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On boundary control of the Poisson equation with the third boundary condition

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This paper studies controllability of the Poisson equation on the unit disk in $\mathbb{C}$ subject to the third boundary condition when the control is imposed on the boundary. We use complex analytic methods to prove existence and uniqueness of the control when the parameter λ is a nonzero complex number but not a negative integer (not an eigenvalue). Otherwise, due to multiplicity of solutions to the underlying problem, when λ is a negative integer, controllability could only be obtained if proper additional conditions on the boundary are imposed.

1. Introduction

We consider the Poisson equation for functions of complex variables on the unit disk $\mathbb{D} \subset \mathbb{C}$

$$\Delta u = f \qquad \text{in} \quad \mathbb{D}, \tag{1.1}$$

subject to the boundary condition

$$\frac{\partial}{\partial n} u + \lambda u = g \quad \text{on} \quad \partial\mathbb{D}, \tag{1.2}$$

where n is the outward unit normal and $\lambda \in \mathbb{C}\backslash\{0\}$. This boundary condition is known as the third boundary condition. We assume throughout that $f \in C(\overline{\mathbb{D}})$ (continuous). We investigate the optimal control problem of finding a control g in an admissible set of functions which minimizes the quadratic functional

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$$J(u,g) = \int_{\mathbb{D}} |u|^2 \, dx \, dy + \int_0^{2\pi} |g(\theta)|^2 \, d\theta, \tag{1.3}$$

for different values of the parameter $\lambda \in \mathbb{C} \backslash \{0\}$.

Most works in the literature treat the case when λ is a positive real parameter [10,15,20] known as the Robin boundary condition [18,24,23,25] for which energy methods can be used to study the problem. The case of $\lambda < 0$, also known as the Steklov problem [28,18], has been considered by [5] and explicit solutions are provided in terms of polar coordinates on a disk and it has been also studied by [3] where explicit solutions are provided in terms of spherical coordinates on a sphere. In [4], the author considered different representations of Green's functions for Laplacian boundary value problems.

The Steklov problem is an eigenvalue problem with the spectral parameter in the boundary condition and has various applications [17]. This boundary condition is relevant to the study of certain physical and biological models such as the electron energy barrier model and oxygen absorption of human lungs [18, 16]. In some special cases, the Steklov spectrum can be explicitly computed as in [17] where the Steklov eigenvalues and eigenfunctions of cylinders and balls are calculated explicitly using separation of variables for a non-negative weight function $\rho \equiv 1$ on the boundary and where the eigenfunctions are given in terms of polar coordinates for the unit disk.

This paper considers however the more general case where λ is any complex valued parameter on the unit disk, and refers to this case by the "third boundary condition". Moreover the explicit solutions are provided in terms of holomorphic and anti holomorphic polynomials/functions from the perspective of complex analysis and the parameter is assumed to be any complex number rather than limiting it to be a positive or a negative real number [25]. The third boundary condition for holomorphic and harmonic functions is studied by [24,23,25] and explicit solutions are provided for the case when λ is a general complex valued function. In this paper, we utilize the explicit solutions provided by [25] for the case of a general complex valued constant λ, to obtain a boundary controllability result on the solution to the BVP.

For $\lambda > 0$, or the Robin problem on a general bounded domain, it is well known that there is a unique solution, given sufficiently regular data g. Energy methods and the Lax–Milgram theorem are usually invoked to establish the existence of a weak solution to the equation. The controllability of this problem and optimal solution in case of the Robin as well as the Dirichlet boundary conditions are also known [13,21,2] for some general domains and in the case of Steklov, there are also some studies available, among others [14,17]. These tools cannot be applied however for more general values of λ when λ is a complex number or when $\lambda(z)$ is a complex valued function. In this paper, we study controllability of the Poisson equation with the third boundary condition when the control is imposed in the boundary condition.

2. Preliminaries

2.1. Function spaces

We introduce the Sobolev space which we will use throughout the paper

$$W^{2,1}(\mathbb{D}) \equiv \{v \in L^2(\mathbb{D}) : \partial_z v \in L^2(\mathbb{D}), \partial_{\overline{z}} v \in L^2(\mathbb{D})\},$$

equipped with the norm

$$\|v\|_{W^{2,1}(\mathbb{D})} \equiv \left(\int_{\mathbb{D}} |v|^2 \, dx \, dy + \int_{\mathbb{D}} |\partial_z v|^2 \, dx \, dy + \int_{\mathbb{D}} |\partial_{\overline{z}} v|^2 \, dx \, dy \right)^{1/2},$$

induced by inner product

$$\langle v, w\rangle = \int\limits_{\mathbb{D}} v\,\overline{w}\,dx\,dy + \int\limits_{\mathbb{D}} \partial_z v\,\partial_{\overline{z}}\overline{w}\,dx\,dy + \int\limits_{\mathbb{D}} \partial_{\overline{z}} v\,\partial_z\overline{w}\,dx\,dy.$$

We also denote by $W^{2,2}$ the Sobolev space

$$W^{2,2}(\mathbb{D}) \equiv \{v \in L^2(\mathbb{D}) : \partial_z v \in W^{2,1}(\mathbb{D}), \partial_{\overline{z}} v \in W^{2,1}(\mathbb{D})\},$$

with the norm

$$\|v\|_{W^{2,2}(\mathbb{D})} \equiv \left(\|v\|^2_{L^2} + \|\partial_z v\|^2_{W^{2,1}(\mathbb{D})} + \|\partial_{\overline{z}} v\|^2_{W^{2,1}(\mathbb{D})}\right)^{1/2}.$$

The fractional Sobolev spaces $W^{k,2}(\mathbb{D})$ for $0 < k < 2$, are defined as real interpolation spaces between $W^{2,2}(\mathbb{D})$ and $L^2(\mathbb{D})$, see [6]. The trace map acts continuously from $W^{m,2}(\mathbb{D})$ to $W^{m-1/2,2}(\partial\mathbb{D})$. We shall also make use of the continuous embedding $W^{2,m}(\mathbb{D}) \subset C^1(\mathbb{D})$ for $m > 2$.

The set of all complex-valued functions on the unit circle of the complex plane with absolutely convergent Fourier series is called the Wiener algebra [22,19] and can be denoted by

$$A(\partial\mathbb{D}) = \left\{h \middle| h(z) = \sum_{-\infty}^{+\infty} h_k z^k, \quad z \in \partial\mathbb{D}, \quad \|h\| := \sum_{-\infty}^{+\infty} |h_k| < \infty\right\}.$$

2.2. Particular/special solution of the Poisson equation

The general solution to the Poisson equation (1.1) can be represented [9–12] as

$$u = \phi(z) + \overline{\psi(z)} + w(z), \tag{2.1}$$

where ϕ and ψ are functions, holomorphic in $\mathbb{D}$ and w is the particular solution given by the expression $w = T_{1,1}(f)$, a generalization of the so-called T operator [7,8,11,12]. We quote some relevant notations and results from [8,11,12] to use in the paper without going into details.

The T operator, also known as the Pompeiu operator is defined [8,29] as

$$Tf(z) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} f(\zeta)\frac{d\xi\,d\eta}{\zeta - z}, \quad z \in \mathbb{D},$$

and the special solution w defined as $T_{1,1}f(z)$ on the unit disk is given in explicit form (cf. [10]) by

$$w(z) = T_{1,1}f(z) = \frac{2}{\pi}\int\limits_{\mathbb{D}} f(\zeta)\ln|z-\zeta|\,d\xi\,d\eta. \tag{2.2}$$

Higher order generalization of the Pompeiu operator provides a representation of functions $w \in C^{m+n}(\overline{\mathbb{D}})$ in terms of an area integral $T_{m,n}(\partial^{m+n}w/\partial z^m\partial\overline{z}^n)$ with $m+n \geq 0$, but $(m,n) \not\equiv (0,0)$. Then $T_{0,1}$ and $T_{1,0}$ are the T and $\overline{T}$ operators respectively. The operators $T_{m,n}$ have interesting properties such as $\partial_z(T_{m,n}w) = T_{m-1,n}w$ and $\partial_{\overline{z}}(T_{m,n}w) = T_{m,n-1}w$. Clearly $\partial_z(T_{1,1}w) = T_{0,1}w$ and $\partial_{\overline{z}}(T_{1,1}w) = T_{1,0}w$. It is important to notice that $\overline{T_{m,n}w} = T_{n,m}\overline{w}$.

