(I, X_{\theta+}^*)$ for which,

$$\langle \langle \xi|_{\theta+}, f \rangle \rangle_{\theta+} = \int_I \langle g(t), f(t) \rangle_{\theta+} dt$$

holds for all $f \in L^{p(\cdot)}(I, X_{\theta+})$. Moreover, for every $J \in \Lambda(I)$, the other restriction $\xi|_{J, \theta+}$ lies in $L^{p(\cdot)}(J, X_{\theta+}^*)$. Likewise, there is another unique function $g_J \in L^{p'(\cdot)}(J, X_{\theta+}^*)$ such that for all $f \in L^{p(\cdot)}(J, X_{\theta+}^*)$,

$$\langle \langle \xi|_{J, \theta+}, f \rangle \rangle_{\theta+} = \int_J \langle g_J(t), f(t) \rangle_{\theta+} dt$$

holds. For each $f \in L^{p(\cdot)}(J, X_{\theta+})$, take $\tilde{f} \in L^{p(\cdot)}(I, X_{\theta+})$ as the zero extension of f out of J , we have

$$\begin{aligned} \int_J \langle g(t), f(t) \rangle_{\theta+} dt &= \int_I \langle g(t), \tilde{f}(t) \rangle_{\theta+} dt \\ &= \langle \langle \xi|_{I, \theta+}, \tilde{f} \rangle \rangle_{\theta+} = \langle \langle \xi|_{J, \theta+}, f \rangle \rangle_{\theta+} = \int_J \langle g_J(t), f(t) \rangle_{\theta+} dt. \end{aligned}$$

Therefore by the uniqueness of representation of $\xi|_{J, \theta+}$ and the density of $L^{p(\cdot)}(J, X_{\theta+})$ in $L^{p(\cdot)}(J, X_{\theta+}^*)$ (see Proposition A.2), we conclude that $g(t) = g_J(t)$ for a.e. $t \in J$. Hence $g \in L_-^0(I, X_{\theta(\cdot)}^*)$ by the arbitrariness of $J \in \Lambda(I)$.

For each $n \in \mathbb{N}$ and $k = 1, 2, \dots$, take $t_{n,k}$, $J_{n,k}$ as in the preceding arguments and let $\theta_{n,k}^+ = \theta_{J_{n,k}}^+$, $\theta_n(t) = \theta_{J_{n,k}}^+$ for $t \in J_{n,k}$. Since $X_{\theta_{n,k}}^+$ is norm attainable, there is a map $F : X_{\theta_{n,k}}^+ \rightarrow 2^{X_{\theta_{n,k}}^+}$ with $\mathcal{D}(F) = X_{\theta_{n,k}}^+$ such that for all $h \in X_{\theta_{n,k}}^+$ and all $f \in F(h)$,

$$\langle h, f \rangle_{\theta_{n,k}^+} = \|h\|_{\theta_{n,k}^+, *}^2 = \|f\|_{\theta_{n,k}^+}^2.$$

Like the usual dual map, F is bounded, monotone, coercive and hemicontinuous. So it has a closed graph in the product space $X_{\theta_{n,k}}^* \times X_{\theta_{n,k}}^+$, and for each $h \in X_{\theta_{n,k}}^*$, its image $F(h)$ is closed in $X_{\theta_{n,k}}^+$ (refer to [14], Ch. 2). Thus for the restriction $g|_{J_{n,k}}$ of g , the multiple function $t \mapsto F(g|_{J_{n,k}}(t))$ has a measurable selection $w_{n,k} : J_{n,k} \rightarrow X_{\theta_{n,k}}^+$ (cf. [13], Ch. 8). Let

$$u_{n,k}(t) = \begin{cases} \|g(t)\|_{\theta_{n,k}^+}^{p'(t)-2} w_{n,k}(t), & \text{if } \|g(t)\|_{\theta_{n,k}^+} \neq 0, \\ 0, & \text{if } \|g(t)\|_{\theta_{n,k}^+} = 0 \end{cases}$$

and $u_n(t) = u_{n,k}(t)$ for $t \in J_{n,k}$, $k = 1, 2, \dots$. Then $u_n \in L^{p(\cdot)}(I, X_{\theta_n(\cdot)})$ satisfies

$$\rho_{p(\cdot)}(\|u_n(\cdot)\|_{\theta_n(\cdot)}) = \rho_{p'(\cdot)}(\|g(\cdot)\|_{\theta_n(\cdot),*}).$$

Consequently,

$$\begin{aligned} \rho_{p'(\cdot)}(\|g(\cdot)\|_{\theta_n(\cdot),*}) &= \int_I \langle g(t), u_n(t) \rangle_{\theta_n(t)} dt = \langle \langle \xi |_{\theta_n(\cdot)}, u_n \rangle \rangle_{\theta_n(\cdot)} = \langle \langle \xi, u_n \rangle \rangle_{\theta(\cdot)} \\ &\leq \|u_n\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})} \leq C \|u_n\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)})} \\ &\leq C \zeta_{p(\cdot)}^+(\rho_{p'(\cdot)}(\|g(\cdot)\|_{\theta_n(\cdot),*})), \end{aligned}$$

which implies by Lemma 3.11 (2) that

$$\rho_{p'(\cdot)}(\|g(\cdot)\|_{\theta_n(\cdot),*}) \leq \sigma_{p'(\cdot)}^+(C).$$

This inequality, together with

$$\rho_{p'(\cdot)}(\|g(\cdot)\|_{\theta_n(\cdot),*}) \geq \sigma_{p'(\cdot)}^-(\|g\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)}^*)}),$$

yields

$$\sigma_{p'(\cdot)}^-(\|g\|_{L^{p(\cdot)}(I, X_{\theta_n(\cdot)}^*)}) \leq \sigma_{p'(\cdot)}^+(C).$$

Since the right hand of the above inequality is independent of n , using Theorem 3.6, we conclude that $g \in L^{p(\cdot)}(I, X_{\theta(\cdot)}^*)$ and

$$\|g\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)}^*)} \leq C \quad (3.1)$$

for another constant $C > 0$.

Now, for every $f \in L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$, take a sequence $\{\varphi_k\} \subseteq \mathcal{S}(I, X_{\theta+})$ such that $\varphi_k \rightarrow f$ in $L_+^{p(\cdot)}(I, X_{\theta(\cdot)})$ as $k \rightarrow \infty$. For each $k \in \mathbb{N}$,

$$\langle \langle \xi, \varphi_k \rangle \rangle_{\theta(\cdot)} = \langle \langle \xi |_{\theta+}, \varphi_k \rangle \rangle_{\theta+} = \int_I \langle g(t), \varphi_k(t) \rangle_{\theta(t)} dt = \int_I \langle g(t), \varphi_k(t) \rangle_{\theta(t)} dt.$$

Let $k \rightarrow \infty$, we obtain the desired equality

$$\langle \langle \xi, f \rangle \rangle_{\theta(\cdot)} = \int_I \langle g(t), f(t) \rangle_{\theta(t)} dt.$$

This means that $\xi = \Psi_g$, and the theorem is finally proved by means of scaling arguments. \square

Remark 3.13. By reviewing the above proof, we get an equivalent relation between g and $\xi = \Psi_g$, that is

$$\frac{1}{C} \|g\|_{L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*)} \leq \|\xi\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})^*} \leq 2 \|g\|_{L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*)},$$

where the constant $C > 0$ is that appearing in (3.1).

