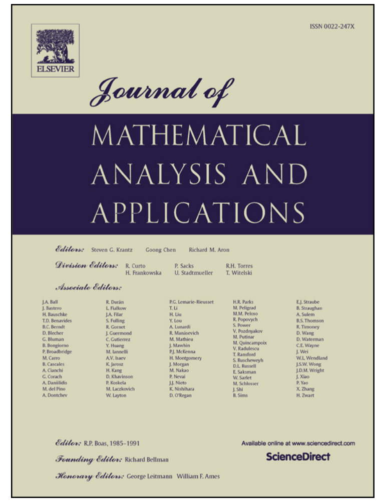

Symplectic rigidity of real and complex polydiscs

Yat-Sen Wong

PII: S0022-247X(17)31001-6
DOI: https://doi.org/10.1016/j.jmaa.2017.11.008
Reference: YJMAA 21806

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 14 June 2017

SYMPLECTIC RIGIDITY OF REAL AND COMPLEX POLYDISCS

YAT-SEN WONG

ABSTRACT. In $\mathbb{R}^{2n}$ with its standard symplectic structure, the complex polydisc, $\mathbb{D}^{2n}_{\mathbb{C}}(r)$, is constructed as the product of n open complex discs of radius r. When $n = 2$, the real polydisc, $\mathbb{D}^4_{\mathbb{R}}(r)$, is constructed as the product of 2 open real/Lagrangian discs of radius r. Sukhov and Tumanov recently showed that $\mathbb{D}^4_{\mathbb{C}}(1)$ and $\mathbb{D}^4_{\mathbb{R}}(1)$ are not symplectically equivalent. We extend this result in two ways. First we give the necessary and sufficient conditions for an orthogonal image of $\mathbb{D}^4_{\mathbb{C}}(1)$ to be symplectically equivalent to $\mathbb{D}^4_{\mathbb{C}}(1)$. Second, we show that for all $r \geq 1$ and $n \geq 1$, $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ is not symplectically equivalent to $\mathbb{D}^4_{\mathbb{C}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$.

1. Introduction

The problem of symplectic rigidity has been studied for a long time. The first striking result was obtained by Gromov [3], which states that one can symplectically embed a sphere into a cylinder only if the radius of the sphere is less than or equal to the radius of the cylinder. Following Gromov's work, many results on symplectic rigidity were obtained for various domains. For example, McDuff [5] studied when a 4-dimensional ellipsoid can be symplectically embedded in a ball; Guth [4] gave an asymptotic result on when a polydisc $\mathbb{D}^2_{\mathbb{C}}(r_1) \times \cdots \times \mathbb{D}^2_{\mathbb{C}}(r_n)$ can be symplectically embedded into another.

Sukhov and Tumanov [7] applied techniques in classical complex analysis to a problem of symplectic rigidity. They showed that the real bi-disc $\mathbb{D}^4_{\mathbb{R}}(1) = \{(z_1, z_2) \in \mathbb{C}^2 : |x_1|^2 + |x_2|^2 < 1, |y_1|^2 + |y_2|^2 < 1\}$ cannot be symplectically embedded into the complex cylinder $\mathbb{D}^2_{\mathbb{C}}(1) \times \mathbb{C}$ of radius 1. If we consider the real bidisc as obtained from a non-holomorphic change of coordinates

$$T_0 : (x_1, y_1, x_2, y_2) \mapsto (x_1, x_2, y_1, y_2)$$

of $\mathbb{D}^4_{\mathbb{C}}(1)$, then the result of Sukhov and Tumanov shows that $T_0(\mathbb{D}^4_{\mathbb{C}}(1))$ is not symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$ itself.

In this paper, we apply the complex analysis techniques used by Sukhov and Tumanov [7] to solve the problem of symplectic rigidity on different domains: real bidisc and its modifications.

Let $x_1, y_1, \ldots, x_n, y_n$ be the standard coordinates on the $2n$-dimensional Euclidean space $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the standard symplectc form on the space is given by $dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. All symplectic embeddings considered in this paper will be with respect to the standard symplectic form on $\mathbb{R}^{2n}$, unless otherwise specified. For $n > 1$, define the real $2n$-dimensional complex n-disc of radius r by

Date: Nov 2, 2014.
2010 *Mathematics Subject Classification.* Primary 53D05.

$\mathbb{D}^{2n}_{\mathbb{C}}(r) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_i| < r \text{ for } i = 1, \cdots, n\}$. When $n = 1$, $\mathbb{D}^2_{\mathbb{C}}(r)$ is the standard real 2-dimensional disc in $\mathbb{C}$ centered at the origin with radius r. To simplify the notation, we will write $\mathbb{D}^2_{\mathbb{C}}(r)$ as $\mathbb{D}(r)$ and $\mathbb{D}^2_{\mathbb{C}}(1)$ as $\mathbb{D}$. When $n = 2$, $\mathbb{D}^4_{\mathbb{C}}(r)$ is called complex bidisc of radius r. We also define the real bidisc in $\mathbb{C}^2$ by $\mathbb{D}^4_{\mathbb{R}} = \{(z_1, z_2) \in \mathbb{C}^2 : |x_1|^2 + |x_2|^2 < r^2, |y_1|^2 + |y_2|^2 < r^2\}$.