Now, the boundary condition (1.2) is written as

$$\zeta\frac{\partial u}{\partial\zeta}+\overline{\zeta}\frac{\partial u}{\partial\overline{\zeta}}+\lambda u=g(\zeta)\quad\text{on}\quad\partial\mathbb{D}. \tag{2.3}$$

Note that throughout the manuscript, we occasionally use the notation $g(z)$ for $z\in\partial\mathbb{D}$ interchangeably with $g(\theta)$ for $\theta\in[0,2\pi]$ for functions defined on the boundary.

Therefore, the boundary condition (2.3) can be expressed in the form

$$(\zeta\phi_\zeta+\lambda\phi)+\overline{(\zeta\psi_\zeta+\overline{\lambda}\psi)}=g^\star(\zeta)\quad\text{on}\quad\partial\mathbb{D}, \tag{2.4}$$

where

$$g^\star(\zeta)\equiv g(\zeta)-\zeta T_{0,1}f(\zeta)-\overline{\zeta}T_{1,0}f(\zeta)-\lambda T_{1,1}f(\zeta),\quad\zeta\in\partial\mathbb{D}. \tag{2.5}$$

Since $|\zeta|=1$ on $\partial\mathbb{D}$ and $\overline{T_{1,0}f}=T_{0,1}\overline{f}$, it suffices to analyze the terms $T_{0,1}f=Tf$ and $T_{1,1}f$.

According to (p. 89) Theorem 30 in [7], the operator T is a completely continuous operator from $C^\nu(\overline{\mathbb{D}})$ to $C^{1+\nu}(\overline{\mathbb{D}})$ and the norm of Tf is bounded by the norm of f. Thus,

$$||Tf||_{C^{\nu+1}(\overline{\mathbb{D}})}\le M||f||_{C^\nu(\overline{\mathbb{D}})}. \tag{2.6}$$

Moreover, by Theorem 4.3 (c) [12] on page 679, for $p=2$, $T_{1,1}$ satisfies

$$|T_{1,1}f(z)|\le M||f||_{L^2(\mathbb{D})}, \tag{2.7}$$

for all $z\in\overline{\mathbb{D}}$ and $f\in L^2(\mathbb{D})$.

On the other hand, if $f\in L^2(\mathbb{D})$ then $Tf\in W^{2,1}(\mathbb{D})$ and the estimate

$$||Tf||_{W^{2,1}(\mathbb{D})}\le M||f||_{L^2(\mathbb{D})} \tag{2.8}$$

holds, see [6]. Consequently, regarding $T_{1,1}f$, the estimate

$$||T_{1,1}f||_{W^{2,2}(\mathbb{D})}\le M||f||_{L^2(\mathbb{D})}, \tag{2.9}$$

holds.

Hence, the norm of w satisfies the following inequality

$$\|w\|_{W^{2,2}(\mathbb{D})}\le M||f||_{L^2(\mathbb{D})}. \tag{2.10}$$

By the Bernstein theorem [19], $C^\nu(\partial\mathbb{D})\subset A(\partial\mathbb{D})$ for $\nu>1/2$. Obviously $C^1(\partial\mathbb{D})\subset A(\partial\mathbb{D})\subset C(\partial\mathbb{D})$ and if $g(\zeta)\in C^1(\partial\mathbb{D})$, then $g^\star(\zeta)\in C^1(\partial\mathbb{D})$.

We can also conclude that the boundary term $g^\star$ satisfies the estimate

$$\begin{aligned}\|g^\star\|_{L^2(\partial\mathbb{D})}&\le C(\|g\|_{L^2(\partial\mathbb{D})}+\|T_{1,1}f\|_{L^2(\partial\mathbb{D})}+\|T_{0,1}f\|_{L^2(\partial\mathbb{D})}+\|T_{1,0}f\|_{L^2(\partial\mathbb{D})})\\&\le C(\|g\|_{L^2(\partial\mathbb{D})}+\|T_{1,1}f\|_{C(\overline{\mathbb{D}})}+\|T_{0,1}f\|_{W^{2,1}(\mathbb{D})}+\|T_{1,0}f\|_{W^{2,1}(\mathbb{D})})\\&\le C(\|g\|_{L^2(\partial\mathbb{D})}+\|f\|_{L^2(\mathbb{D})}),\end{aligned} \tag{2.11}$$

where C throughout the paper denotes a constant that depends on λ.

3. Main results

Our main results in this paper are the following four theorems. The first two address the case when the complex number λ is not a negative integer.

Theorem 3.1. *Suppose $\lambda \in \mathbb{C}$, but $-\lambda \notin \mathbb{N}$, then given $f \in C(\overline{\mathbb{D}})$ and $g \in C^1(\partial\mathbb{D})$, the Poisson equation (1.1) with the third boundary condition (1.2) has a unique solution u explicitly given by the formula*

$$u = (T_{1,1}f)(z) + \sum_{k=0}^{\infty} \frac{g_k^\star}{k+\lambda} z^k + \sum_{k=0}^{\infty} \frac{g_{-k}^\star}{k+\lambda} \overline{z}^k, \tag{3.1}$$

where $G^\star(z) := \sum_{k=0}^{\infty} g_k^\star z^k$ and $H^\star(z) := \sum_{k=0}^{\infty} \overline{g}_{-k}^\star z^k$ are holomorphic functions obtained by applying the Schwarz operator on $g^\star(\zeta)$ and $\overline{g}^\star(\zeta)$ as defined in the expressions given in (4.6) and (4.7) respectively.

On the other hand, if $g \in L^2(\partial\mathbb{D})$ and $f \in L^2(\mathbb{D})$ only, there exists a unique weak solution $u \in W^{2,1}(\mathbb{D})$ satisfying

$$\begin{aligned} &-2\int_{\mathbb{D}} \partial_z u\, \partial_{\overline{z}} \overline{v}\, dx\, dy - 2\int_{\mathbb{D}} \partial_{\overline{z}} u\, \partial_z \overline{v}\, dx\, dy - \lambda \int_0^{2\pi} u(e^{i\theta})\, \overline{v}(e^{i\theta})\, d\theta + \int_0^{2\pi} g(\theta)\, \overline{v}(e^{i\theta})\, d\theta \\ &= \int_{\mathbb{D}} f\, \overline{v}\, dx\, dy, \quad \forall v \in W^{2,1}(\mathbb{D}). \end{aligned} \tag{3.2}$$

The solution satisfies the estimate

$$\|u\|_{W^{2,1}(\mathbb{D})} \le C\|g\|_{L^2(\partial\mathbb{D})} + C\|f\|_{L^2(\mathbb{D})}, \tag{3.3}$$

with C being a constant depending on λ.

The second theorem is on the existence and uniqueness of the optimal control g which minimizes the functional (1.3).

