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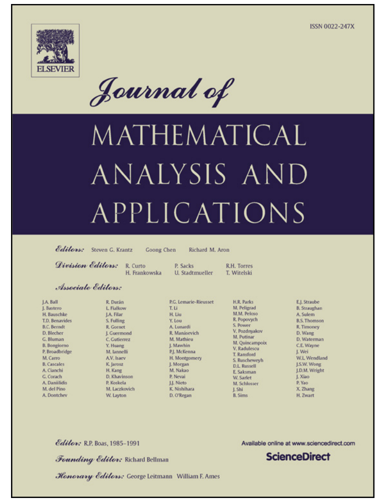

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Global boundedness in a fully parabolic quasilinear chemotaxis system with singular sensitivity*

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Abstract. In this paper we study the global boundedness of solutions to the quasilinear fully parabolic chemotaxis system: $u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla\varphi(v))$, $v_t = \Delta v - v + u$, where bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ subject to the non-flux boundary conditions, the diffusivity fulfills $D(u) = a_0(u+1)^{-\alpha}$ with $a_0 > 0$ and $\alpha \geq 0$, while the density-signal governed sensitivity satisfies $0 \leq S(u) \leq b_0(u+1)^{\beta}$ and $0 < \varphi'(v) \leq \frac{\chi}{v^k}$ for $b_0, \chi > 0$ and $\beta, k \in \mathbb{R}$. It is shown that the solution is globally bounded provided $\alpha + \beta < 1$ and $k \leq 1$. This result demonstrates the effect of signal-dependent sensitivity on the blow-up prevention.

Keywords: Chemotaxis; Nonlinear diffusion; Global boundedness; Signal-dependent sensitivity

Mathematics Subject Classification: 92C17; 35K55; 35B35; 35B40

1 Introduction

In this article, we study the following parabolic-parabolic Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla\varphi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary, $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. u and v stand for the cell density, the concentration of an attractive signal produced by cells themselves respectively. Initial data u_0 and v_0 fullfill

$$\begin{cases} u_0 \in C(\bar{\Omega}), \quad u_0 \geq 0 \ \text{ in } \bar{\Omega}, \quad u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega), \quad v_0 > 0 \ \text{ in } \bar{\Omega}. \end{cases} \tag{1.2}$$

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$D(u)$ measures the nonlinear cell self-diffusion, $S(u)$ and $\varphi(v)$ represent the density-dependent sensitivity and the signal-dependent sensitivity.

Recall related results obtained in previous literatures of the field.

For the parabolic-parabolic case, Winkler [18] established that problem (1.1) has a unique globally bounded classical solution provided $0<\varphi(v)\le\frac{\chi}{(1+\alpha v)^k}$ with $\alpha,\chi>0$ and $k>1$. However this result is not completely correct for all $\chi>0$, and it was repaired by Mizukami and Yokota [14] under a smallness condition for χ. Moreover, Fujie and Yokota [5] dealt with the singular case that $0<\varphi(s)\le\frac{\chi}{s^k}$ with $k>1$, $\chi>0$ and presented global existence and boundedness of classical solutions. While for $\varphi(v)=\frac{1}{v}$, Winkler [19] proved the global existence of classical solutions to (1.1) if $\chi<\sqrt{\frac{2}{n}}$, and the global existence of weak solutions is ensured by the condition $\chi<\sqrt{\frac{n+2}{3n-4}}$, Fujie [8] obtained the boundedness result for (1.1) with $D(u)\equiv 1$ under the condition $0<\chi<\sqrt{\frac{2}{n}}$. When $D(u)$ takes algebraic form and $\varphi(v)=v$, the problem (1.1) with Ω being a ball possesses smooth solutions that blow up either in finite or infinite time if $\frac{S(u)}{D(u)}\ge cu^{\frac{2}{n}+\varepsilon}$ for $u>1$ [20]. Whereas all solutions are globally bounded if Ω is convex and $\frac{S(u)}{D(u)}\le cu^{\frac{2}{n}-\varepsilon}$ for large u [16], this result was extended to the case without convexity assumption by Ishida etal. [10], and Fujie etal. [7] established that the same result holds also for $\varphi(v)=\log v$.