Corollary 3.14. Suppose that in addition to Theorem 3.12, $L^0(I, X_{\theta(\cdot)})$ equals to $L_+^0(I, X_{\theta(\cdot)})$, then

$$L^{p(\cdot)}(I, X_{\theta(\cdot)})^* = L^{p'(\cdot)}(I, X_{\theta(\cdot)}^*)$$

in the sense of isomorphism.

Recall that, for a reflexive space X , both X and its dual space X^* are norm attainable, and own the Radon–Nikodym’s property w.r.t. all finite measure spaces. These facts, along with Theorem 3.12 and 3.14, leads to the following corollary.

Corollary 3.15. Under the following conditions:

- (1) $p \in \mathcal{P}_0(I)$,
- (2) X_α is reflexive for all $\alpha \in \mathcal{A}$,
- (3) both $\{X_\alpha : \alpha \in \mathcal{A}\}$ and $\{X_\alpha^* : \alpha \in \mathcal{A}\}$ are dense and regular \mathcal{BSN} s, and
- (4) $L^0(I, X_{\theta(\cdot)}) = L_+^0(I, X_{\theta(\cdot)})$, $L^0(I, X_{\theta(\cdot)}^*) = L_+^0(I, X_{\theta(\cdot)}^*)$,

$L^{p(\cdot)}(I, X_{\theta(\cdot)})$ is a reflexive space.

Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be a uniformly bounded and norm-continuous \mathcal{BSN} . In many situations, we will encounter the following space

$$C^-(I, X_{\theta(\cdot)}) = \{f \in L^0(I, X_{\theta(\cdot)}) : f|_J \in C(J, X_{\theta_J^-}) \text{ for all } J \in \Lambda(I), \\ \text{and } \sup_{t \in I} \|f(t)\|_{\theta(t)} < \infty\}.$$

If we endow it with the norm

$$\|f\|_{C^-(I, X_{\theta(\cdot)})} := \sup_{t \in I} \|f(t)\|_{\theta(t)},$$

then $C^-(I, X_{\theta(\cdot)})$ becomes a Banach space. Furthermore, we have

Theorem 3.16. Under the regularity assumption of $\{X_\alpha\}$, $C^-(I, X_{\theta(\cdot)})$ is exactly equal to the space

$$\{f \in C(I, X) : f|_J \in C(J, X_{\theta_J^-}) \text{ for all } J \in \Lambda(I) \text{ and } \sup_{J \in \Lambda(I)} \|f|_J\|_{C(J, X_{\theta_J^-})} < \infty\},$$

while its norm is equivalent to $\sup_{J \in \Lambda(I)} \|f|_J\|_{C(J, X_{\theta_J^-})}$.

All the properties listed above can be verified straightly, and here we omit the whole proofs.

Remark 3.17. Unlike the range-fixed type, in $C^-(I, X_{\theta(\cdot)})$, continuity of the norm function $t \rightarrow \|f(t)\|_{\theta(t)}$ could not be expected.

Next proposition reveals the relation between $C^-(I; X_{\theta(\cdot)})$ and $L^{p(\cdot)}(I; X_{\theta(\cdot)})$. For its proof, please refer to [34].

Proposition 3.18. *Suppose that $I = [0, T]$ and $p \in \mathcal{P}_b(I)$, then $C^-(I; X_{\theta(\cdot)})$ can be embedded continuously into $L^{p(\cdot)}(I; X_{\theta(\cdot)})$, and*

$$\|f\|_{L^{p(\cdot)}(I; X_{\theta(\cdot)})} \leq \varsigma_{p(\cdot)}^+(T) \|f\|_{C^-(I; X_{\theta(\cdot)})}, \quad \forall f \in C^-(I; X_{\theta(\cdot)}).$$

Let Ω be a bounded domain of \mathbb{R}^N and $Q = I \times \Omega$. It is well known that, for each $u \in L^0(Q)$, its $L^0(\Omega)$ -realization Pu defined by $Pu(t)(x) = u(x, t)$ for a.e. $x \in \Omega$ and a.e. $t \in I$ is a $L^0(\Omega)$ -valued measurable function. Conversely, for each $u \in L^0(I, L^0(\Omega))$, its scalar realization \tilde{u} defined by $\tilde{u}(x, t) = u(t)(x)$ for a.e. $(x, t) \in Q$ is also measurable. Moreover, P is a linear isometrical isomorphism from $L^q(Q)$ onto $L^q(I, L^q(\Omega))$ for all $1 \leq q \leq \infty$ ($I = [0, T]$ in case that $1 \leq q < \infty$) with the inverse $P^{-1}u = \tilde{u}$ for all $u \in L^q(I, L^q(\Omega))$. If $q \in \mathcal{P}_b(\Omega)$ is a variable exponent with $I = [0, T]$, then $P : L^{q(\cdot)}(Q) \rightarrow L^{q^-}(I, L^{q(\cdot)}(\Omega))$ is also a continuous operator. For the detailed discussions, see for example the appendix of [5].

Suppose that $p \in \mathcal{P}_b(Q)$ is a Caratheodory type exponent, i.e.

- (1) $p(t, \cdot)$ is measurable for every $t \in I$,
- (2) $p(\cdot, x)$ is continuous for a.e. $x \in \Omega$, and
- (3) there is a nonnegative and nondecreasing function ϱ , continuous at point 0 with $\varrho(0) = 0$ such that

$$|p(t, x) - p(s, x)| \leq \varrho(|t - s|) \quad (3.2)$$

for all $t, s \in I$ and almost all $x \in \Omega$.

Let $\underline{p} = p_Q^-$, $\bar{p} = p_Q^+$, and \mathcal{A}_b be that introduced in Example 2.3. It is easy to check that, under present situations, $\theta = Pp : I \rightarrow \mathcal{A}_b$ is a \mathcal{DC} exponent with $X_{\theta(t)} = L^{p(x,t)}(\Omega)$ for all $t \in I$, and two other exponents $p_t^- = \text{ess inf}_{x \in \Omega} p(x, t)$, $p_t^+ = \text{ess sup}_{x \in \Omega} p(x, t)$ both lie in $\mathcal{P}_b(I)$. Hence by invoking Example 2.3 and Theorem 3.3, we conclude that, $L^{p_t^\pm}(I, X_{\theta(\cdot)}) =: L^{p_t^\pm}(I, L^{p(x,t)}(\Omega))$ are both Banach spaces.

Remark 3.19. Take $p_\Omega(t) = |\Omega|^{-1} \int_\Omega p(x, t)$ the average of $p(x, t)$ on Ω , then we obtain another exponent $p_\Omega \in \mathcal{P}_b(I)$, for which the following imbedding

$$L^{p_t^+}(I, L^{p(x,t)}(\Omega)) \hookrightarrow L^{p_\Omega(t)}(I, L^{p(x,t)}(\Omega)) \hookrightarrow L^{p_t^-}(I, L^{p(x,t)}(\Omega))$$

holds on $I = [0, T]$.