Theorem 3.2. *Suppose $\lambda \in \mathbb{C}$, but $-\lambda \notin \mathbb{N}$, then given $f \in C(\overline{\mathbb{D}})$, there exists a unique minimizer $g^0 \in L^2(\partial\mathbb{D})$ to the functional (1.3) subject to boundary value problem (1.1)–(1.2). Moreover, the optimal control $g^0 \in C^1(\partial\mathbb{D})$.*

The next two theorems address the case when $-\lambda = k_0 \in \mathbb{N}$ wherein additional conditions are required to guarantee uniqueness of solution to the boundary value problem and hence the well-posedness of the control problem. In particular, we have

Theorem 3.3. *Suppose $-\lambda = k_0 \in \mathbb{N}$, and suppose $f \in C(\overline{\mathbb{D}})$ and $g \in C^1(\partial\mathbb{D})$ satisfy the compatibility conditions*

$$\int_{|\zeta|=1} g^\star(\zeta)\overline{\zeta}^{\pm k_0} \frac{d\zeta}{\zeta} = 0 \tag{3.4}$$

(where $g^\star$ is defined in (2.5)), then the Poisson equation (1.1) with boundary condition (1.2) has the general solution u explicitly given by the formula

$$u = (T_{1,1}f)(z) + \sum_{k=0, k\ne k_0}^{\infty} \frac{g_k^\star}{k-k_0}\, z^k + \sum_{k=0, k\ne k_0}^{\infty} \frac{g_{-k}^\star}{k-k_0}\, \overline{z}^k + Az^{k_0} + B\overline{z}^{k_0}, \tag{3.5}$$

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where A and B are any constants, while $G^\star(z) := \sum_{k=0}^\infty g_k^\star z^k$ and $H^\star(z) := \sum_{k=0}^\infty \overline{g}_{-k}^\star z^k$ are given by (4.6) and (4.7) respectively. If the additional conditions

$$\int_{\partial\mathbb{D}} u(\zeta)\ \overline{\zeta}^{k_0+1}\, d\zeta = u_1 \tag{3.6}$$

$$\int_{\partial\mathbb{D}} u(\zeta)\ \zeta^{k_0-1}\, d\zeta = u_2, \tag{3.7}$$

are imposed, where u_1 and u_2 are given complex constants, then there is a unique classical solution.

On the other hand, suppose that $u_1 = u_2 = 0$, while $g \in L^2(\partial\mathbb{D})$ only and satisfies compatibility conditions (3.4), then there exists a unique weak solution $u \in W^{2,1}(\mathbb{D})$ satisfying (3.2) and conditions (3.6) and (3.7). Moreover, the solution u satisfies the estimate

$$\|u\|_{W^{2,1}(\mathbb{D})} \le C\|g\|_{L^2(\partial\mathbb{D})} + C\|f\|_{L^2(\mathbb{D})} + C(|u_1| + |u_2|). \tag{3.8}$$

As for the optimal control, we have the following theorem

Theorem 3.4. *Suppose $-\lambda = k_0 \in \mathbb{N}$, $u_1 = u_2 = 0$, then given $f \in C(\overline{\mathbb{D}})$, there exists a unique minimizer g^0 to the functional (1.3) satisfying compatibility condition (3.4), subject to boundary value problem (1.1)–(1.2) and conditions (3.6)–(3.7).*

4. Proof of Theorem 3.1

Using the decomposition (2.1) of the solution to the Poisson equation, it suffices to find holomorphic functions ϕ and ψ satisfying the boundary condition (2.4) on $\partial\mathbb{D}$. In other words, ϕ and ψ satisfy

$$\Re(z\partial\phi + \lambda\phi) + \Re\big(z\partial\psi + \overline{\lambda}\psi\big) = \Re(g^\star(z)), \quad z \in \partial\mathbb{D}, \tag{4.1}$$

and

$$\Im(z\partial\phi + \lambda\phi) - \Im\big(z\partial\psi + \overline{\lambda}\psi\big) = \Im(g^\star(z)), \quad z \in \partial\mathbb{D}, \tag{4.2}$$

where $g^\star$ is given in (2.5).

The solutions for ϕ and ψ are obtained through the Schwarz operator S which furnishes a solution G to the problem

$$\overline{\partial} G = 0 \quad \text{in} \ \ \mathbb{D}$$

$$\Re(G) = \tau(z) \quad \text{on} \ \ \partial\mathbb{D},$$

given a function $\tau(z) \in C(\partial\mathbb{D})$. In particular, $G(z) = S(\tau(\cdot))(z)$ where the Schwarz operator S is defined by

$$S(\tau(\cdot))(z) \equiv \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \tau(\zeta)\left(\frac{2\zeta}{\zeta - z} - 1\right)\frac{d\zeta}{\zeta} + ic, \tag{4.3}$$

and $c \in \mathbb{R}$ is arbitrary.

Therefore, the solution to boundary problem (4.1) is

$$(z\partial\phi + \lambda\phi) + \big(z\partial\psi + \overline{\lambda}\psi\big) = S(\Re[g^\star(.)])(z), \quad z \in \mathbb{D}, \tag{4.4}$$

while the solution to (4.2) is given by

$$(z\partial\phi + \lambda\phi) - \big(z\partial\psi + \overline{\lambda}\psi\big) = iS(\Im[g^\star(.)])(z), \quad z \in \mathbb{D}. \tag{4.5}$$

Adding and subtracting the two equations, we obtain the conditions

$$(z\partial\phi + \lambda\phi) = \frac{1}{2}S(\Re[g^\star(.)])(z) + \frac{1}{2}iS(\Im[g^\star(.)])(z) \equiv \frac{1}{2}S(g^\star(.))(z) := G^\star(z), \quad z \in \mathbb{D}, \tag{4.6}$$

and

$$\big(z\partial\psi + \overline{\lambda}\psi\big) = \frac{1}{2}S(\Re[g^\star(.)])(z) - \frac{1}{2}iS(\Im[g^\star(.)])(z) \equiv \frac{1}{2}S(\overline{g}^\star(.))(z) := H^\star(z), \quad z \in \mathbb{D}. \tag{4.7}$$

Due to the fact that the right hand sides of equations (4.6) and (4.7) are associated with the Schwarz kernel, they are holomorphic functions and they can be written as

$$z\phi' + \lambda\phi = G^\star(z), \quad z \in \mathbb{D}, \tag{4.8}$$

and

$$z\psi' + \overline{\lambda}\psi = H^\star(z), \quad z \in \mathbb{D} \tag{4.9}$$

which are clearly linear equations for holomorphic functions ϕ and ψ. To the corresponding homogeneous equations

$$z\phi' + \lambda\phi = 0, \quad z \in \mathbb{D},$$

and

$$z\psi' + \overline{\lambda}\psi = 0, \quad z \in \mathbb{D}$$

solutions can be given in the form of z^r. Clearly $r = -\lambda$ and $r = -\overline{\lambda}$ respectively and thus the solutions are $\phi(z) = C_1 z^{-\lambda}$ and $\psi(z) = C_2 z^{-\overline{\lambda}}$ with C_1 and C_2 being arbitrary complex numbers.

However, the terms $z^{-\lambda}$ and $z^{-\overline{\lambda}}$ are the nontrivial solutions of the corresponding homogeneous equations of (4.8) and (4.9) only if they are holomorphic, i.e., only if $-\lambda = k_0 \in \mathbb{N}$ (automatically $-\overline{\lambda} = k_0$). When $-\lambda \notin \mathbb{N}$, the terms $z^{-\lambda}$ and $z^{-\overline{\lambda}}$ are no longer holomorphic for $z \in \mathbb{D}$, so the only solutions of the homogeneous equations are the trivial solution, i.e., $C_1 = C_2 = 0$.

4.1. Fourier series solution

Since $g^\star(\zeta) \in C^1(\partial\mathbb{D}) \subset A(\partial\mathbb{D})$, functions $G^\star(z)$ and $H^\star(z)$ can be expressed in terms of absolutely convergent Fourier series as (see Remarks 6.3)

$$G^\star(z) = \sum_{k=0}^{\infty} g_k^\star z^k, \quad H^\star(z) = \sum_{k=0}^{\infty} \overline{g}_{-k}^\star z^k, \quad z \in \mathbb{D}; \quad g^\star(\zeta) = \sum_{k=0}^{\infty} g_{-k}^\star \zeta^{-k} + \sum_{k=0}^{\infty} g_k^\star \zeta^k, \quad \zeta \in \partial\mathbb{D}. \tag{4.10}$$

Since ϕ and ψ are holomorphic in $\mathbb{D}$, one can assume that

$$\phi(z) = \sum_{k=0}^{\infty} \phi_k z^k, \quad \psi(z) = \sum_{k=0}^{\infty} \psi_k z^k, \quad z \in \mathbb{D}. \tag{4.11}$$

By substituting (4.10) and (4.11) into (4.8), it is easy to get that

$$\sum_{k=0}^{\infty}[(k+\lambda)\phi_k - g_k^\star]z^k = 0, \quad z \in \mathbb{D}. \tag{4.12}$$