For the parabolic-elliptic counterpart (which takes place in the situation where chemicals diffuse much faster than cells move [11]), the corresponding problem

$$\begin{cases} u_t=\nabla\cdot(D(u)\nabla u-S(u)\nabla\varphi(v)), & x\in\Omega,\ t>0,\\ 0=\Delta v-v+u, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega \end{cases}\tag{1.3}$$

has also been studied extensively. With $D(u)\equiv 1$ in (1.3), Biler [1] proved the global existence of weak solutions under $0<\chi<\frac{2}{n}$. Independently, on the condition $D(u)\equiv 1$, Nagai and Senba [13] obtained that the radial solutions are globally bounded when either $n\ge 3$ with $0<\chi<\frac{n}{n-2}$, or $n=2$ with any $\chi>0$, whereas if $n\ge 3$ with $\chi\ge\frac{2n}{n-2}$, then the finite time blow-up of solutions may occur. Without the symmetric assumption, for $D(u)\equiv 1$ and $k\ge 1$, Fujie etal. got the result that there exists a unique globally bounded classical solution to (1.3) under some additional conditions on parameters χ and k [4]. When the second equation in (1.3) is replaced by $0=\Delta v-m+u$ with $m:=\frac{1}{|\Omega|}\int_\Omega u_0dx$ and $\varphi(v)=v$, Winkler and Djie [21] proved that the solutions remain uniformly bounded in time if $\alpha+\beta<2/n$, under the assumption that $D(u)\simeq u^{-\alpha}$ and $S(u)\simeq u^{\beta}$ for large u with $\alpha\ge 0$ and $\beta\in\mathbb{R}$, whereas there exist solutions that blow up in either finite or infinite time when $\alpha+\beta>2/n$ [2]. Recently, [17] proved that (1.3) possesses a unique globally bounded classical solution if $\alpha+\beta<1$ and $k\ge 1$, which exhibited the effect of the signal-dependent sensitivity on the interaction between the self-diffusivity and the density-dependent sensitivity.

In the present paper, we will find a criterion that ensures the existence of gobally bounded solution and explore the effect of the signal-dependent sensitivity. We assume that the signal-dependent sensitivity $\varphi \in C^{2+\omega}_{\text{loc}}((0,\infty))$ $(0<\omega<1)$ fulfills

$$0 < \varphi'(v) \le \frac{\chi}{v^k} \quad \text{for any } v > 0 \tag{1.4}$$

with $\chi > 0$ and $k \ge 1$. In addition, we suppose that the diffusivity $D \in C^2([0,\infty))$ and the density-dependent sensitivity $S \in C^2([0,\infty))$ with $S(0)=0$ satisfy

$$D(u) = a_0(u+1)^{-\alpha}, \quad 0 \le S(u) \le b_0(u+1)^{\beta} \quad \text{for all } u \ge 0, \tag{1.5}$$

where $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$ are constants.

Under these hypotheses, we have following result about solution to the problem (1.1).

Theorem 1.1 *Let $\Omega \subseteq \mathbf{R}^n (n \ge 2)$ be a bounded domain with smooth boundary, (1.4)-(1.5) hold with $\alpha \ge 0$. Suppose that $k \ge 1$ and $\alpha + \beta < 1$, then for any initial data u_0 and v_0 fulfilling (1.2), the problem (1.1) possesses a globally bounded and classical solution (u,v).*

Remark 1 For the corresponding problem (1.3), the solutions remain bounded globally provided $\alpha + \beta < 1$ and $k \ge 1$ [17], thus it is obvious that our result coincides with the one for the parabolic-elliptic model.

Remark 2 Suppose that $\varphi(v) = v$ in (1.1), we can deduce from [20] that there exist radial blow-up solutions if $\frac{2}{n} < \alpha + \beta < 1$ with $n \ge 3$ in symmetric setting. Comparably when $\varphi(v) = \log v$, Theorem 1.1 asserts the global boundedness of solutions to (1.1) under the condition $\alpha + \beta < 1$, this result is different from that for quasilinear fully parabolic system (1.1) with $k = 0$ and $\chi = 1$, which shows the signal-dependent sensitivity benefiting the global boundedness of solutions. Meanwhile our theorem solves the open problem that Fujie etal. proposed in [7].

2 Preliminaries

In this section, we assert some basic facts that will be used later. It is easy to check the mass conservation of u in the model, namely,

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx =: M, \qquad \text{for all } t > 0. \tag{2.1}$$

Let $v(x,t)$ solve the second equation of (1.1) with $u \in C^0(\bar{\Omega} \times (0,T))$ satisfying (2.1). We know from [6, Lemma 2.2] that

$$v(x,t) \ge \eta > 0, \quad \text{for all } x \in \ \Omega \text{ and } t > 0, \tag{2.2}$$

where η is independent of t. Now we begin with the local existence of classical solutions to (1.1), which is established in [7].