The first main result of this paper generalizes the result of Sukhov and Tumanov [7] mentioned above: if $T \in O(4)$ is any orthogonal transformation on $\mathbb{R}^4 = \mathbb{C}^2$, then $T(\mathbb{D_C}^4(1))$ is symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$ if and only if the image of $\mathbb{D}^4_{\mathbb{C}}(1)$ under T agrees with the image of $\mathbb{D}^4_{\mathbb{C}}(1)$ under a unitary transformation. We will give a more precise statement in Section 3.

The second result of this paper considers a high dimensional analogy of the first result. We will show that for $r \geq 1$ and $n \geq 2$, $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ is not symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$.

There are a lot of open problems concerning symplectic rigidity, for instance it is not known that whether $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ is symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ when $r < 1$. The results in this paper only show that such symplectomorphism does not exist when $r \geq 1$. Another interesting open problem is to characterize when two polydiscs $\mathbb{D}^2_{\mathbb{C}}(r_1) \times \cdots \times \mathbb{D}^2_{\mathbb{C}}(r_n)$ and $\mathbb{D}^2_{\mathbb{C}}(s_1) \times \cdots \times \mathbb{D}^2_{\mathbb{C}}(s_n)$ are symplectomorphic.

2. J-holomorphic discs and symplectic manifolds

In this section we will recall some basic properties of J-holomorphic discs and symplectic manifolds.

Definition 2.1. A smooth map $\phi : (M, J) \to (M', J')$ from one almost complex manifold to another is said to be (J, J')-holomorphic if its derivative $d\phi$ is complex linear, that is

$$d\phi \circ J = J' \circ d\phi. \tag{2.1}$$

Denote by J_{st} the standard complex structure of $\mathbb{C}^n$, that is, J_{st} is $\mathbb{R}$-linear and $J^2_{\text{st}} = -I$ where I is the identity map. A J-holomorphic disc, also known as a pseudo-holomorphic disc, is a (J_{st}, J)-holomorphic map

$$u : \mathbb{D} \to M$$

from $\mathbb{D}$ to an almost complex manifold (M, J).

In local coordinates $z \in \mathbb{C}^n$, an almost complex structure J is represented by an $\mathbb{R}$-linear operator $J(z) : \mathbb{C}^n \to \mathbb{C}^n$ such that $J(z)^2 = -I$. Now the Cauchy-Riemann equations (2.1) for a J-holomorphic disc $z : \mathbb{D} \to \mathbb{C}^n$ can be written in the form

$$z_\eta = J(z)z_\xi,\ \zeta = \xi + i\eta \in \mathbb{D}.$$

It is possible to represent J by a complex $n \times n$ matrix function $A = A(z)$ so that we get the equivalent equations

$$z_{\overline{\zeta}} = A(z)\overline{z}_{\overline{\zeta}},\ \zeta \in \mathbb{D}. \tag{2.2}$$

We now recall how one constructs A from J for fixed z. Let $J : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{R}$-linear map so that $\det(J_{\text{st}} + J) \neq 0$, where $J_{\text{st}}v = iv$. Here we only consider J

such that $\det(J_{\mathrm{st}}+J) \neq 0$, for example when J is tamed by the standard symplectic structure on $\mathbb{C}^n$ (the notion of tamed is defined below). Set

$$Q = (J_{\mathrm{st}} + J)^{-1}(J_{\mathrm{st}} - J).$$

Lemma 2.2. *(*See, for example, Chapter 2 Section 1.1 of [1]*) $J^2 = -I$ if and only if $QJ_{st} + J_{st}Q = 0$.*

Notice that $QJ_{\mathrm{st}} + J_{\mathrm{st}}Q = 0$ is equivalent to Q being a linear anti-complex operator. Therefore Lemma 2.2 implies that there is a unique matrix $A \in \mathrm{Mat}(n, \mathbb{C})$ such that

$$Av = Q\overline{v}, v \in \mathbb{C}^n.$$

A symplectic manifold, (M, ω), consists of a smooth manifold M of dimension $2n$ and a closed, non-degenerate 2-form ω. A basic example is $M = \mathbb{C}^n$ with the coordinates $z_j = x_j + iy_j$, $j = 1, \ldots, n$, equipped with the standard symplectic form $\omega_{\mathrm{st}} = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z_j}$.

A symplectic form ω tames an almost complex structure J on M if $\omega(u, Ju) > 0$, for all $u \neq 0$. A basic example is $(M, \omega, J) = (\mathbb{C}^n, \omega_{\mathrm{st}}, J_{\mathrm{st}})$.