If $\lambda \in \mathbb{C}$, but $-\lambda \notin \mathbb{N} \cup \{0\}$, then $k+\lambda \neq 0$, $\forall k \in \mathbb{N} \cup \{0\}$. Thus (4.12) means that

$$\phi_k = \frac{g_k^\star}{k+\lambda}, \quad k \in \mathbb{N} \cup \{0\}, \quad \phi(z) = \sum_{k=0}^{\infty} \frac{g_k^\star}{k+\lambda} z^k, \quad z \in \mathbb{D}. \tag{4.13}$$

Similarly, we conclude

$$\psi(z) = \sum_{k=0}^{\infty} \frac{\overline{g_{-k}^\star}}{k+\overline{\lambda}} z^k. \tag{4.14}$$

4.2. Uniqueness of solution

Considering the homogeneous problem ($f = 0$ and $g = 0$) arising from taking the difference of two solutions, the solution has the form $u = \phi + \overline{\psi}$ where ϕ and ψ are two holomorphic functions satisfying the homogeneous version of conditions (4.6) and (4.7)

$$(z\partial\phi + \lambda\phi) = 0 \quad \text{on} \quad \mathbb{D} \tag{4.15}$$

$$\left(z\partial\psi + \overline{\lambda}\psi\right) = 0 \quad \text{on} \quad \mathbb{D}, \tag{4.16}$$

which implies that

$$\phi = C_1 z^{-\lambda}, \tag{4.17}$$

$$\psi = C_2 z^{-\overline{\lambda}}. \tag{4.18}$$

Since, ϕ and ψ are holomorphic we either have $-\lambda = k_0 \in \mathbb{N}$ or $C_1 = C_2 \equiv 0$. But λ is nonzero and is not a negative integer by assumption and thus $C_1 = C_2 \equiv 0$ and uniqueness of a classical solution follows. Note, the case of $\lambda = 0$ is the Neumann condition which is no longer the third boundary condition.

4.3. Continuous dependence on data

We next establish the estimate (3.3) showing continuous dependence of the solution u on the data f and g in the prescribed norms. In particular, estimating the $W^{2,1}$ norm of ϕ, $\overline{\psi}$ and w we have

$$\begin{aligned}
\|\phi\|^2_{W^{2,1}(\mathbb{D})} &= \|\phi\|^2_{L^2(\mathbb{D})} + \|\partial\phi\|^2_{L^2(\mathbb{D})} \\
&= 2\pi \int_0^1 \sum_{k=0}^{\infty} \frac{|g_k^\star|^2}{|k+\lambda|^2} r^{2k+1}\, dr + 2\pi \int_0^1 \sum_{k=0}^{\infty} \frac{k^2|g_k^\star|^2}{|k+\lambda|^2} r^{2k-1}\, dr \\
&= 2\pi \sum_{k=0}^{\infty} \frac{|g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k+2} + 2\pi \sum_{k=0}^{\infty} \frac{k^2|g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k}
\end{aligned}$$

Now, if $\Im(\lambda) \neq 0$, then $|k+\lambda| \geq |\Im(\lambda)|$. If $\Im(\lambda) = 0$ and $\Re(\lambda) > 0$ then $|\lambda + k| \geq \lambda > 0$. On the other hand, if $\Im(\lambda) = 0$ and $\Re(\lambda) = -m - d < 0$ where m is a nonnegative integer and $0 < d < 1$ then $|\lambda + k| \geq \min\{d, 1-d\}$. Hence, the first sum satisfies

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$$2\pi \sum_{k=0}^{\infty} \frac{|g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k+2} \leq C_1(\lambda) 2\pi \sum_{k=0}^{\infty} |g_k^\star|^2 \frac{1}{2k+2} = C_1(\lambda) \left\| \frac{1}{2} S(g^\star) \right\|_{L^2(\mathbb{D})}^2,$$

where

$$C_1(\lambda) = \begin{cases} \frac{1}{|\Im(\lambda)|}, & \Im(\lambda) \neq 0 \\ \frac{1}{\lambda}, & \Im(\lambda) = 0, \quad \Re(\lambda) > 0 \\ \max\{\frac{1}{d}, \frac{1}{1-d}\}, & \lambda = -m-d < 0 \end{cases}$$

To estimate the second sum, note that if $\Re(\lambda) \geq 0$, then

$$\frac{k^2}{|k+\lambda|^2} \leq 1.$$

On the other hand, if $\Re(\lambda) = -m-d < 0$, where m is a nonnegative integer and $0 \leq d < 1$, then we bound the first m terms

$$\frac{k^2}{|k+\lambda|^2} \leq \frac{m^2}{d^2 + |\Im(\lambda)|^2}, k = 0, 1, 2, \ldots m$$

noting that d and $\Im(\lambda)$ are not both zero. The remaining terms are bounded since $\frac{k^2}{(k-m-d)^2}$ is a decreasing function of k for $k \geq m+1$ and asymptotically converges to 1. Hence, we have

$$\begin{aligned} 2\pi \sum_{k=0}^{\infty} \frac{k^2 |g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k} &= 2\pi \sum_{k=0}^{m} \frac{k^2 |g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k} + 2\pi \sum_{k=m+1}^{\infty} \frac{k^2 |g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k} \\ &\leq 2\pi \frac{m^2}{d^2 + |\Im(\lambda)|^2} \sum_{k=0}^{m} |g_k^\star|^2 \frac{1}{2k} + 2\pi \frac{(m+1)^2}{(1-d)^2} \sum_{k=m+1}^{\infty} |g_k^\star|^2 \frac{1}{2k} \end{aligned}$$

Therefore,

$$\begin{aligned} 2\pi \sum_{k=0}^{\infty} \frac{k^2 |g_k^\star|^2}{|k+\lambda|^2} \frac{1}{2k} &\leq 4\pi C_2(\lambda) \sum_{k=0}^{\infty} |g_k^\star|^2 \frac{1}{2k+2} \\ &= 2C_2(\lambda) \left\| \frac{1}{2} S(g^\star) \right\|_{L^2(\mathbb{D})}^2, \end{aligned}$$

where

$$C_2(\lambda) = \begin{cases} 1, & \Re(\lambda) \geq 0 \\ \max\left\{ \left(\frac{m^2}{d^2 + |\Im(\lambda)|^2} \right), \left(\frac{m+1}{1-d} \right)^2 \right\}, & \Re(\lambda) = -m-d < 0. \end{cases}$$

Therefore, we conclude

$$\|\phi\|_{W^{2,1}(\mathbb{D})}^2 \leq C(\lambda) \|S(g^\star)\|_{L^2(\mathbb{D})}^2, \tag{4.19}$$

where $C(\lambda) = \frac{1}{4} C_1(\lambda) + \frac{1}{2} C_2(\lambda)$.

We now appeal to the properties of the Schwarz operator S defined in (4.3) which is bounded from $L^p(\partial\mathbb{D})$ to $L^p(\mathbb{D})$ for $1 < p < \infty$ [26]. In fact, the real part of the kernel in (4.3) is the Poisson kernel and the map $\tau \to \Re(S(\tau))$ is bounded from $L^p(\partial\mathbb{D})$ to $L^p(\mathbb{D})$ for all p (it is easy to see this is true for $p = \infty$ by the maximum principle). The imaginary part of the kernel is called the conjugate Poisson kernel, and the boundedness of mapping $\tau \to \Im(S(\tau))$ is equivalent to boundedness of the Hilbert transform on $L^p(\mathbb{D})$ which holds for $p \in (1, \infty)$, a classical result due to Riesz (1924) [27].