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with smooth boundary. Assume that $\varphi \in C^{2+\omega}_{\text{loc}}((0,\infty))$ with $0 < \omega < 1$, and $D, S \in C^2([0,\infty))$ with $D(u) > 0$ for $u \geq 0$ and $S(0) = 0$. Furthermore, suppose that the initial data u_0 and v_0 satisfy (1.2). Then there exist the maximal existence time $T_{\max} \in (0,\infty]$ and a pair (u,v) of nonnegative functions from $C^{2,1}(\bar{\Omega} \times (0,T_{\max})) \cap C(\bar{\Omega} \times [0,T_{\max}))$ solving problem (1.1) classically in $\Omega \times [0,T_{max})$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.3}$$

Lemma 2.2 *([6, Lemma 2.4]) Let $1 \leq \theta, \mu \leq \infty$ and (u,v) satisfies the problem (1.1),*

(i) *If $\frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$, then there exists a positive constant C independent of t such that*

$$\|v(\cdot,t)\|_{L^\mu(\Omega)} \leq C(1 + \sup \|u(\cdot,s)\|_{L^\theta(\Omega)}), \ \text{for all } t > 0.$$

(ii) *If $\frac{1}{2} + \frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$, then there exists a positive constant C independent of t such that*

$$\|\nabla v(\cdot,t)\|_{L^\mu(\Omega)} \leq C(1 + \sup \|u(\cdot,s)\|_{L^\theta(\Omega)}), \ \text{for all } t > 0.$$

Lastly, we will use an extended version of the Gagliardo-Nirenberg inequality (see e.g.[3, 12]).

Gagliardo-Nirenberg inequality. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $\phi \in W^{1,2}(\Omega) \cap L^r(\Omega)$ with $0 < r \leq \infty$. If $q \in (0,\infty]$ is such that

$$a := \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} + \frac{1}{n} - \frac{1}{2}} \in (0,1),$$

then

$$\|\phi\|_{L^q(\Omega)} \leq C_{GN}\big(\|\nabla\phi\|^a_{L^2(\Omega)}\|\phi\|^{1-a}_{L^r(\Omega)} + \|\phi\|_{L^r(\Omega)}\big)$$

with $C_{GN} > 0$ depending on n, r, a and Ω.

3 A priori estimates

In this section, we will give a priori estimates for the local classical solution (u,v) of (1.1) ensured by Lemma 2.1.

Lemma 3.1 *Let (u,v) be a solution of the problem (1.1), φ satisfies (1.4) and (1.5) is valid for D and S. If $\alpha + \beta < 1$, then for any $p > \max\{1, 1-\alpha, 2-2\alpha-2\beta\}$ and $\varepsilon_1 > 0$, we can find a positive constant $C_1 = C_1(p,\alpha,\beta,\eta,k,\varepsilon_1)$ such that*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^{p+\alpha}dx + \frac{a_0(p+\alpha)(p+\alpha-1)}{2}\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx \\
&\leq \varepsilon_1 \int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx + C_1\int_\Omega |\nabla v|^2 dx, \qquad \text{for all } t > 0
\end{aligned}$$

with $\gamma = \frac{2k(1-\alpha-\beta)}{p+2\alpha+2\beta-2} > 0$.

Proof. Multiplying the first equation in (1.1) by $(p+\alpha)(u+1)^{p+\alpha-1}$, integrating by parts over Ω, we obtain with considering (1.4) and (1.5) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^{p+\alpha}dx &= -(p+\alpha)(p+\alpha-1)\int_\Omega D(u)(u+1)^{p+\alpha-2}|\nabla u|^2dx \\ &+ (p+\alpha)(p+\alpha-1)\int_\Omega S(u)(u+1)^{p+\alpha-2}\nabla u\nabla\varphi(v)dx \\ &\le -(p+\alpha)(p+\alpha-1)a_0\int_\Omega (u+1)^{p-2}|\nabla u|^2dx \\ &+ (p+\alpha)(p+\alpha-1)b_0\int_\Omega (u+1)^{p+\alpha+\beta-2}\frac{\chi}{v^k}|\nabla u||\nabla v|dx. \qquad (3.1)\end{aligned}$$

Cauchy's inequality indicates that

$$\begin{aligned}\int_\Omega (u+1)^{p+\alpha+\beta-2}\frac{\chi}{v^k}|\nabla u||\nabla v|dx &\le \frac{a_0}{2b_0}\int_\Omega (u+1)^{p-2}|\nabla u|^2dx \\ &+ \frac{b_0\chi^2}{2a_0}\int_\Omega \frac{|\nabla v|^2}{v^{2k}}(u+1)^{p+2\alpha+2\beta-2}dx, \qquad (3.2)\end{aligned}$$

where the last integral can be estimated by Young's inequality

$$\begin{aligned}&\frac{b_0\chi^2}{2a_0}\int_\Omega \frac{|\nabla v|^2}{v^{2k}}(u+1)^{p+2\alpha+2\beta-2}dx \\ &\le \varepsilon_1\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx + C_1\int_\Omega |\nabla v|^2dx. \qquad (3.3)\end{aligned}$$