For the sake of simplicity, in the remaining parts of this section, we always assume $I = [0, T]$, and use p_t representing p_t^+ , p_t^- and $p_\Omega(t)$, without any other comments.

Proposition 3.20. *Under all the assumptions as above,*

$$L^{p_t}(I; L^{p(x,t)}(\Omega)) = L_+^{p_t}(I; L^{p(x,t)}(\Omega)).$$

Consequently $L^{p_t^\pm}(I; L^{p(x,t)}(\Omega))$ is separable. Moreover, if in addition $\underline{p} > 1$, then $L^{p_t}(I; L^{p(x,t)}(\Omega))$ is reflexive, together with

$$L^{p_t}(I; L^{p(x,t)}(\Omega))^* = L^{(p_t)'}(I; L^{p'(x,t)}(\Omega))$$

in the sense of isomorphism.

Proof. From Theorem 3.10, 3.12 and Corollary 3.14, 3.15, it suffices to prove the first assertion. For this purpose, notice that $\{L^q(\Omega) : q \in \mathcal{A}_b\}$ is a dense \mathcal{BSN} , so by Proposition 3.7, we need only show the density of $S(I, L^{\bar{p}}(\Omega))$ in $L^{p_t}(I; L^{p(x,t)}(\Omega))$. Take any function $u \in L^{p_t}(I; L^{p(x,t)}(\Omega))$, then there exist a sequence of simple functions $\{h_k\}$ such that $h_k(t, x) \rightarrow u(t, x)$ as $k \rightarrow \infty$ and $|h_k(t, x)| \leq |u(t, x)|$, $k = 1, 2, \dots$, for a.e. $(t, x) \in Q$. It is easy to see that, complement of the set

$$E = \{t \in I : |h_k(t, x)| \leq |u(t, x)| \text{ for a.e. } x \in \Omega\}$$

has a zero measure in I . Thus by the solidity of $L^{p(\cdot, t)}(\Omega)$ and dominated convergence theorem (cf. [19], §2.3), it follows that $h_k(t, \cdot) \rightarrow u(t, \cdot)$ in $L^{p(\cdot, t)}(\Omega)$ as $k \rightarrow \infty$ and $\|h_k(t, \cdot)\|_{L^{p(\cdot, t)}(\Omega)} \leq \|u(t, \cdot)\|_{L^{p(\cdot, t)}(\Omega)}$ for a.e. $t \in I$. Now using dominated convergence theorem again, we obtain

$$\lim_{k \rightarrow \infty} \int_I \|h_k(t, \cdot) - u(t, \cdot)\|_{L^{p(\cdot, t)}(\Omega)}^{p_t} dt = 0,$$

and consequently $h_k \rightarrow u$ in $L^{p_t}(I; L^{p(x,t)}(\Omega))$. Since every simple function h_k can be viewed as an element of $S(I, L^{\bar{p}}(\Omega))$, we conclude that $S(I, L^{\bar{p}}(\Omega))$ is dense in $L^{p_t}(I; L^{p(x,t)}(\Omega))$, and the proposition has been proved. \square

Lemma 3.21. Suppose that the boundary of Ω is of C^1 type, $p(\cdot, \cdot)$ is a continuous exponent satisfying (3.2), and for all $t \in I$, $p(t, \cdot)$ lies in \mathcal{A}_ω for some ω as in (2.2). Then

$$L^0(I; W^{1,p(x,t)}(\Omega)) = L_+^0(I; W^{1,p(x,t)}(\Omega)).$$

Proof. Since Ω is a bounded C^1 domain, it is of (ε, ∞) type for some $\varepsilon > 0$ (cf. [19], §8.5 or [30]). Then for each $p \in \mathcal{P}_{\log}^\omega(\Omega)$, every function $u \in W^{1,p(\cdot)}(\Omega)$ has an extension $\tilde{u} \in W^{1,\tilde{p}(\cdot)}(\mathbb{R}^N)$ with the estimate

$$\|\tilde{u}\|_{W^{1,\tilde{p}(\cdot)}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p(\cdot)}(\Omega)} \quad (3.3)$$

holding. Here \tilde{p} is the uniformly log-continuous extension of p outside Ω (see [19], §4.1 or [20] for references), and the constant $C > 0$ depends on n , Ω , p^\pm and C_ω arising in (2.3).

For each $\delta > 0$, denote by j_δ the normalized and bell-shaped mollifier and consider the convolution $\tilde{u} * j_\delta$. It is well known that $\tilde{u} * j_\delta \in C^\infty(\mathbb{R}^N)$ and $\partial_{x_i}(\tilde{u} * j_\delta) = (\partial_{x_i} \tilde{u}) * j_\delta$, $i = 1, 2, \dots, N$. Furthermore, in view of [19], §4.6, we have

$$\lim_{\delta \rightarrow 0} \tilde{u} * j_\delta = \tilde{u} \text{ in } L^{\tilde{p}(\cdot)}(\mathbb{R}^N), \quad \lim_{\delta \rightarrow 0} (\nabla \tilde{u}) * j_\delta = \nabla \tilde{u} \text{ in } L^{\tilde{p}(\cdot)}(\mathbb{R}^N)^N, \quad (3.4)$$

and

$$\|\tilde{u} * j_\delta\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^N)} \leq C \|\tilde{u}\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^N)}, \quad \|(\nabla \tilde{u}) * j_\delta\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^N)} \leq C \|\nabla \tilde{u}\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^N)}, \quad (3.5)$$

where the constant $C > 0$ depends on N and C_ω . Combining (3.3), (3.4) and (3.5), we obtain

$$\lim_{\delta \rightarrow 0} \tilde{u} * j_\delta = u \text{ in } W^{1,p(\cdot)}(\Omega)$$

and

$$\|\tilde{u} * j_\delta\|_{W^{1,p(\cdot)}(\Omega)} \leq C \|u\|_{W^{1,p(\cdot)}(\Omega)} \quad (3.6)$$

for some $C > 0$ independent of u .