Therefore, taking $p = 2$ and appealing to the expression for $\frac{1}{2}S(g^\star)$ in (4.6), we have that

$$\|S(g^\star)\|^2_{L^2(\mathbb{D})} \le C\|g^\star\|^2_{L^2(\partial\mathbb{D})}.$$

Thus, by the inequality (2.11)

$$\|S(g^\star)\|^2_{L^2(\mathbb{D})} \le C\|g\|^2_{L^2(\partial\mathbb{D})} + C\|f\|^2_{L^2(\mathbb{D})}.$$

A similar estimate for $\overline{\psi}$ also applies since

$$\begin{aligned}
\|\overline{\psi}\|^2_{W^{2,1}(\mathbb{D})} &= \|\overline{\psi}\|^2_{L^2(\mathbb{D})} + \|\overline{\partial\psi}\|^2_{L^2(\mathbb{D})} \\
&= 2\pi \int_0^1 \sum_{k=0}^\infty \frac{|\overline{g^\star_{-k}}|^2}{|k+\lambda|^2} r^{2k+1}\, dr + 2\pi \int_0^1 \sum_{k=0}^\infty \frac{k^2|\overline{g^\star_{-k}}|^2}{|k+\lambda|^2} r^{2k-1}\, dr \\
&= 2\pi \sum_{k=0}^\infty \frac{|\overline{g^\star_{-k}}|^2}{|k+\lambda|^2} \frac{1}{2k+2} + 2\pi \sum_{k=0}^\infty \frac{k^2|\overline{g^\star_{-k}}|^2}{|k+\lambda|^2} \frac{1}{2k} \\
&\le C(\lambda)\|S(\overline{g^\star})\|^2_{L^2(\mathbb{D})},
\end{aligned} \tag{4.20}$$

and $S(\overline{g^\star})$ satisfies the same estimate as $S(g^\star)$.

Moreover, since $w = T_{1,1}f$, it satisfies the estimate

$$\begin{aligned}
\|w\|^2_{W^{2,1}(\mathbb{D})} &\le C(\|T_{1,1}f\|^2_{L^2(\mathbb{D})} + \|T_{1,0}f\|^2_{L^2(\mathbb{D})} + \|T_{0,1}f\|^2_{L^2(\mathbb{D})}) \\
&\le C\|f\|^2_{L^2(\mathbb{D})},
\end{aligned}$$

where we used (2.10). Hence, the estimate (3.3) is established.

4.4. Existence and uniqueness of a weak solution

We now use a density argument to pass through the limit in the weak formulation of the problem (3.2). In particular, given g in $L^2(\partial\mathbb{D})$ and f in $L^2(\mathbb{D})$ one can find a sequence of functions $g_n \in C^1(\partial\mathbb{D})$ converging to g in $L^2(\partial\mathbb{D})$, and a sequence of functions $f_n \in C(\overline{\mathbb{D}})$ converging to f in $L^2(\mathbb{D})$. The corresponding solution u_n to the equation satisfies the inequality (3.3), and thus we can extract a weakly convergent subsequence u_{n_j} corresponding to data $\{g_{n_j}, f_{n_j}\}$ and converging weakly to an element u in $W^{2,1}(\mathbb{D})$.

Taking the inner product of (1.1) with a test function v and integrating by parts using Green's identity over the domain $\mathbb{D}$ (see appendix), we obtain the weak formulation for the problem

$$2\int_{\mathbb{D}} \partial u_{n_j} \overline{\partial v}\, dx\, dy + 2\int_{\mathbb{D}} \partial_{\overline{z}} u_{n_j}\, \partial_z \overline{v}\, dx\, dy + \lambda \int_0^{2\pi} u_{n_j}(e^{i\theta})\overline{v}(e^{i\theta})\, d\theta - \int_0^{2\pi} g_{n_j}(\theta)\, \overline{v}(e^{i\theta})\, d\theta = -\int_{\mathbb{D}} f_{n_j}\overline{v}\, dx\, dy. \tag{4.21}$$

Hence, passing to the limit in (4.21), we have that $u \in W^{2,1}(\mathbb{D})$ is a weak solution corresponding to the data $g \in L^2(\partial\mathbb{D})$ and $f \in L^2(\mathbb{D})$. That the solution is unique follows again from considering the homogeneous problem satisfied by the difference of two solutions. Since the domain is regular and the problem is homogeneous, the weak solution is in fact a classical solution and thus it is the zero solution.

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5. Controllability: proof of Theorem 3.2

Let $m \geq 0$ be the infimum of J over $L^2(\partial\mathbb{D})$. We now consider a minimizing sequence $g_n \in L^2(\partial\mathbb{D})$ of the functional (1.3), or in other words

$$m = \inf_{g \in L^2(\partial\mathbb{D})} J(u(g), g) = \lim_{n\to\infty} J(u(g_n), g_n), \tag{5.1}$$

and the corresponding sequence of weak solutions $u_n = u(g_n) \in W^{2,1}(\mathbb{D})$. Then, there exist constants $M > 0$ and $N \in \mathbb{N}$, such that $J(u(g_n), g_n) \leq M$ for all $n \geq N$. Therefore, we can extract a weakly convergent subsequence $g_n \to g^0 \in L^2(\partial\mathbb{D})$. Moreover, using the inequality (3.3) we can extract a weakly convergent subsequence u_{n_j} in $W^{2,1}(\mathbb{D})$ converging to an element $u \in W^{2,1}(\mathbb{D})$. By passing to the limit as $n_j \to \infty$ in the weak formulation (4.21), we conclude that u is the unique weak solution to the problem corresponding to data f and boundary data g^0. In addition, u_{n_j} converges to u strongly in $L^2(\mathbb{D})$ by compactness.

Now, since the norm is convex and lower semicontinuous it is weakly lower semicontinuous which implies

$$\begin{aligned} J(u, g^0) &= \int_{\mathbb{D}} |u|^2 \, dx\, dy + \int_0^{2\pi} |g^0(\theta)|^2 \, d\theta \\ &\leq \liminf_n \int_{\mathbb{D}} |u_n|^2 \, dx\, dy + \liminf_n \int_0^{2\pi} |g_n(\theta)|^2 \, d\theta \\ &\leq \liminf_n J(u_n(g_n), g_n) = m. \end{aligned}$$

Therefore, the minimum of J is realized by the function g^0. Uniqueness follows from strict convexity of $J(u(g), g)$ in g which is a consequence of strict convexity of the norm and the convexity of $u(g)$.

We next characterize the optimal control and show that in fact $u(g^0)$ is a classical solution. In fact, we can express $u(g)$ as $u = Lg^\star + T_{1,1}f$, where $L : L^2(\partial\mathbb{D}) \to W^{2,1}(\mathbb{D})$ denotes the solution map defined by

$$Lg = u \iff -2\int_{\mathbb{D}} \partial_z u \, \partial_{\overline{z}} \overline{v} \, dx\, dy - 2\int_{\mathbb{D}} \partial_{\overline{z}} u \, \partial_z \overline{v} \, dx\, dy - \lambda \int_0^{2\pi} u(e^{i\theta}) \, \overline{v}(e^{i\theta}) \, d\theta + \int_0^{2\pi} g(\theta) \, \overline{v}(e^{i\theta}) \, d\theta = 0, \tag{5.2}$$

$\forall v \in W^{2,1}(\mathbb{D})$, which is well-defined by Theorem 3.1.

We next derive the expression of the optimal control g^0 by taking the variation of the functional J with respect to g.

First, using the expression $u = Lg^\star + T_{1,1}f$ we have

$$J = \langle Lg^\star + T_{1,1}f, Lg^\star + T_{1,1}f \rangle_{\mathbb{D}} + \langle g^\star, g^\star \rangle_{\partial\mathbb{D}},$$

which after taking the variation, and denoting by $L^\star$ the adjoint of L with respect to the $L^2(\mathbb{D})$ inner product, we obtain

$$\begin{aligned} 0 = \delta J = &\langle L^\star L g^\star, \delta g^\star \rangle_{\partial\mathbb{D}} + \langle \delta g^\star, L^\star L g^\star \rangle_{\partial\mathbb{D}} + \langle L^\star T_{1,1} f, \delta g^\star \rangle_{\partial\mathbb{D}} + \langle \delta g^\star, L^\star T_{1,1} f \rangle_{\partial\mathbb{D}} + \langle g^\star, \delta g^\star \rangle_{\partial\mathbb{D}} \\ &+ \langle \delta g^\star, g^\star \rangle_{\partial\mathbb{D}}. \end{aligned}$$

Therefore

$$2\Re\langle L^\star L g^\star, \delta g^\star \rangle_{\partial\mathbb{D}} + 2\Re\langle L^\star T_{1,1} f, \delta g^\star \rangle_{\partial\mathbb{D}} + 2\Re\langle g^\star, \delta g^\star \rangle_{\partial\mathbb{D}} = 0,$$