Given a matrix $B \in \mathbb{R}^{m\times n}$, the matrix norm of B is defined by

$$\|B\| = \sup\left\{\frac{|Bv|_{\mathbb{R}^m}}{|v|_{\mathbb{R}^n}} \text{ with } v \neq 0\right\}.$$

Lemma 2.3. *(*See, for example, Chapter 2 Section 1.1 of [1]*) Let J be an almost complex structure on $\mathbb{C}^n$, then J is tamed by ω_{st} if and only if the complex matrix A of J satisfies the condition*

$$\|A(z)\| < 1, \text{ for all } z \in \mathbb{C}^n. \tag{2.3}$$

For a map $u : \mathbb{D} \to \mathbb{C}^n$, the (symplectic) area of u is given by

$$\mathrm{Area}(u) = \int_{\mathbb{D}} u^*\omega_{\mathrm{st}}. \tag{2.4}$$

If J is ω_{st} tamed, we can consider the canonical Riemannian metric $g_J(X, Y) = \frac{1}{2}(\omega_{\mathrm{st}}(X, JY) + \omega_{\mathrm{st}}(Y, JX))$ determined by J and ω_{st}. Suppose u is a J-holomorphic disc. The symplectic area of u coincides with the area induced by g_J; in particular, it coincides with the Euclidean area if $J = J_{\mathrm{st}}$ (see [1] for more details).

3. Orthogonal transformation of complex bidisc

Let $T \in O(4)$ be an orthogonal transformation on $\mathbb{R}^4 \cong \mathbb{C}^2$. In this section we will give a necessary and sufficient condition for $T(\mathbb{D}^4_{\mathbb{C}}(1))$ to be symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$ with respect to the standard symplectic form on $\mathbb{C}^2$.

First of all, we define the notion of holomorphic radius and state a theorem proved by A. Sukhov and A. Tumanov [7] which provides a necessary condition on holomorphic radius for the existence of symplectic embedding.

Definition 3.1. Let Ω be a complex manifold. A closed set $A \subset \Omega$ is called a **(complex) analytic set** if it is, in a neighborhood of each of its points, the set of common zeros of a certain finite family of holomorphic functions.

In this paper we only consider closed analytic sets.

Definition 3.2. A point p of an analytic set A in a complex manifold Ω is called **regular** if there is a neighborhood U in Ω containing p such that $A \cap U$ is a complex submanifold of U. The **complex dimension** of this submanifold is said to be the dimension of A at its regular point p, and is denoted by $\dim_p A$. The set of all regular points of A is denoted by $\mathrm{reg}A$.

It is a fundamental result of complex analytic sets that the set of all regular points of an analytic set A is dense in A (see, for example, Section 2.3 of [2]).

Definition 3.3. A **purely m-dimensional analytic set** A is an analytic set such that for every $p \in \mathrm{reg}A$, we have $\dim_p A = m$.

Definition 3.4. Let G be a domain in $\mathbb{C}^n$ containing the origin. Denote by $\mathcal{O}_0^1(G)$ the set of purely one-dimensional analytic sets in G passing through the origin. Denote by $E(X)$ the Euclidean area of $X \in \mathcal{O}_0^1(G)$. The **holomorphic radius** $\mathrm{rh}(G)$ of G is defined as

$$\mathrm{rh}(G) = \inf\{\lambda > 0 : \exists X \in \mathcal{O}_0^1(G), E(X) = \pi\lambda^2\}.$$

Example 3.5. Let $\mathbb{B}^4(r)$ be the Euclidean ball of $\mathbb{C}^2$ with radius r, then $\mathrm{rh}(\mathbb{B}^4(r)) = r$. In fact the area $E(X)$ of $X \in \mathcal{O}_0^1(\mathbb{B}^4(r))$ is bounded from below by the area πr^2 of a section of the ball by a complex line through the origin (Lelong, 1950; see, for example, Section 15.3 of [2]).

The following theorem is known as Bishop's convergence theorem (see, for example, Section 15.5 of [2]). It will be used in the rest of the paper:

Theorem 3.6. *Let $\{A_j\}$ be a sequence of purely p-dimensional analytic subsets in a complex manifold Ω with locally uniformly bounded volumes:*

$$\mathrm{Vol}_{2p}(A_j \cap K) \leq M_K < \infty$$

for any compact set $K \subset \Omega$. Here M_K is a constant depending only on K. Then we can extract a subsequence from $\{A_j\}$ converging on compact subsets in Ω (in Hausdorff sense) to a purely p-dimensional analytic subset or to the empty set.

The following result is due to A. Sukhov and A. Tumanov [7]. It provides a necessary condition on holomorphic radius for the existence of symplectic embedding. This result will be used in the proof of Theorem 3.9.