Taking any $u \in L^0(I; W^{1,p(x,t)}(\Omega))$, let $u_\delta(\cdot, t)$ be the restriction of the point-wise modified function $\tilde{u}(\cdot, t) * j_\delta$ on Ω . From the above arguments, it follows that, $u_\delta(\cdot, t)$ is valued in $C^\infty(\overline{\Omega})$, but strongly measurable in $W^{1,p^-}(\Omega)$, and

$$\lim_{\delta \rightarrow 0} u_\delta(\cdot, t) = u(\cdot, t) \text{ in } W^{1,p(\cdot,t)}(\Omega)$$

for a.e. $t \in I$. Since $C^\infty(\overline{\Omega}) \subset W^{1,p^+}(\Omega)$, while the latter is separable, we conclude that $u_\delta(\cdot, t)$ is strongly measurable in $W^{1,p^+}(\Omega)$, i.e. $u \in L^0(I, W^{1,p^+}(\Omega))$. Thus $u \in L^0_+(I; W^{1,p(x,t)}(\Omega))$ and the equality $L^0(I; W^{1,p(x,t)}(\Omega)) = L^0_+(I; W^{1,p(x,t)}(\Omega))$ holds. \square

Lemma 3.22. *Under the same assumptions upon the doubly variable exponent p ,*

$$L^0(I; W^{-1,p'(x,t)}(\Omega)) = L^0_+(I; W^{-1,p'(x,t)}(\Omega)) \quad (3.7)$$

and

$$L^0(I; W^{1,p(x,t)}(\Omega)^*) = L^0_+(I; W^{1,p(x,t)}(\Omega)^*). \quad (3.8)$$

Proof. Taking any $\xi \in L^0(I; W^{-1,p'(x,t)}(\Omega))$, by virtue of Example 2.5, there is a function $F = (f_0, f_1, \dots, f_n) \in L^0(I, L^{(p_+)' }(\Omega))^{N+1}$ such that $F(t, \cdot) \in L^{p'(\cdot,t)}(\Omega)^N$ and

$$\xi(t) = f_0(t, \cdot) - \sum_{i=1}^N \partial_{x_i} f_i(t, \cdot) \quad \text{in } \mathcal{D}'(\Omega)$$

for a.e. $t \in I$.

Denote by $\tilde{F}(\cdot, t)$ the zero extension of $F(\cdot, t)$ to \mathbb{R}^N . Similar to the proof of the above lemma, for the modified functions $\tilde{F}(\cdot, t) * j_\delta$, we have that

$$\begin{aligned} \|\tilde{F}(\cdot, t) * j_\delta\|_{L^{p'(\cdot,t)}(\Omega)} &\leq C \|F(\cdot, t)\|_{L^{p'(\cdot,t)}(\Omega)}, \\ \lim_{\delta \rightarrow 0} \|\tilde{F}(\cdot, t) * j_\delta - F(\cdot, t)\|_{L^{p'(\cdot,t)}(\Omega)} &= 0 \end{aligned} \quad (3.9)$$

and

$$\xi_\delta(t) = \tilde{f}_0(t, \cdot) * j_\delta - \sum_{i=1}^n \partial_{x_i} \tilde{f}_i(t, \cdot) * j_\delta \in W^{-1,(p_-)'}(\Omega)$$

for a.e. $t \in I$. It is easy to check that, strong measurability of the function $t \mapsto \tilde{F}(\cdot, t) * j_\delta$ in $L^{(p_+)' }(\Omega)$ leads to the strong measurability of ξ_δ in $W^{-1,(p_+)' }(\Omega)$, hence the weak measurability in $W^{-1,(p_-)'}(\Omega)$. Take the separability of $W^{-1,(p_-)'}(\Omega)$ into account, we drive the strong measurability of ξ_δ in $W^{-1,(p_-)'}(\Omega)$. This fact, combined with (3.9), leads to the first equality (3.7). And the second equality (3.8) can be dealt with in the same way. \square

Combining with the above two lemmas, we can derive that

Proposition 3.23. *In the same situations,*

$$L^{p_t}(I; W^{1,p(x,t)}(\Omega)) = L^0_+(I; W^{1,p(x,t)}(\Omega))$$

and

$$L^{(p_t)'}(I; W^{1,p(x,t)}(\Omega)^*) = L_+^{(p_t)'}(I; W^{1,p(x,t)}(\Omega)^*).$$

Consequently, $L^{p_t}(I; W^{1,p(x,t)}(\Omega))$ is a separable and reflexive space, and

$$L^{p_t}(I; W^{1,p(x,t)}(\Omega))^* = L^{(p_t)'}(I; W^{1,p(x,t)}(\Omega)^*)$$

in the sense of isomorphism.

Corollary 3.24. Under the same assumptions upon p ,

$$L^{p_t}(I; W_0^{1,p(x,t)}(\Omega)) = L_+^{p_t}(I; W_0^{1,p(x,t)}(\Omega)). \quad (3.10)$$

Moreover, $L^{p_t}(I; W_0^{1,p(x,t)}(\Omega))$ is both separable and reflexive together with

$$L^{p_t}(I; W_0^{1,p(x,t)}(\Omega))^* = L^{(p_t)'}(I; W^{-1,p'(x,t)}(\Omega))$$

in the sense of isomorphism.

Proof. Firstly, as a closed subspace of $L_+^{p_t}(I; W^{1,p(x,t)}(\Omega))$, $L_+^{p_t}(I; W_0^{1,p(x,t)}(\Omega))$ is also reflexive, hence its double dual space is itself exactly. Notice that

$$\begin{aligned} L_+^{p_t}(I; W_0^{1,p(x,t)}(\Omega))^* &= L^{(p_t)'}(I; W^{-1,p'(x,t)}(\Omega)), \\ L^{(p_t)'}(I; W^{-1,p'(x,t)}(\Omega)) &= L_+^{(p_t)'}(I; W^{-1,p'(x,t)}(\Omega)) \end{aligned}$$

which is derived by (3.8), and

$$L_+^{(p_t)'}(I; W^{-1,p'(x,t)}(\Omega))^* = L^{p_t}(I; W_0^{1,p(x,t)}(\Omega)),$$

equality (3.10) follows. And other results come naturally. \square

4. Modular-modular space

Suppose that \mathcal{W} is a topological linear space, in which there is also defined an order \prec . Consider the set $V = \{u \in \mathcal{W} : 0 \prec u\}$. We say V is a closed cone, we mean that V is closed, and for all $u, v \in V$, $\lambda \geq 0$, inclusions $\lambda u \in V$ and $u + v \in V$ always hold. Under this situation, \mathcal{W} is called an ordered topological linear space.

Let \mathcal{X} be a topological linear space, and $M : \mathcal{X} \rightarrow V$ be a continuous operator. We say M is a V -modular, if the following hypotheses are satisfied:

- (1) $M(cu) = |c|M(u)$ for all $c \in \mathbb{R}$ and all $u \in \mathcal{X}$;
- (2) $M(u + v) \prec M(u) + M(v)$ for all $u, v \in \mathcal{X}$; and
- (3) $M(0) = 0$ if and only if $u = 0$.

Like the B -valued one, a function $f : I \rightarrow \mathcal{X}$ is called (strongly) measurable, if there exists a sequence of simple functions $\{s_k\}$ such that $s_n(t) \rightarrow f(t)$ in \mathcal{X} a.e. on I . All the measurable \mathcal{X} -valued functions constitute a linear space, denoted by $L^0(I, \mathcal{X})$, according to the linear operations. The cone V is called proper w.r.t. \mathcal{X} , or in other words \mathcal{X} -proper, if there exist a constant $K > 0$ such that for every $f \in L^0(I, \mathcal{X})$, there is a sequence of simple functions $\{s_n\}$ verifying $s_n(t) \rightarrow f(t)$ and $M(s_n(t)) \prec KM(f(t))$ a.e. on I for all $n \in \mathbb{N}$.