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which holds for all $\delta g^\star \in L^2(\partial\mathbb{D})$ and thus

$$(I + L^\star L)g^\star = -L^\star T_{1,1} f. \tag{5.3}$$

Now, the operator $I + L^\star L$ is bounded on $L^2(\partial\mathbb{D})$ and coercive since

$$\langle (I + L^\star L)h, h\rangle_{\partial\mathbb{D}} = \|h\|^2_{L^2(\partial\mathbb{D})} + \|L^\star h\|^2_{L^2(\partial\mathbb{D})} \geq \|h\|^2_{L^2(\partial\mathbb{D})},$$

for all $h \in L^2(\partial\mathbb{D})$. Therefore, by the Lax–Milgram theorem, $(I + L^\star L)$ has a bounded inverse on $L^2(\partial\mathbb{D})$ and hence we conclude

$$g^0(z) = -(I + L^\star L)^{-1} L^\star T_{1,1} f + zTf + \overline{z}\overline{T}f + \lambda T_{1,1} f, \quad z \in \partial\mathbb{D}. \tag{5.4}$$

Moreover, we can explicitly compute the adjoint of the map L. Let $h \in L^2(\mathbb{D})$ and set v to be the solution to the problem

$$\Delta v = h \qquad \text{in} \quad \mathbb{D}, \tag{5.5}$$

subject to the boundary condition

$$\frac{\partial}{\partial n} v + \lambda v = 0 \quad \text{on} \quad \partial\mathbb{D}, \tag{5.6}$$

then, given any $g \in L^2(\partial\mathbb{D})$, we have using Green's identity

$$\begin{aligned}
\langle Lg, h\rangle_{\mathbb{D}} &= \langle Lg, \Delta v\rangle_{\mathbb{D}} \\
&= -2 \int_{\mathbb{D}} \partial_z Lg\, \partial_{\overline{z}} \overline{v}\, dx\, dy - 2 \int_{\mathbb{D}} \partial_{\overline{z}} Lg\, \partial_z \overline{v}\, dx\, dy + \int_0^{2\pi} Lg(e^{i\theta})\, \frac{\partial}{\partial n} \overline{v}(e^{i\theta})\, d\theta \\
&= \lambda \int_0^{2\pi} Lg(e^{i\theta})\, \overline{v}(e^{i\theta})\, d\theta - \int_0^{2\pi} g(\theta)\, \overline{v}(e^{i\theta})\, d\theta + \int_0^{2\pi} Lg(e^{i\theta})\, \frac{\partial}{\partial n} \overline{v}(e^{i\theta})\, d\theta \\
&= -\int_0^{2\pi} g(\theta)\, \overline{v}(e^{i\theta})\, d\theta \\
&= \langle g, L^\star h\rangle_{\partial\mathbb{D}},
\end{aligned}$$

where we used the definition of L as the solution map defined in (5.2).

Hence, the adjoint $L^\star : L^2(\mathbb{D}) \to L^2(\partial\mathbb{D})$ is defined as the solution map to the problem

$$\begin{aligned}
& L^\star h = q \iff q = -v|_{\partial\mathbb{D}} \quad \text{and} \\
& -2 \int_{\mathbb{D}} \partial_z v\, \partial_{\overline{z}} \overline{u}\, dx\, dy - 2 \int_{\mathbb{D}} \partial_{\overline{z}} v\, \partial_z \overline{u}\, dx\, dy - \lambda \int_0^{2\pi} v(e^{i\theta})\, \overline{u}(e^{i\theta})\, d\theta = \int_{\mathbb{D}} h\overline{u}\, dx\, dy \quad \forall u \in W^{2,1}(\mathbb{D}).
\end{aligned} \tag{5.7}$$

In other words, the adjoint is the trace of the solution to the problem (5.5)–(5.6) and is thus bounded $L^2(\mathbb{D}) \to W^{2,1/2}(\partial\mathbb{D})$. Using a higher regularity argument via the Agmon–Douglis–Nirenberg method [1], we can also deduce the regularity property $L^\star : W^{2,s}(\mathbb{D}) \to W^{2,s+1/2}(\partial\mathbb{D})$.

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Now, from (5.3) we have that

$$g^\star = -L^\star L g^\star - L^\star T_{1,1} f. \tag{5.8}$$

The above relation shows that the optimal control g^0 is in fact more regular, and thus the corresponding solution u to this control is a classical solution. In particular, we conclude that $Lg^\star \in W^{2,1}(\mathbb{D})$. On the other hand, $L^\star$ is the solution map $L^\star h = v|_{\partial\mathbb{D}}$ to the BVP (5.5)–(5.6), and thus $L^\star L g^\star \in W^{2,3/2}(\partial\mathbb{D}) \subset C(\partial\mathbb{D})$. Moreover, $T_{1,1}f \in W^{2,2}(\mathbb{D})$, and accordingly $L^\star T_{1,1} f \in W^{2,5/2}(\partial\mathbb{D}) \subset C^1(\partial\mathbb{D})$. Thus, $g^\star \in W^{2,3/2}(\partial\mathbb{D}) \cap C(\partial\mathbb{D})$. Iterating again using (5.8), we conclude $L^\star L g^\star \in W^{2,3}(\partial\mathbb{D}) \subset C^1(\partial\mathbb{D})$. Therefore, $g^\star$ is C^1 while the optimal control g^0 may be expressed as

$$g^0(\zeta) \equiv g^\star(\zeta) + \zeta T_{0,1} f + \overline{\zeta} T_{1,0} f - \lambda T_{1,1} f, \quad \zeta \in \partial\mathbb{D},$$

using (2.5). If we further assume $f \in C(\overline{\mathbb{D}})$ then we conclude g^0 is also $C^1(\partial\mathbb{D})$ and thus the corresponding optimal solution u^0 is a classical solution in $C^2(\overline{\mathbb{D}})$. This completes the proof of Theorem 3.3.

6. The case of $-\lambda = k_0 \in \mathbb{N}$: proof of Theorems 3.3, 3.4

If $-\lambda = k_0 \in \mathbb{N}$, then (4.12) implies that

$$\phi_k = \frac{g_k^\star}{k+\lambda}, \quad k \in \mathbb{N} \setminus \{k_0\}, \quad \phi_{k_0} = A \text{ if } g_{k_0}^\star = 0; \quad \phi(z) = \sum_{k=0,\ k\neq k_0}^{\infty} \frac{g_k^\star}{k+\lambda} z^k + A z^{k_0}, \quad z \in \mathbb{D} \tag{6.1}$$

where $A \in \mathbb{C}$ is an arbitrary constant. In this case,

$$\phi(z) = \sum_{k=0, k\neq k_0}^{\infty} \frac{g_k^\star}{k-k_0} z^k + A z^{k_0}. \tag{6.2}$$

If the compatibility condition $g_{k_0}^\star = 0$ is not satisfied, the problem (4.12) is not solvable for the case $-\lambda \in \mathbb{N}$ due to incompatibility. This means that under the assumption $-\lambda = k_0 \in \mathbb{N}$, if $\phi = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic solution to (4.8) then we must have $g_{k_0}^\star = 0$. In other words, due to the fact that $G^\star(z)$ is holomorphic, we may express $g_{k_0}^\star$ as

$$g_{k_0}^\star = \frac{1}{k_0!} G^{*(k_0)}(0)$$

and the compatibility condition $g_{k_0}^\star = 0$ can be expressed using the Cauchy integral formula as

$$\int\limits_{|\zeta|=1} G^\star(\zeta)\overline{\zeta}^{k_0} \frac{d\zeta}{\zeta} = 0, \quad \zeta \in \partial\mathbb{D}$$

and the following Theorem can be deduced.