Theorem 3.7. *([7]) Let G_1 be a domain in $\mathbb{C}^2$ containing the origin and let G_2 be a domain in $\mathbb{D}(R) \times \mathbb{C}$ for some $R > 0$. Assume there exists a symplectomorphism $\phi : G_1 \to G_2$, then $\mathrm{rh}(G_1) \leq R$.*

For $v = (v_1, \ldots, v_4), w = (w_1, \ldots, w_4) \in \mathbb{R}^4$, we denote the real inner product by $\langle v, w\rangle_{\mathbb{R}^4} = \sum_{j=1}^4 v_j w_j$. Similarly for $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{C}^2$, we denote the complex inner product by $\langle v, w\rangle_{\mathbb{C}^2} = \sum_{j=1}^2 v_j \overline{w_j}$. Notice that $\langle v, w\rangle_{\mathbb{R}^4} = \mathrm{Re}\langle v, w\rangle_{\mathbb{C}^2}$.

Let $L \subset \mathbb{C}^2$ be a real two dimensional plane. Denote by $L^{\perp_{\mathbb{R}^4}}$ the orthogonal complement of L with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}^4}$ and by $L^{\perp_{\mathbb{C}^2}}$ the orthogonal complement of L with respect to $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$. L is called a complex line when $v \in L$ if and only if $iv \in L$ for all $v \in \mathbb{C}^2$. By using the properties of inner product, the following lemma can be proved easily.

Lemma 3.8. *If $L \subset \mathbb{C}^2$ is a complex line, then:*

(1) $L^{\perp_{\mathbb{R}^4}} = L^{\perp_{\mathbb{C}^2}}$.
(2) $L^{\perp_{\mathbb{C}^2}}$ *is also a complex line.*

We denote by $\mathfrak{I}$ the set consisting of four diagonal matrices:

$$\mathfrak{I} = \left\{ \begin{pmatrix} 1 & & & \\ & a & & \\ & & 1 & \\ & & & b \end{pmatrix} : a = \pm 1, b = \pm 1 \right\}$$

The following is the main theorem of this section. We used the canonical identification between complex matrices on $\mathbb{C}^2$ and real matrices on $\mathbb{R}^4$:

Theorem 3.9. *Let $T \in O(4)$ be an orthogonal transformation. Then in $(\mathbb{R}^4, \omega_{std.})$, $T(\mathbb{D}^4_{\mathbb{C}}(1))$ is symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$ if and only if there exists $U \in U(2)$ such that $T(\mathbb{D}^4_{\mathbb{C}}(1)) = U(\mathbb{D}^4_{\mathbb{C}}(1))$ as a set.*

Idea of the proof. The if-part of the proof is straight forward. For the only-if-part, we start by considering the complex lines $H_1 = \{z_2 = 0\}$ and $H_2 = \{z_1 = 0\}$ in $\mathbb{C}^2$. The key argument is to show that $T(H_1)$ and $T(H_2)$ are also complex lines in $\mathbb{C}^2$. It is then obvious that $\widetilde{U}T \in \mathfrak{I}$ for some unitary matrix $\widetilde{U} \in U(2)$. Hence the result follows immediately.

Proof. ($\Leftarrow$) Suppose there exists a $U \in U(2)$ such that $T(\mathbb{D}^4_{\mathbb{C}}(1)) = U(\mathbb{D}^4_{\mathbb{C}}(1))$, then since $U \in U(2)$ is a linear symplectomorphism on $\mathbb{C}^2$, we can conclude that $T(\mathbb{D}^4_{\mathbb{C}}(1))$ is symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$.

($\Rightarrow$) Let (z_1, z_2) be the coordinate on $\mathbb{C}^2$. First of all, let $\partial\mathbb{D}^4_{\mathbb{C}}(1) \cap \partial\mathbb{B}^4(1) = S_1 \cup S_2$ where $S_1 = \{|z_1| = 1, z_2 = 0\}$ and $S_2 = \{z_1 = 0, |z_2| = 1\}$. Therefore S_1 and S_2 are contained in the complex line H_1 and H_2 respectively. For $i = 1, 2$, let $u_i, v_i \in \mathbb{C}^2$ be an orthonormal basis of $T(H_i)$ under the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ on $\mathbb{R}^4$. Note that $T(S_i)$, the image of S_i under T, can be parameterized by

$$\frac{1}{2}\left(t + \frac{1}{t}\right) u_i + \frac{1}{2i}\left(t - \frac{1}{t}\right) v_i$$

for $|t| = 1$ in $\mathbb{C}$. The complexification of $T(S_i)$, denoted by $\widetilde{T(S_i)}$, is given by the same parametrization but allowing $t \in \mathbb{CP}^1$. Here $\mathbb{CP}^n$ is the complex projective space of complex dimension n. Hence $\widetilde{T(S_i)}$ is a complex algebraic curve in $\mathbb{CP}^2$ parameterized by $t \in \mathbb{CP}^1$.