Example 4.1. Let Ω be a measurable subset of \mathbb{R}^N with finite measure, and let $\mathcal{W} = \mathcal{X} = L^0(\Omega)$. It is easy to check that, according to the linear operation, the natural order: $f(x) \prec g(x)$ provided $f(x) \leq g(x)$ a.e. on Ω , and the distance

$$d(f, g) = \int_{\Omega} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx,$$

$L^0(\Omega)$ becomes an ordered linear metric space. Let $(Mf)(x) = |f(x)|$, then the positive cone

$$V = \{f \in L^0(\Omega) : f(x) \geq 0 \text{ for a.e. } x \in \Omega\}$$

is proper with the constant $K = 1$. As a matter of fact, for each $u \in L^0(I, L^0(\Omega))$, its scalar realization $\hat{u}(t, x) = u(t)(x)$ can be regarded as a measurable function on $I \times \Omega$. Thus there exists $\{s_n\} \subseteq S(I \times \Omega)$ satisfying $|s_n(t, x)| \leq |\hat{u}(t, x)|$ and $\lim_{n \rightarrow \infty} s_n(t, x) = \hat{u}(t, x)$ a.e. on $I \times \Omega$. Note that for each $n \in \mathbb{N}$, the $L^0(\Omega)$ -realization $Ps_n(t) = s_n(t, \cdot)$ is also a $L^0(\Omega)$ -valued simple function, and $M(Ps_n) \prec M(u)$. Since the set $\{t \in I : s_n(t, x) \rightarrow \hat{u}(t, x) \text{ a.e. on } \Omega\}$ has a measure zero complement, we claim that $d(Ps_n(t), u(t)) \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in I$. Therefore V is a $L^0(\Omega)$ -proper cone.

Example 4.2. Let $\mathcal{W} = \mathbb{R}$, $V = [0, \infty)$. Then for every Banach space X , V is X -proper with the constant $K = 2$ if we take $M(u) = \|u\|$ the norm of $u \in X$. This fact was shown in many textbooks of real analysis, for example, see [7], §10.2.

Given a semimodular ρ defined on V , it is easy to check that, the multifunction $\rho \circ M$ is also a semimodular defined on \mathcal{X} . It is lower semi-continuous on the corresponding modular space, denoted by X_ρ . We say ρ satisfies the weak Δ_2 -condition, we mean that convergence $\rho(u_n) \rightarrow 0$ and $\rho(2u_n) \rightarrow 0$ are equivalent. It is well known that a semimodular fulfilling the weak Δ_2 -condition is a modular definitely. We say ρ is M -monotone, if

- $M(u) \prec M(v)$ implies $\rho(M(u)) \leq \rho(M(v))$.

ρ is called satisfying the dominated convergence property, provided

- $u_n \rightarrow u$ in \mathcal{X} ($n \rightarrow \infty$) and $M(u_n) \prec M(w)$ for all $n \in \mathbb{N}$ with $w \in X_\rho$ infers $\rho(M(u_n - u)) \rightarrow 0$ as $n \rightarrow \infty$.

Remark 4.1. Under all the conditions mentioned above, if $u_n \rightarrow u$ in \mathcal{X} and $M(u_n) \prec M(w)$ for some $w \in X_\rho$, then we have $u_n, u \in X_\rho$, and $u_n \rightarrow u$ in X_ρ as $n \rightarrow \infty$.

For the sake of applications, in the coming arguments, we lay a strong continuous assumption on ρ , that is

- $u_n \rightarrow u$ in \mathcal{X} as $n \rightarrow \infty$ with $M(u_n) \prec KM(u)$ for some $K > 0$ and all $n \in \mathbb{N}$ means $\lim_{n \rightarrow \infty} \rho(M(u_n)) = \rho(M(u))$.

Remark 4.2. In Example 4.1, the above continuous assumption can be weakened as

- If $u_n \rightarrow u$ in \mathcal{X} as $n \rightarrow \infty$, then $\rho(M(u)) \leq \liminf_{n \rightarrow \infty} \rho(M(u_n))$.

And in Example 4.2, it can also be changed by

- ρ is a continuous Φ function on $[0, \infty)$.

Proposition 4.3. Suppose that V is an \mathcal{X} -proper cone and $\rho : V \rightarrow [0, \infty]$ is a strongly continuous semimodular. Then for each $f \in L^0(I, \mathcal{X})$, the multifunction $t \mapsto \rho(M(f(t)))$ is measurable, hence the functional

$$\Phi_\rho(f) = \int_I \rho(M(f(t))) dt$$

is a semimodular on $L^0(I, \mathcal{X})$. Moreover, if in addition, ρ is M -monotone, and satisfies both the weak Δ_2 -condition and the dominated convergence property, then Φ_ρ is a modular, and the modular space $L^\rho(I, X_\rho)$, derived by Φ_ρ , is contained in $L^0(I, X_\rho)$.

Proof. Take any $f \in L^0(I, \mathcal{X})$, by the \mathcal{X} -proper assumption upon V , one can find a constant $K > 0$ and a sequence of simple functions $\{s_n\}$ such that $M(s_n(t)) \prec KM(f(t))$ and $s_n(t) \rightarrow f(t)$ in \mathcal{X} a.e. on I . Since for every $n \in \mathbb{N}$, the multifunction $\rho(M(s_n(\cdot)))$ is also a scalar simple function (perhaps has value ∞ on a measurable set with nonzero measure), it is certainly measurable. Thus using the strong continuity of ρ , we get $\rho(M(f(t))) = \lim_{n \rightarrow \infty} \rho(M(s_n(t)))$ for a.e. $t \in I$, hence the measurability of $\rho(M(f(\cdot)))$.

Suppose further $f \in L^\rho(I, X_\rho)$, i.e. $\Phi_\rho(f/\lambda) < \infty$ for some $\lambda > 0$. Then the integrand $\rho(M(f(t)/\lambda))$ is finite, or equivalently $f(t) \in X_\rho$ for a.e. $t \in I$. Using the facts $s_n(t) \rightarrow f(t)$ in \mathcal{X} , $M(s_n(t)) \prec M(Kf(t))$ and the dominated convergence assumption on ρ , we claim that $\rho(M(s_n(t) - f(t))) \rightarrow 0$ and consequently $s_n(t) \rightarrow f(t)$ in X_ρ a.e. on I , which leads to the inclusion $f \in L^0(I, X_\rho)$.

Other claims in this proposition can be examined directly. \square

Let \mathcal{A} be a \mathcal{BTL} , and $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be a family of modulars on V . Suppose that for each $\alpha \in \mathcal{A}$, modular ρ_α is M -monotone, strongly continuous and verifies the weak Δ_2 -condition. Suppose also the collection of multifunctions $\{\rho_\alpha \circ M : \alpha \in \mathcal{A}\}$ forms a \mathcal{CMN} . Then for a \mathcal{DC} exponent $\theta : I \rightarrow \mathcal{A}$, we can define

$$\Phi_{\theta(\cdot)}(f) = \int_I \rho_{\theta(t)}(M(f(t))) dt, \quad f \in L^0(I, \mathcal{X}).$$

Theorem 4.4. $\Phi_{\theta(\cdot)}$ is a modular defined on $L^0(I, \mathcal{X})$, and the corresponding modular space, denoted by $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, is a Banach space contained in $L^0(I, X_{\theta(\cdot)})$.