Theorem 6.1. *Let $-\lambda = k_0 \in \mathbb{N}$. If $\phi = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic solution to (4.8), then $G^\star$ must satisfy the compatibility condition*

$$g_{k_0}^\star = 0 \quad \texttt{or} \quad \int\limits_{|\zeta|=1} G^\star(\zeta)\overline{\zeta}^{k_0} \frac{d\zeta}{\zeta} = 0. \tag{6.3}$$

If the compatibility condition (6.3) is satisfied, then the problem (4.8) has the solution (6.2) for $-\lambda \in \mathbb{N}$ with A being an arbitrary complex number. If the compatibility condition (6.3) is not satisfied, then the problem (4.8) is not solvable when $-\lambda \in \mathbb{N}$.

Similarly, one can conclude that

$$\psi(z) = \sum_{k=0, k\neq k_0}^{\infty} \frac{\overline{g^\star_{-k}}}{k-k_0}\, z^k + Bz^{k_0}, \tag{6.4}$$

where B is an arbitrary constant, assuming the compatibility condition $g^\star_{-k} = 0$ is satisfied by $H^\star$. An analogous theorem can be stated for the problem (4.9):

Theorem 6.2. *Let $-\lambda = k_0 \in \mathbb{N}$. If $\psi = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic solution to (4.9), then $H^\star$ must satisfy the compatibility condition*

$$\overline{g}^\star_{-k_0} = 0 \quad \texttt{or} \quad \int\limits_{|\zeta|=1} H^\star(\zeta)\overline{\zeta}^{k_0}\frac{d\zeta}{\zeta} = 0. \tag{6.5}$$

If the compatibility condition (6.5) is satisfied, then the problem (4.9) has the solution (6.4) with B being an arbitrary complex number. If the compatibility condition (6.5) is not satisfied, then the problem (4.9) is not solvable when $-\lambda \in \mathbb{N}$.

Note that compatibility conditions (6.3) and (6.5) can be expressed as

$$g^\star_{\pm k_0} = 0 \quad \texttt{or} \quad \int\limits_{|\zeta|=1} g^\star(\zeta)\zeta^{\mp k_0}\frac{d\zeta}{\zeta} = 0. \tag{6.6}$$

Moreover, since $-\lambda = k_0 \in \mathbb{N}$, the problem

$$\Delta u = 0 \quad \text{in} \quad \mathbb{D} \tag{6.7}$$

subject to the boundary condition

$$\frac{\partial}{\partial n}u + \lambda u = 0 \quad \text{on} \quad \partial\mathbb{D}, \tag{6.8}$$

admits a family of solutions of the form

$$u(z) = Az^{k_0} + B\overline{z}^{k_0}, \tag{6.9}$$

for any $A, B \in \mathbb{C}$, cf. [25]. This means that the optimal control problem of minimizing (1.3) subject to the BVP (1.1)–(1.2) is not well-posed in this case. In fact, for any choice of g, the lack of uniqueness of solution u implies a lack of controllability. In particular, for any $g \in L^2(\partial\mathbb{D})$, we may choose A, and B so that the value of the functional is arbitrarily large.

Therefore, we impose additional conditions (3.6) and (3.7) on the boundary data u which determine the values of the coefficients A and B uniquely

$$\int\limits_{\partial\mathbb{D}} u(\zeta)\,\overline{\zeta}^{k_0+1}\,d\zeta = u_1,$$

and

$$\int_{\partial\mathbb{D}} u(\zeta)\ \zeta^{k_0-1}\, d\zeta = u_2,$$

where u_1 and u_2 are given complex numbers.

This is justified by the following computations. If we take the inner product of z^{k_0} with u while paying attention to $\overline{z}^{\ k_0} = z^{-k_0}$ on $\partial\mathbb{D}$, then we have

$$u(z)\ \overline{z}^{k_0} = w(z)\overline{z}^{\ k_0} + \sum_{k=0,k\neq k_0}^{\infty} a_k z^{k-k_0} + \sum_{k=0,k\neq -k_0}^{\infty} b_k \overline{z}^{\ k+k_0} + A\overline{z}^{2k_0} + B \quad \text{on} \quad \partial\mathbb{D}.$$

Hence, by the Cauchy complex integral formula, we have

$$\begin{aligned}\int_{\partial\mathbb{D}} u(\zeta)\ \overline{\zeta}^{k_0+1}\, d\zeta &= \int_{\partial\mathbb{D}} \left(w(z)\overline{\zeta}^{k_0+1} + \sum_{k=0,k\neq k_0}^{\infty} a_k \zeta^{k-k_0-1} + \sum_{k=0,k\neq -k_0}^{\infty} b_k \overline{\zeta}^{\ k+k_0+1} + A\overline{\zeta}^{2k_0+1} + \frac{B}{\zeta} \right) d\zeta, \\ &= \int_{\partial\mathbb{D}} w(z)\overline{\zeta}^{k_0+1}\, d\zeta + 2\pi i B.\end{aligned}$$

Thus,

$$2\pi i B = u_1 - \int_{\partial\mathbb{D}} w(z)\overline{\zeta}^{k_0+1}\, d\zeta.$$

Similarly,

$$u(z)\ z^{k_0} = w(z)z^{k_0} + \sum_{k=0,k\neq -k_0}^{\infty} a_k z^{k+k_0} + \sum_{k=0,k\neq k_0}^{\infty} b_k \overline{z}^{\ k-k_0} + A + Bz^{2k_0} \quad \text{on} \quad \partial\mathbb{D}.$$

Therefore, using the Cauchy integral formula, we have

$$\begin{aligned}\int_{\partial\mathbb{D}} u(\zeta)\ \zeta^{k_0-1}\, d\zeta &= \int_{\partial\mathbb{D}} \left(w(z)\zeta^{k_0-1} + \sum_{k=0,k\neq -k_0} a_k \zeta^{k+k_0-1} + \sum_{k=0,k\neq k_0} b_k \overline{\zeta}^{\ k-k_0+1} + \frac{A}{\zeta} + B\zeta^{2k_0-1} \right) d\zeta \\ &= \int_{\partial\mathbb{D}} w(z)\zeta^{k_0-1}\, d\zeta + 2\pi i A,\end{aligned}$$

from which, we get

$$2\pi i A = u_2 - \int_{\partial\mathbb{D}} w(z)\zeta^{k_0-1}\, d\zeta.$$

Hence, the choice of u_1 determines the coefficient B and the choice of u_2 determines the coefficient A, which guarantees uniqueness of solution.

Repeating the estimates on the $W^{2,1}$ norm of the solution u as in the proof of Theorem 3.1 and using the expressions for A and B, the estimate (3.8) easily follows.

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Remark. Notice for example that when $f(z) = 0$, $\lambda = -2$, $g(z) = c_1\overline{z}^2 + c_2 z^2 + c_3 z^3$, problem (1.1)–(1.2) is not solvable for $|c_1|^2 + |c_2|^2 \neq 0$, $\forall c_3 \in \mathbb{C}$ due to the fact that the compatibility condition is violated and it is solvable for $|c_1|^2 + |c_2|^2 = 0$, $\forall c_3 \in \mathbb{C}$. Therefore, if the compatibility conditions are not satisfied, there is no solution to the original problem.

6.1. *Existence of a weak solution*

Given $f \in C(\overline{\mathbb{D}})$, and $g \in L^2(\partial\mathbb{D})$ satisfying the compatibility condition

$$\int\limits_{|\zeta|=1} g^{\star}(\zeta)\zeta^{\pm k_0}\frac{d\zeta}{\zeta} = 0,$$

we can find a sequence of functions $g_n \in C^1(\partial\mathbb{D})$ converging to $g \in L^2(\partial\mathbb{D})$ as $n \to \infty$, and such that the same compatibility condition

$$\int\limits_{|\zeta|=1} g_n^{\star}(\zeta)\zeta^{\pm k_0}\frac{d\zeta}{\zeta} = 0$$

is satisfied. Such a sequence can be constructed by truncating the Fourier series for $g^\star$ to construct an approximate sequence $g_n^\star$, which guarantees that the coefficients corresponding to z^{k_0} and $\overline{z}^{k_0}$ remain zero. Accordingly, $g_n(\zeta)$ may be selected to be $g_n^\star(\zeta) + \zeta T_{0,1}f(\zeta) + \overline{\zeta}T_{1,0}f(\zeta) - k_0 T_{1,1}f(\zeta)$.