Notice that for $i = 1, 2$, $\widetilde{T(S_i)}$ passes through the origin in $\mathbb{C}^2$ if and only if u_i and v_i are $\mathbb{C}$-dependent.

Suppose $T\mathbb{D}^4_{\mathbb{C}}(1)$ is symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$, then Theorem 3.7 implies that $\mathrm{rh}(T(\mathbb{D}^4_{\mathbb{C}}(1))) \leq 1$. By the definition of $\mathrm{rh}(T(\mathbb{D}^4_{\mathbb{C}}(1)))$, consider a sequence of real numbers $\lambda_n \geq \mathrm{rh}(T(\mathbb{D}^4_{\mathbb{C}}(1)))$ that converges to $\mathrm{rh}(T(\mathbb{D}^4_{\mathbb{C}}(1)))$, and consider a sequence of $X_n \in \mathcal{O}^1_0(T(\mathbb{D}^4_{\mathbb{C}}(1)))$ with $E(X_n) = \pi\lambda_n^2$. By applying Theorem 3.6 to the sequence $\{X_n\}$, there exists $X \in \mathcal{O}^1_0(T(\mathbb{D}^4_{\mathbb{C}}(1)))$ such that $E(X) = \pi * \mathrm{rh}(T(\mathbb{D}^4_{\mathbb{C}}(1)))^2$. Suppose there exists $p \in \partial X \cap \partial\mathbb{B}^4(1)$ such that $p \in \mathrm{Int}(T(\mathbb{D}^4_{\mathbb{C}}(1)))$; then X is not entirely contained in $\mathbb{B}^4(1)$. Hence $E(X) > E(X \cap \mathbb{B}^4(1)) \geq \pi$ (see Example 3.5), which implies $\mathrm{rh}(T(\mathbb{D}^4_{\mathbb{C}}(1))) > 1$, a contradiction. Therefore $\partial X \subset \partial\mathbb{B}^4(1) \cap \partial T(\mathbb{D}^4_{\mathbb{C}}(1)) = T(S_1) \cup T(S_2)$ and X is a complex one dimensional analytic subset in $\mathbb{C}^2 \backslash (T(S_1) \cup T(S_2))$. Since $T(S_1) \cup T(S_2)$ is a real one dimensional curve, it is totally real. Hence, by the reflection principle for analytic sets (see, for example, Section 20.5 of [2]), X extends as a complex one dimensional analytic set to a neighborhood of $T(S_1) \cup T(S_2)$. By the uniqueness theorem X is contained in the complex algebraic curve $\widetilde{T(S_1)} \cup \widetilde{T(S_2)}$.

Since X contains the origin in $\mathbb{C}^2$, without loss of generality we can assume $\widetilde{T(S_1)}$ contains the origin. By the discussion above, we know that u_1 and v_1 are $\mathbb{C}$-dependent. Hence $T(H_1) = \mathrm{span}_{\mathbb{R}}\{u_1, v_1\} = \mathrm{span}_{\mathbb{R}}\{u_1, iu_1\} = \mathrm{span}_{\mathbb{C}}\{u_1\} = \widetilde{T(S_1)}$. This shows that $T(H_1)$ is a complex line.

By Lemma 3.8, $H_2 = H_1^{\perp_{\mathbb{C}^2}} = H_1^{\perp_{\mathbb{R}^4}}$. Since T is an orthogonal matrix, we have $T(H_2) = (T(H_1))^{\perp_{\mathbb{R}^4}} = (T(H_1))^{\perp_{\mathbb{C}^2}}$ where the last equality follows from Lemma 3.8 and the fact that $T(H_1) = \widetilde{T(S_1)} = \mathrm{span}_{\mathbb{C}}\{u_1\}$ is a complex line. Therefore Lemma 3.8 implies that $T(H_2)$ is a complex line.

We've shown that if T is orthogonal and $T(\mathbb{D}^4_{\mathbb{C}}(1))$ is symplectomorphic to $\mathbb{D}^4_{\mathbb{C}}(1)$, then T maps the complex lines $H_1 = \{z_2 = 0\}, H_2 = \{z_1 = 0\}$ to complex lines $T(H_1), T(H_2)$. Therefore there exists a unitary matrix $\widetilde{U} \in U(2)$ such that $\widetilde{U}T \in \mathfrak{I}$, it then follows that $T(\mathbb{D}^2) = U(\mathbb{D}^2)$ as a set for some $U \in U(2)$. □

4. Rigidity persisting through stabilization

The following is the main theorem of this section:

Theorem 4.1. *For $r \geq 1$ and $n \geq 2$, the domains $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ and $\mathbb{D}^4_{\mathbb{C}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ in $\mathbb{C}^n$ equipped with the standard symplectic form are not symplectomorphic.*

We will first give the proof for the case $r > 1$ by adapting the idea in the proof of Theorem 2.2 in [9]. We will then develop a new method to prove Theorem 4.1 for the case $r = 1$.