Proof. For each $n \in \mathbb{N}$, define the step function θ_n as in front of Remark 3.4, then by virtue of Proposition 4.3, we assert that for every $f \in L^0(I, \mathcal{X})$, the multifunction $\rho_{\theta_n(\cdot)}(M(f(\cdot)))$ is measurable. Let $n \rightarrow \infty$, using the facts $\theta_n(t) \prec \theta(t)$ and $\lim_{n \rightarrow \infty} \theta_n(t) = \theta(t)$ for all $t \in I$, we obtain

$$\lim_{n \rightarrow \infty} \rho_{\theta_n(t)}(M(f(t))) = \rho_{\theta(t)}(M(f(t)))$$

a.e. on I , which leads to the measurability of $\rho_{\theta(\cdot)}(M(f(\cdot)))$. Hence functional $\Phi_{\theta(\cdot)}$ is well defined. Furthermore, convexity and left continuity of $\Phi_{\theta(\cdot)}$ are both inherited from $\rho_{\theta(\cdot)} \circ M$. And $\Phi_{\theta(\cdot)}(f) = 0$ means that $\rho_{\theta(t)}(M(f(t))) = 0$, which in turn produces $f(t) = 0$ in \mathcal{X} for a.e. $t \in I$. Therefore $\Phi_{\theta(\cdot)}$ is a modular.

Suppose that $\Phi_{\theta(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$, then $\rho_{\theta(t)}(M(\lambda f(t))) < \infty$ and consequently $f(t) \in X_{\theta(t)}$ for a.e. $t \in I$. Furthermore, for each $J \in \Lambda(I)$, by inequality (2.1), we have

$$\int_J \rho_{\theta_J^-}(M(\lambda f(t))) dt \leq C_1 \int_J \rho_{\theta(t)}(M(\lambda f(t))) dt + C_2 |J| < \infty,$$

which means $f \in L^{\rho_{\theta_J^-}}(J, X_{\theta_J^-})$. By the arbitrariness of J and f , we claim that $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \subseteq L^0(I, X_{\theta(\cdot)})$.

For the completeness of $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$, assume that $\{f_n\}$ is a Cauchy sequence of $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ in norm, then for every $\lambda > 0$, limit

$$\lim_{n,k \rightarrow \infty} \int_I \rho_{\theta(t)}(M(\lambda^{-1}(f_n(t) - f_k(t)))) dt = 0 \quad (4.1)$$

holds. Thus the integrands $\rho_{\theta(t)}(M(\lambda^{-1}(f_n(t))))$, $n = 1, 2, \dots$ constitute a Cauchy sequence in measure, consequently by Riesz's theorem, there exists a subsequence, say $\{\rho_{\theta(t)}(M(\lambda^{-1}(f_{n_j}(t))))\}$, such that

$$\lim_{i,j \rightarrow \infty} \rho_{\theta(t)}(M(\lambda^{-1}(f_{n_i}(t) - f_{n_j}(t)))) = 0$$

a.e. on I , which in turn yields

$$\lim_{i,j \rightarrow \infty} \|f_{n_i}(t) - f_{n_j}(t)\|_{\theta(t)} = 0$$

for a.e. $t \in I$ since $\rho_{\theta(t)} \circ M$ satisfies the weak Δ_2 -condition. By the completeness of $X_{\theta(t)}$, one can find a function $f : I \rightarrow \mathcal{X}$ such that $f(t) \in X_{\theta(t)}$ and $\|f_{n_j}(t) - f(t)\|_{\theta(t)} \rightarrow 0$ for a.e. $t \in I$. In addition, for every $J \in \Lambda(I)$, by the uniform boundedness of $\{X_\alpha\}$, we have $f_{n_j}(t) \rightarrow f(t)$ in $X_{\theta_J^-}$ a.e. on J . Hence $f \in L^0(I, X_{\theta(\cdot)})$. Moreover, by the lower semicontinuity of $\rho_{\theta(t)} \circ M$ and Fatou's lemma, we obtain

$$\begin{aligned} \int_I \rho_{\theta(t)}(M(\lambda^{-1}(f_n(t) - f(t)))) dt &\leq \int_I \liminf_{j \rightarrow \infty} \rho_{\theta(t)}(M(\lambda^{-1}(f_n(t) - f_{n_j}(t)))) dt \\ &\leq \liminf_{j \rightarrow \infty} \int_I \rho_{\theta(t)}(M(\lambda^{-1}(f_n(t) - f_{n_j}(t)))) dt, \end{aligned}$$

which combined with (4.1), leads to

$$\lim_{n \rightarrow \infty} \int_I \rho_{\theta(t)}(M(\lambda^{-1}(f_n(t) - f(t)))) dt = 0.$$

Therefore, $f \in L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ and $f_n \rightarrow f$ in $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$. This completes the proof. \square

Remark 4.5. As a special case, in Proposition 4.3, the modular space $L^\rho(I, X_\rho)$ is also a Banach space provided the underlying space X_ρ is complete.

Given a semimodular $\rho : \mathcal{X} \rightarrow [0, \infty]$. We say ρ is uniformly convex, we mean that for every $\varepsilon \in (0, 1)$, there is a $\delta \in (0, 1)$, for which

$$\rho\left(\frac{u-v}{2}\right) \leq \varepsilon \frac{\rho(u) + \rho(v)}{2} \quad \text{or} \quad \rho\left(\frac{u-v}{2}\right) \leq (1-\delta) \frac{\rho(u) + \rho(v)}{2} \quad (4.2)$$

holds. And ρ is said to satisfy the Δ_2 -condition, if there exists a constant $C_2 > 0$ such that

$$\rho(2u) \leq C_2 \rho(u) \quad \text{for all } u \in \mathcal{X}.$$

Recall that, under the Δ_2 -condition, a semimodular modular turns to be a continuous modular. And every uniformly convex semimodular satisfying the Δ_2 -condition generates a uniformly convex space (cf. [19], §2.4).

Theorem 4.6. Besides the assumptions in Theorem 4.4, assume that for each $\alpha \in \mathcal{A}$, ρ_α satisfies the Δ_2 -condition with a common constant C_2 , and the multifunction $\rho_\alpha \circ M$ is uniformly convex uniformly in α , i.e. for each $\varepsilon \in (0, 1)$, the corresponding constant $\delta \in (0, 1)$ appearing in (4.2) is independent of

$\alpha \in \mathcal{A}$. Then functional $\Phi_{\theta(\cdot)}$ is uniformly convex, together with the Δ_2 -condition verified. Consequently, the semimodular space $L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ is uniformly convex.

Proof of this theorem is much similar to that of Theorem 2.4.11 in [19], and here we omit it.

Now we pay attention to the relations between Modular-Norm and Modular-Modular spaces.