Therefore, if we impose the conditions (3.6) and (3.7), by the result of the previous section there exists a unique solution u_n corresponding to the data g_n and f to the BVP (1.1) with the boundary condition (1.2) when $-\lambda = k_0 \in \mathbb{N}$. Integrating by parts via Green's formula, we have that u_n satisfies (4.21). We now invoke the a priori estimate (3.8), to extract a weakly convergent subsequence u_{n_j} converging to an element $u \in W^{2,1}(\mathbb{D})$. Passing to the limit in (4.21) as $n \to \infty$, we obtain the weak formulation (3.2) satisfied by u. The same uniqueness argument as in the case of Theorem 3.1 also holds.

6.2. *Proof of Theorem 3.4*

The proof of Theorem 3.4 on existence of the optimal control follows the same argument of proof of Theorem 3.2, using the inequality (3.8). Under the assumptions $u_1 = u_2 = 0$, the solution map $L : g \to u$ is a linear map. Here, the admissible set of controls includes the compatibility conditions (3.4). In other words, given $f \in C(\overline{\mathbb{D}})$ we minimize the functional J over the closed subset K of $L^2(\partial\mathbb{D})$:

$$K := \left\{ g \in L^2(\partial\mathbb{D}) : \int\limits_{|\zeta|=1} g(\zeta)\overline{\zeta}^{\pm k_0}\frac{d\zeta}{\zeta} = \int\limits_{|\zeta|=1}\left(\zeta T_{0,1}f(\zeta) + \overline{\zeta}T_{1,0}f(\zeta) - k_0 T_{1,1}f(\zeta)\overline{\zeta}^{\pm k_0}\right)\frac{d\zeta}{\zeta} \right\}$$

By finding a minimizing sequence $\{g_n\} \in K$ of the functional J, we can extract a weakly convergent subsequence $\{g_{n_j}\}$ converging weakly to an element $g^0 \in L^2(\partial\mathbb{D})$. Moreover, g^0 must satisfy the same compatibility conditions and thus belongs to K. From (3.8) we have a uniform bound on $u(g_n)$, and thus we can extract a corresponding subsequence of weak solutions $u_{n_j} = u(g_{n_j})$ converging weakly to u in $W^{2,1}(\mathbb{D})$. It is easy to see again that this element u is the weak solution corresponding to g^0 and indeed satisfies the conditions (3.6) and (3.7). In particular, the trace map is compact $W^{2,1}(\mathbb{D}) \to L^2(\partial\mathbb{D})$, and thus u_{n_j} converges strongly to u on the boundary in $L^2(\partial\mathbb{D})$, and thus we can pass to the limit in conditions (3.6) and (3.7) satisfied by u_{n_j}, which implies u also satisfies (3.6) and (3.7). The same argument as before applies to show that g^0 is the unique minimizer of J over the set K.

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6.3. Some remarks

The following identities hold

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\frac{d\zeta}{\zeta}=\frac{1}{2\pi i}\int\limits_0^{2\pi}\frac{d(e^{i\theta})}{e^{i\theta}}=\frac{i}{2\pi i}\int\limits_0^{2\pi}d\theta=1 \tag{6.10}$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\zeta^{\pm k}\frac{d\zeta}{\zeta}=\frac{1}{2\pi i}\int\limits_0^{2\pi}e^{\pm ik\theta}\frac{d(e^{i\theta})}{e^{i\theta}}=\frac{i}{2\pi i}\int\limits_0^{2\pi}e^{\pm ik\theta}d\theta=0,\ \forall k\in\mathbb{N} \tag{6.11}$$

By (4.10),

$$g^\star(\zeta)=\sum_{k=0}^{\infty}g_k^\star\zeta^k+\sum_{k=0}^{\infty}g_{-k}^\star\zeta^{-k}=\sum_{k=0}^{\infty}g_k^\star\zeta^k+\sum_{k=1}^{\infty}g_{-k}^\star\overline{\zeta}^k,\quad \zeta\in\partial\mathbb{D}$$

and by (4.6),

$$G^\star(z)=\frac{1}{2}S(g^\star(.))(z)=\frac{1}{2}\Big\{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}(g^\star(\zeta))\left(\frac{2\zeta}{\zeta-z}-1\right)\frac{d\zeta}{\zeta}+ic\Big\}=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}g^\star(\zeta)\left(\frac{\zeta}{\zeta-z}-\frac{1}{2}\right)\frac{d\zeta}{\zeta}+\frac{i}{2}c$$

$$=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}g^\star(\zeta)\left(\frac{1}{1-z\overline{\zeta}}-\frac{1}{2}\right)\frac{d\zeta}{\zeta}+\frac{i}{2}c$$

$$=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\left(\sum_{k=0}^{\infty}g_k^\star\zeta^k+\sum_{k=0}^{\infty}g_{-k}^\star\zeta^{-k}\right)\left(\sum_{h=0}^{\infty}(z\overline{\zeta})^h-\frac{1}{2}\right)\frac{d\zeta}{\zeta}+\frac{i}{2}c.$$

Here, only terms $k=h$ are of particular interest, and the rest all vanish due to (6.10) and (6.11) so that

$$G^\star(z)=\sum_{k=1}^{\infty}g_k^\star z^k+g_0^\star+\frac{i}{2}c.$$

Clearly, $G^\star(z)$ is fully determined by the holomorphic coefficients of $g^\star(z)$ up to a constant term.

Appendix A. Green's identity

Let $u(z)$ and $v(z)$ be two complex valued functions defined on the unit disk and in $C^2(\mathbb{D})$. Green's identity states that on any smooth domain in the plane, we have

$$\int\limits_{\mathbb{D}}\Delta u\,\overline{v}\,dx\,dy=-\int\limits_{\mathbb{D}}\nabla u\cdot\nabla\overline{v}\,dx\,dy+\int\limits_{\partial\mathbb{D}}\frac{\partial u}{\partial n}\overline{v}\,ds$$

where $ds=\sqrt{x'^2(\theta)+y'^2(\theta)}\,d\theta$ is the infinitesimal arclength and $(x(\theta),y(\theta))$ is a parametrization of the boundary $\partial\mathbb{D}$, and n is the outward unit normal to the boundary.

Since the domain is the unit disk and $\nabla u\cdot\nabla\overline{v}=2\partial u\overline{\partial}\overline{v}+2\overline{\partial}u\partial\overline{v}$ we have

$$\int\limits_{\mathbb{D}}\Delta u\overline{v}\,dx\,dy=-2\int\limits_{\mathbb{D}}\partial u\overline{\partial}\overline{v}\,dx\,dy-2\int\limits_{\mathbb{D}}\overline{\partial}u\partial\overline{v}\,dx\,dy+\int\limits_0^{2\pi}\frac{\partial u}{\partial n}(e^{i\theta})\overline{v}(e^{i\theta})d\theta.$$

Moreover, since our boundary condition on $\partial\mathbb{D}$ is

$$\frac{\partial u}{\partial n} + \lambda u = g,$$

the relation becomes

$$\int_{\mathbb{D}} \Delta u \overline{v}\, dx\, dy = -2 \int_{\mathbb{D}} \partial u \overline{\partial} \overline{v}\, dx\, dy - 2 \int_{\mathbb{D}} \overline{\partial} u \partial \overline{v}\, dx\, dy - \lambda \int_0^{2\pi} u(e^{i\theta}) \overline{v}(e^{i\theta})\, d\theta + \int_0^{2\pi} g(e^{i\theta}) \overline{v}(e^{i\theta})\, d\theta.$$

Using $\Delta u = f$, we finally have

$$-2 \int_{\mathbb{D}} \partial u \overline{\partial} \overline{v}\, dx\, dy - 2 \int_{\mathbb{D}} \overline{\partial} u \partial \overline{v}\, dx\, dy - \lambda \int_0^{2\pi} u(e^{i\theta}) \overline{v}(e^{i\theta})\, d\theta + \int_0^{2\pi} g(e^{i\theta}) \overline{v}(e^{i\theta})\, d\theta = \int_{\mathbb{D}} f \overline{v}\, dx\, dy,$$

which is the weak formulation of the problem given in (3.2).

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