4.1. The case $r > 1$.

In the case $r > 1$, theorem 4.1 follows from a more general result:

Theorem 4.2. *If there exists a symplectic embedding $\phi : \mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r) \to \mathbb{D}(R) \times \mathbb{C}^{n-1}$ and $r > 1$, then $R > 1$.*

Proof. For $R > 0$, suppose there exists a symplectic embedding from $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(r)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$. It is proved in [9] that for every $1 \leq r_1 < \frac{2}{\sqrt{\pi}}$, there is a symplectic embedding from $\mathbb{B}^4(r_1)$ into $\mathbb{D}^4_{\mathbb{R}}(1)$. Take $1 < r_1 < \frac{2}{\sqrt{\pi}}$, then by combining these two embeddings, we obtain an embedding from $\mathbb{B}^{2n}(a)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$ where $a = \min(r, r_1) > 1$. Therefore, by Gromov's non-squeezing theorem [3], we have $R > 1$. □

4.2. The case $r = 1$.

In order to prove the case $r = 1$, we need the following theorem by A. Sukhov and A. Tumanov [6] regarding the existence of J-holomorphic discs. The original statement was about the triangular cylinder $\Delta \times \mathbb{C}^{n-1}$ where $\Delta = \{z \in \mathbb{C} : 0 < \mathrm{Im} z < 1 - |\mathrm{Re} z|\}$ instead of the circular cylinder $\mathbb{D} \times \mathbb{C}^{n-1}$. However, one can see the result still holds for the circular cylinder by applying an area preserving map from the triangle to the disc.

Theorem 4.3. *(A. Sukhov and A. Tumanov [6]) Let A be a continuous $n \times n$ matrix function on $\mathbb{C}^n$ with compact support in $\mathbb{D} \times \mathbb{C}^{n-1}$. Suppose there is a constant $0 < a < 1$ such that*

$$\|A(z)\| \le a, \ \forall z \in \mathbb{D} \times \mathbb{C}^{n-1}. \tag{4.1}$$

Then there exists $p > 2$ such that for every point $x \in \mathbb{D}\times\mathbb{C}^{n-1}$ there is a solution $Z \in W^{1,p}(\mathbb{D})$ (Sobolev space) of equation (2.2)

$$Z_{\overline{\zeta}} = A(Z)\overline{Z_{\zeta}}$$

such that $Z(\overline{\mathbb{D}}) \subset \overline{\mathbb{D} \times \mathbb{C}^{n-1}}, x \in Z(\mathbb{D})$, $\mathrm{Area}(Z) = \pi$ and

$$Z(\partial\mathbb{D}) \subset \partial(\mathbb{D} \times \mathbb{C}^{n-1}) = (\partial\mathbb{D}) \times \mathbb{C}^{n-1}.$$

Furthermore, if we denote the components of Z by $Z = (f_1, \dots, f_n)$, then we have the following area property

$$\mathrm{Area}(f_1) = \pi, \ \mathrm{Area}(f_j) = 0, \ \textit{for } j = 2, \dots, n.$$

For $1 \le j \le n$, let M_j be the holomorphic disc $M_j = (m_1, \dots, m_n) : \mathbb{D} \to \mathbb{C}^n$ where $m_k(z) = 0$ if $k \neq j$ and $m_j(z) = z$. Notice that the minimal area of an analytic set passing through the origin in $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}$ is π: this is because $\mathbb{B}^{2n} \subset \mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}$ and the minimal area of analytic set of $\mathbb{B}^{2n}$ passing through the origin is π (Lelong 1950; see, for example, Section 15.3 of [2]).

Lemma 4.4. *The analytic set of minimal area of $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ through the origin is given by one of the $n-2$ distinct holomorphic discs $M_3, \dots, M_n$.*

Proof. Let $S_1 = \{x_1^2 + x_2^2 = 1, y_1 = y_2 = 0, z_3 = \cdots = z_n = 0\}, S_2 = \{y_1^2 + y_2^2 = 1, x_1 = x_2 = 0, z_3 = \cdots = z_n = 0\}, S_j = \{|z_j| = 1, z_k = 0 \text{ for } k \neq j\}$ for $3 \le j \le n$. By using Lelong's result (see, for example, Section 15.3 of [2]) and the argument in proof of Theorem 3.9, we conclude that the boundary of the analytic set E of minimal area in $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ through the origin must lie in the intersection of the boundary of $\mathbb{B}^{2n}$ and the boundary of $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$. Notice that this intersection consists of n circles $S_1, \dots, S_n$. Suppose a boundary point of E lies in $S_1 \cup S_2$, then E must have a component lying in the complexification of $S_1 \cup S_2$, which is given by $\{z_1^2 + z_2^2 = 1, z_3 = \cdots = z_n = 0\}$. In fact all of E lies in this set since E is of minimal area. However $\{z_1^2 + z_2^2 = 1, z_3 = \cdots = z_n = 0\}$ does not pass through the origin, so the boundary of E is contained in the circles $S_3 \cup \cdots \cup S_n$. Hence E is one of the discs $M_3, \dots, M_n$. □

The following lemma is a consequence of Lemma 4.4 and Theorem 3.6:

Lemma 4.5. *Let E_j be a convergent sequence of analytic sets in $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ passing through the origin so that*

$$\lim_{j\to\infty} \mathrm{Area}(E_j) = \pi.$$

Then the limiting analytic set E_∞ is one of the $n-2$ distinct holomorphic discs $M_3, \dots, M_n$.