Theorem 4.7. Suppose that all the assumptions in Theorem 4.4 are satisfied. Let $p, q \in \mathcal{P}_b(I)$ such that $p(t) \leq q(t)$ a.e. on I and

$$\min \{ \|f(t)\|_{\theta(t)}^{p(t)}, \|f(t)\|_{\theta(t)}^{q(t)} \} \leq \rho_{\theta(t)}(M(f(t))) \leq \max \{ \|f(t)\|_{\theta(t)}^{p(t)}, \|f(t)\|_{\theta(t)}^{q(t)} \} \quad (4.3)$$

for a.e. $t \in I$ and all $f \in L^0(I, X_{\theta(\cdot)})$. Then

$$\begin{aligned} L^{p(\cdot)}(I, X_{\theta(\cdot)}) \cap L^{q(\cdot)}(I, X_{\theta(\cdot)}) &\hookrightarrow L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \\ &\hookrightarrow L^{p(\cdot)}(I, X_{\theta(\cdot)}) + L^{q(\cdot)}(I, X_{\theta(\cdot)}). \end{aligned} \quad (4.4)$$

Proof. Take a function $f \in L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})$ with $\|f\|_{L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \leq 1$, then by the unit ball property, we have $\Phi_{\theta(\cdot)}(f) \leq 1$. Let

$$f_1(t) = \begin{cases} f(t), & \text{if } \|f\|_{X_{\theta(t)}} > 1, \\ 0, & \text{if } \|f(t)\|_{X_{\theta(t)}} \leq 1 \end{cases}$$

and $f_2(t) = f(t) - f_1(t)$. Apparently, both of f_1 and f_2 lie in $L^0(I, X_{\theta(\cdot)})$, and $f = f_1 + f_2$. Moreover, from (4.3), we derive that

$$\begin{aligned} \int_I \|f_1(t)\|_{X_{\theta(t)}}^{p(t)} dt + \int_I \|f_2(t)\|_{X_{\theta(t)}}^{q(t)} dt &= \int_I \min \{ \|f(t)\|_{X_{\theta(t)}}^{p(t)}, \|f(t)\|_{X_{\theta(t)}}^{q(t)} \} dt \\ &\leq \int_I \rho_{\theta(t)}(M(f(t))) dt \leq 1, \end{aligned}$$

which yields $\|f_1\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq 1$ and $\|f_2\|_{L^{q(\cdot)}(I, X_{\theta(\cdot)})} \leq 1$ by the unit ball property. Consequently, $f \in L^{p(\cdot)}(I, X_{\theta(\cdot)}) + L^{q(\cdot)}(I, X_{\theta(\cdot)})$ and $\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)}) + L^{q(\cdot)}(I, X_{\theta(\cdot)})} \leq 2$ by the decomposition $f = f_1 + f_2$.

Conversely, suppose $\|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)}) \cap L^{q(\cdot)}(I, X_{\theta(\cdot)})} \leq 1/2$, that is

$$\max \{ \|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})}, \|f\|_{L^{q(\cdot)}(I, X_{\theta(\cdot)})} \} \leq 1/2.$$

Thus, by the unit ball property, we have

$$\int_I \|f(t)\|_{X_{\theta(t)}}^{p(t)} dt \leq \|f\|_{L^{p(\cdot)}(I, X_{\theta(\cdot)})} \leq 1/2$$

and

$$\int_I \|f(t)\|_{X_{\theta(t)}}^{q(t)} dt \leq \|f\|_{L^{q(\cdot)}(I, X_{\theta(\cdot)})} \leq 1/2.$$

The above two inequalities, combined with (4.3), yield

$$\int_I \rho_{\theta(t)}(M(f(t)))dt \leq \int_I \|f(t)\|_{\theta(t)}^{p(t)} dt + \int_I \|f(t)\|_{\theta(t)}^{q(t)} dt \leq 1,$$

which in turn leads to $\|f\|_{L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)})} \leq 1$.

Finally, using the scaling arguments, we obtain the desired imbedding relations in (4.4). \square

Remark 4.8. Under the condition (4.3), if additionally, $I = [0, T]$ is bounded, then (4.4) turns to be

$$L^{q(\cdot)}(I, X_{\theta(\cdot)}) \hookrightarrow L^{\rho_{\theta(\cdot)}}(I, X_{\theta(\cdot)}) \hookrightarrow L^{p(\cdot)}(I, X_{\theta(\cdot)}).$$

Let Ω be an bounded domain of \mathbb{R}^N and $Q = I \times \Omega$. Suppose that $p \in \mathcal{P}_b(Q)$ is a Caratheodory type exponent satisfying (3.2). Let $\mathcal{W} = \mathcal{X} = L^0(\Omega)$ and V, M be defined as in Example 4.1. For each $q \in \mathcal{A}_b$, define

$$\rho_q(f) = \int_{\Omega} f(x)^{q(x)} dx, \quad f \in V.$$

It is easy to check that, ρ_q is strongly continuous (refer to Remark 4.2), and satisfies the Δ_2 -condition with the constant $2^{\bar{q}}$. Moreover, the multifunction $\rho_q \circ M$ is uniformly convex uniformly for $q \in \mathcal{A}_0$.

Example 4.3. Take any $u \in L^0(I, L^0(\Omega))$, and define

$$\Phi_{Pp(\cdot)}(u) = \int_I \rho_{Pp(t)}(M(u(t)))dt = \int_Q |\hat{u}(x, t)|^{p(x, t)} dx dt, \quad (4.5)$$

where $Pp : I \rightarrow \mathcal{A}_b$ is a DC map as shown in Section 3. By invoking Theorem 4.4, 4.6, we conclude that $\Phi_{Pp(\cdot)}$ is a modular on $L^0(I, L^0(\Omega))$, it derives a Banach space $L^{\rho_{Pp(t)}}(I, L^{p(t, x)}(\Omega))$, which is uniformly convex under condition $\underline{p} > 1$. In view of (4.5), we can also find that the map $P : L^0(Q) \rightarrow L^0(I, L^0(\Omega))$ is an isomorphism from $L^{p(x, t)}(Q)$ to $L^{\rho_{Pp(t)}}(I, L^{p(t, x)}(\Omega))$ with

$$\|Pu\|_{L^{\rho_{Pp(t)}}(I, L^{p(t, x)}(\Omega))} = \|u\|_{L^{p(x, t)}(Q)} \quad (4.6)$$

holding. Thus under condition (3.2), spaces $L^{\rho_{Pp(t)}}(I, L^{p(t, x)}(\Omega))$ and $L^{p(x, t)}(Q)$ are no longer distinguished in the sense of isomorphism.

Remark 4.9. Equality (4.6) is an extension of the isometrical property of P from the constant exponent case to the variable exponent case.

The following proposition, which is a natural corollary of Theorem 4.7, exhibits the connections between $L^{p(x, t)}(Q)$ and $L^{p_t^{\pm}}(I, L^{p(x, t)}(\Omega))$, and extends the corresponding result in [5] from the case $p(x)$ to the case $p(x, t)$.

Proposition 4.10. Under all the assumptions upon p , imbedding relations

$$\begin{aligned} & L^{p_t^-}(I, L^{p(x, t)}(\Omega)) \cap L^{p_t^+}(I, L^{p(x, t)}(\Omega)) \\ & \hookrightarrow L^{p(x, t)}(Q) \hookrightarrow L^{p_t^-}(I, L^{p(x, t)}(\Omega)) + L^{p_t^+}(I, L^{p(x, t)}(\Omega)) \end{aligned} \quad (4.7)$$

together with the following estimates

$$2^{-1}\|f\|_{L^{p_t^-}(I, L^{p(x,t)}(\Omega)) + L^{p_t^+}(I, L^{p(x,t)}(\Omega))} \leq \|f\|_{L^{p(x,t)}(Q)} \leq 2\|f\|_{L^{p_t^-}(I, L^{p(x,t)}(\Omega)) \cap L^{p_t^+}(I, L^{p(x,t)}(\Omega))}$$

hold. Furthermore, if $I = [0, T]$, then (4.7) becomes

$$L^{p_t^+}(I, L^{p(x,t)}(\Omega)) \hookrightarrow L^{p(x,t)}(Q) \hookrightarrow L^{p_t^-}(I, L^{p(x,t)}(\Omega)).$$

Example 4.4. Suppose that $p_i \in \mathcal{P}(Q)$, $i = 1, 2, \dots, N$ are all continuous exponents satisfying (3.2) with $Pp_i(t) \in \mathcal{A}_0$ for all $t \in I$. Then the anisotropic space

$$W(Q) = \{u \in L^0(I, W_0^{1,1}(\Omega)) : u \in L^2(I, L^2(\Omega)), \partial_i u \in L^{Pp_i(t)}(I, L^{p_i(x,t)}(\Omega)), i = 1, 2, \dots, N\}$$

with the norm

$$\|u\|_{W(Q)} = \|u\|_{L^2(I, L^2(\Omega))} + \sum_{i=1}^N \|\partial_i u\|_{L^{Pp_i(t)}(I, L^{p_i(x,t)}(\Omega))}$$

is a Banach space. This space was used in the work [10].

Appendix A

At the end of the paper, let us review briefly on the X -valued Lebesgue spaces with variable exponents. For the detailed discussions, please refer to [16,19] etc.

Suppose that (A, Σ, μ) is a σ -finite complete measure space, and X is a Banach space. For each $p \in \mathcal{P}(A, \mu)$, define the semimodular on $L^0(A, X)$ through

$$\rho_{p(\cdot)}(f) = \int_A \varphi_{p(t)}(\|f(t)\|_X) d\mu.$$

This semimodular induces a Banach space, denoted by $L^{p(\cdot)}(A, X)$. If $p \in \mathcal{P}_b(A, \mu)$, then $\rho_{p(\cdot)}$ is a continuous modular, consequently $f \in L^{p(\cdot)}(A, X)$ if and only if $\rho_{p(\cdot)}(f) < \infty$, and

$$\sigma^-(L^{p(\cdot)}(A, X)) \leq \rho_{p(\cdot)}(f) \leq \sigma^+(L^{p(\cdot)}(A, X)).$$

Proposition A.1. Suppose that μ is a separable measure (cf. [19], p. 50) and X is a separable space, then under the condition $p \in \mathcal{P}_b(A, \mu)$, $L^{p(\cdot)}(A, X)$ is also a separable space.

Proposition A.2. Let X and Y be two Banach spaces. If Y is densely injected in X , then $L^{p(\cdot)}(A, Y)$ is also dense in $L^{p(\cdot)}(A, X)$.

Definition A.3. A map $\Gamma : \Sigma \rightarrow X$ is called an abstract vector-valued measure, if for any sequence of disjoint sets $\{E_n\} \subseteq \Sigma$ satisfying $E = \bigcup_{n=1}^\infty E_n$, equality $\Gamma(E) = \sum_{n=1}^\infty \Gamma(E_n)$ holds with the right series absolutely convergent in X .

Given an X -valued measure Γ as above, for each $E \in \Sigma$, define

$$|\Gamma|(E) = \sup_{\pi} \sum_{B \in \pi} \|\Gamma(B)\|_X$$

as the variation of Γ on E . Here the supremum is taken over the set of all finite measurable decompositions π of E . If $|\Gamma|(A) < \infty$, or in other words, Γ has a bounded total variation, then Γ is called totally bounded.

Remark A.4. Recall that, for a totally bounded abstract-valued measure Γ , the corresponding set function $|\Gamma| : \Sigma \rightarrow \mathbb{R}^+$ is a positive measure.

Definition A.5. Suppose that Γ is an X -valued measure attached to the measure space (A, Σ, μ) . We say that Γ is absolutely continuous w.r.t. the positive measure μ , we mean that $\Gamma(E) = 0$ provided $\mu(E) = 0$.

Definition A.6. A Banach space X is called having the Radon–Nikodym’s property w.r.t. (A, Σ, μ) , if for every X -valued measure Γ owning a bounded total variation and absolutely continuous about μ , there exists a Bochner integrable X -valued function f , for which the equality

$$\Gamma(E) = \int_E f d\mu$$

holds for all $E \in \Sigma$.

Proposition A.7. Let (A, Σ, μ) be a σ -finite complete measure space, and X be a Banach space, whose dual space X^* has the Radon–Nikodym’s property w.r.t. (A, Σ, μ) . Then for each variable exponent $p \in \mathcal{P}_b(A, \mu)$, we have that

$$L^{p(\cdot)}(A, X)^* = L^{p'(\cdot)}(A, X^*)$$

in the sense of isomorphism, where $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$. More precisely, for each $\xi \in L^{p(\cdot)}(A, X)^*$, there is a unique function $g \in L^{p'(\cdot)}(A, X^*)$ such that

$$\langle \langle \xi, f \rangle \rangle = \int_A \langle g(t), f(t) \rangle dt$$

for all $f \in L^{p(\cdot)}(A, X)$. And norms of ξ and g are equivalent, i.e.

$$\|g\|_{L^{p'(\cdot)}(A, X^*)} \leq \|\xi\|_{L^{p(\cdot)}(A, X)^*} \leq 2\|g\|_{L^{p'(\cdot)}(A, X^*)}.$$

Proposition A.8. Assume that X is a reflexive space, and $p \in \mathcal{P}_0(A, \mu)$, then $L^{p(\cdot)}(A, X)$ is also reflexive. In this situation, the norm conjugate formula

$$\|f\|_{L^{p(\cdot)}(A, X)} \leq \sup \left\{ \int_A \langle f(t), g(t) \rangle d\mu : \|g\|_{L^{p'(\cdot)}(A, X^*)} \leq 1 \right\} \leq 2\|f\|_{L^{p(\cdot)}(A, X)}$$

also holds for all $f \in L^0(A, X)$.

Proposition A.9. Let X be a uniformly convex (smooth) space, then under the condition $1 < p^- \leq p^+ < \infty$, $L^{p(\cdot)}(A, X)$ is also uniformly convex (smooth).

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