Our proof of Theorem 4.1 in the case $r = 1$ is based on the fact that the domains $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ and $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ have a different number of analytic sets of minimum area through the origin. We are now ready to prove the main theorem of this section.

Theorem 4.6. *The domains $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ and $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic.*

Proof. Suppose on the contrary that $\psi : \mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}} \to \mathbb{D}^{2n}_{\mathbb{C}}(1)$ is a symplectomorphism. By composing with a symplectomorphism of $\mathbb{D}^{2n}_{\mathbb{C}}(1)$, we can assume that $\psi(0) = 0$.

Consider the standard almost complex structure J_{st} on $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ and let $J = \psi_* J_{\mathrm{st}}$ be the complex structure on $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ given by the push-forward of J_{st} by ψ. Since $\psi^*\omega_{\mathrm{st}} = \omega_{\mathrm{st}}$, the almost complex structure J is tamed by ω_{st}. Then, by Lemma 2.3, the complex matrix $\widetilde{A}$ of J satisfies $\|\widetilde{A}(z)\| < 1$ for $z \in \mathbb{D}^{2n}_{\mathbb{C}}(1)$.

Let $\{K_l\}_{l=1}^{\infty}$ be a compact exhaustion of $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ so that each K_l is a closed polydisc with radius less than 1, that is, $K_l \subset K_{l+1}$, K_l is a compact subset of $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ for all l and $\cup_{l=1}^{\infty} K_l = \mathbb{D}^{2n}_{\mathbb{C}}(1)$. For each l, let χ_l be a smooth cut-off function on $\mathbb{C}^n$ with support in $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ and equals to 1 on K_l. Consider the $n \times n$ matrix function $A_l = \chi_l \widetilde{A}$, which agrees with 0 outside $\mathbb{D}^{2n}_{\mathbb{C}}(1)$. Since $\|\widetilde{A}\| < 1$ on $\mathbb{D}^{2n}_{\mathbb{C}}(1)$, there is a constant $0 < a_l < 1$ such that (4.1) holds for A_l. Let J_l be the almost complex structure on $\mathbb{C}^n$ corresponding to the complex matrix A_l.

By considering $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ as a subset of $\mathbb{D} \times \mathbb{C}^{n-1}$, we can apply Theorem 4.3 so that for each l, there exists a J_l-holomorphic disc $f_l : \mathbb{D} \to \mathbb{D} \times \mathbb{C}^{n-1}$ such that the image of f_l passes through the origin. Also if we write $f_l = (f_{l,1}, \ldots, f_{l,n})$, then we have $\mathrm{Area}(f_{l,j}) = \delta_{j1}\pi$ for all l where δ_{j1} is the Kronecker delta.

Fix an integer N, for each $l \geq N$, $\psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ is an analytic set in $\psi^{-1}(K_N) \subset \mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ passing through the origin. Since ψ is a symplectomorphism, we have

$$\mathrm{Area}(\psi^{-1}(f_l(\mathbb{D}) \cap K_N)) \leq \mathrm{Area}(f_l(\mathbb{D}) \cap K_N) \leq \pi.$$

Therefore by Theorem 3.6, after passing to a subsequence,

$$F_N = \lim_{l \to \infty} \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$$

exists and $\mathrm{Area}(F_N) \leq \pi$. Since $0 \in \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ for all $l \geq N$, it follows that F_N is not an empty set when N is sufficiently large.

The above argument holds for all N that is sufficiently large, so we can apply Theorem 3.6 again to the sequence of analytic set F_N as $N \to \infty$. After passing to a subsequence, denote the limit of F_N by F. Now F is an analytic set in $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ passing through the origin with $\mathrm{Area}(F) \leq \pi$ and $\partial F \subset \partial(\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1))$. Since the minimal area of an analytic set in $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ through the origin is π, we must have $\mathrm{Area}(F) = \pi$. Therefore F is one of the holomorphic discs M_j for $3 \leq j \leq n$ by Lemma 4.5.

Let $E = \psi(F)$. We now know that $\mathrm{Area} f_l = \pi$ for all l and $f_l(\mathbb{D}) \cap \mathbb{D}^{2n}_{\mathbb{C}}(1) \to E$ as $l \to \infty$. We also know that $\mathrm{Area}(E) = \pi$. We want to show that $f_l(\mathbb{D}) \to E$ as $l \to \infty$. Let $X_l = f_l(\mathbb{D}) \setminus \mathbb{D}^{2n}_{\mathbb{C}}(1)$, which is the image of f_l that is not in $\mathbb{D}^{2n}_{\mathbb{C}}(1)$. By the construction of A_l and J_l, we know that $J_l = J_{\mathrm{st}}$ outside $\mathbb{D}^{2n}_{\mathbb{C}}(1)$, hence X_l is a usual analytic set in $(\mathbb{D} \times \mathbb{C}^{n-1}) \setminus \mathbb{D}^{2n}_{\mathbb{C}}(1)$. Since $\mathrm{Area} X_l \leq \mathrm{Area} f_l = \pi$ for all l, we can apply Theorem 3.6 to conclude that, after passing to a subsequence, X_l converges to an analytic set X. However $f_l(\mathbb{D}) \cap \mathbb{D}^{2n}_{\mathbb{C}}(1) \to E$ as $l \to \infty$ and $\mathrm{Area}(E) = \pi$ implies that

$$\lim_{l \to \infty} \mathrm{Area}(f_l(\mathbb{D}) \cap \mathbb{D}^{2n}_{\mathbb{C}}(1)) = \pi.$$

By construction $\mathrm{Area}(f_l) = \pi$ for all l, hence we have $\mathrm{Area}(X_l) \to 0$ as $l \to \infty$. Therefore X is an empty set and we can conclude that

$$\lim_{l\to\infty} f_l(\mathbb{D}) \subset \mathbb{D}^{2n}_{\mathbb{C}}(1),$$

and hence

$$\lim_{l\to\infty} f_l(\mathbb{D}) = E.$$

Since $\mathrm{Area}(f_{l,j}) = \delta_{j1}\pi$, if we write $\omega_{\mathrm{st}} = \omega_1 + \cdots + \omega_n$ where $\omega_j = dx_j \wedge dy_j$ for $j = 1, \dots, n$, then we have

$$\int_E \omega_j = \delta_{j1}\pi.$$

Now for $1 \le k \le n$, by considering $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ as a subset of the cylinder $\mathbb{C}^{k-1} \times \mathbb{D} \times \mathbb{C}^{n-k} \cong \mathbb{D} \times \mathbb{C}^{n-1}$, we can apply the above argument to obtain a real 2-dimensional set E_k in $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ passing through the origin, satisfying the following conditions:

(1) $\int_{E_k} \omega_j = \delta_{jk}\pi$ for $1 \le j \le n$, hence all E_k's are distinct for $1 \le k \le n$.
(2) The preimage $F_k = \psi^{-1}(E_k)$ is an analytic set in $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ passing through the origin.
(3) F_k's are distinct analytic sets for $1 \le k \le n$ since E_k's are distinct and ψ is a bijection.
(4) $\mathrm{Area}(F_k) = \pi$ for $1 \le k \le n$.

Hence for each $1 \le k \le n$, F_k must be one of the holomorphic discs M_j for $3 \le j \le n$ according to Lemma 4.5, but this is impossible since all F_k's are distinct, so we arrived at a contradiction. Therefore $\mathbb{D}^4_{\mathbb{R}}(1) \times \mathbb{D}^{2n-4}_{\mathbb{C}}(1)$ and $\mathbb{D}^{2n}_{\mathbb{C}}(1)$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic.

□

Acknowledgments. The author is grateful to A. Tumanov for formulating this problem and providing useful suggestions. The author would also like to thank the anonymous referee for many useful remarks to improve the quality of the paper.

References

1. M. Audin, J. Lafontaine (Eds.), *Holomorphic curves in symplectic geometry*, Birkhauser, Progress in Mathematics, V. 117 (1994)
2. E. Chirka, *Complex analytic sets*, Kluwer Academic Publishers, 1989
3. M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307 - 347
4. L. Guth, *Symplectic embeddings of polydisks*, Invent. Math. 172 (2008), 477–489. MR 2393077
5. D. McDuff, *Symplectic embeddings of 4-dimensional ellipsoids*, Journ. of Top. 2 (2009), 1 - 22. MR 2499436, Zbl 1166.5305
6. A. Sukhov and A. Tumanov, *Gromov's Non-Squeezing Theorem and Beltrami type equation*, Comm. Partial Differential Equations 39 (2014), 1898-1905.
7. A. Sukhov and A. Tumanov, *Hartogs figure and symplectic non-squeezing*, Illinois J. Math. (Special issue in honor of John D'Angelo on the occasion of his 60th birthday) 56 (2012), 221-233
8. A. Sukhov and A. Tumanov, *Regularization of almost complex structures and gluing holomorphic discs to tori*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. X (2011), 389-411
9. Y. Wong, *Symplectic embedding of real bi-disc*, Complex Variables and Elliptic Equations, DOI:10.1080/17476933.2011.605448

1409 W. Green Street, Urbana IL 61801
E-mail address: sfxcwys@gmail.com