Hence the claim follows by (3.1)-(3.3). $\square$

Lemma 3.2 *Let (u,v) be a solution of (1.1), φ satisfies (1.4) and (1.5) is valid for D and S with $\alpha \ge 0$. If $k \ge 1$ and $\alpha+\beta<1$, then for any $\varepsilon_2>0$, $\gamma>0$ and $p>\max\{2, 2-2\alpha-2\beta\}$, we can find a positive constant $C_2 = C_2(p,\alpha,\beta,\eta,k,\varepsilon_2,\gamma)$ fulfilling*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx \\ &\le \frac{\varepsilon_2}{2}\int_\Omega (u+1)^{p-2}|\nabla u|^2dx + C_2\int_\Omega \frac{|\nabla v|^2}{v^{2k}}dx + C_2\int_\Omega (u+1)^pdx, \qquad \textit{for all } t>0.\end{aligned}$$

Proof. Differentiate $\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx$ respect with to t, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx &= (p+\alpha+\beta-1)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-2}}u_tdx \\ &- (2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-1}}v_tdx \\ &= (p+\alpha+\beta-1)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-2}}\nabla\cdot(D(u)\nabla u - S(u)\nabla\varphi(v))dx\end{aligned}$$

$$-(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-1}}(\Delta v - v + u)dx. \tag{3.4}$$

Integrating by parts and considering (1.4)-(1.5) tell that

$$\begin{aligned}
&\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-2}}\nabla\cdot(D(u)\nabla u - S(u)\nabla\varphi(v))dx\\
&\le -a_0(p+\alpha+\beta-2)\int_\Omega \frac{|\nabla u|^2}{v^{2k+\gamma-2}}(u+1)^{p+\beta-3}dx\\
&\quad + a_0(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|dx\\
&\quad - \chi(2k+\gamma-2)\int_\Omega \frac{|\nabla v|^2}{v^{3k+\gamma-1}}(u+1)^{p+\alpha+\beta-2}S(u)dx\\
&\quad + \chi b_0(p+\alpha+\beta-2)\int_\Omega \frac{(u+1)^{p+\alpha+2\beta-3}}{v^{3k+\gamma-2}}|\nabla u||\nabla v|dx
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-1}}(\Delta v - v + u)dx &= (2k+\gamma-1)\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx\\
&\quad - (p+\alpha+\beta-1)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-1}}\nabla u\nabla v dx\\
&\quad - \int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx + \int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}u}{v^{2k+\gamma-1}}dx.
\end{aligned} \tag{3.6}$$

It follows from (3.4)–(3.6) that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx + (2k+\gamma-1)(2k+\gamma-2)\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx\\
&\le a_0(p+\alpha+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|dx\\
&\quad + \chi b_0(p+\alpha+\beta-1)(p+\alpha+\beta-2)\int_\Omega \frac{(u+1)^{p+\alpha+2\beta-3}}{v^{3k+\gamma-2}}|\nabla u||\nabla v|dx\\
&\quad + (p+\alpha+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-1}}\nabla u\nabla v dx\\
&\quad + (2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.7}$$

where

$$I_1 := a_0(p+\alpha+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|dx,$$

$$I_2 := \chi b_0(p+\alpha+\beta-1)(p+\alpha+\beta-2)\int_\Omega \frac{(u+1)^{p+\alpha+2\beta-3}}{v^{3k+\gamma-2}}|\nabla u||\nabla v|dx,$$

$$I_3 := (p+\alpha+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-1}}\nabla u\nabla v dx,$$

$$I_4 := (2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx.$$

Now we estimate I_1-I_4 respectively. For any $\varepsilon_2 > 0$, it can be obtained upon the fact $\alpha \geq 0$ and Young's inequality that

$$\begin{aligned}
I_1 &\leq a_0(p+\alpha+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|dx \\
&\leq \frac{\varepsilon_2}{8}\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx + C_3\int_\Omega \frac{|\nabla v|^2}{v^{4k+2\gamma-2}}(u+1)^{p+2(\alpha+\beta)-2}dx \\
&\leq \frac{\varepsilon_2}{8}\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx + \frac{C_3}{\eta^{2k+\gamma-2}}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+2(\alpha+\beta)-2}dx \\
&\leq \frac{\varepsilon_2}{8}\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx \\
&\quad + C_4\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}dx
\end{aligned} \tag{3.8}$$

with $C_3 = C_3(p,\alpha,\beta,k,\varepsilon_2,\gamma) > 0$, $C_4 = C_4(p,\alpha,\beta,\eta,k,\varepsilon_2,\gamma) > 0$. Next we do the term I_2 through applying Young's inequality

$$\begin{aligned}
I_2 &\leq \chi b_0(p+\alpha+\beta-1)(p+\alpha+\beta-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{3k+\gamma-2}}|\nabla u||\nabla v|dx \\
&\leq \frac{\chi b_0(p+\alpha+\beta-1)(p+\alpha+\beta-2)}{\eta^{k-1}}\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|dx \\
&\leq \frac{\varepsilon_2}{8}\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx \\
&\quad + C_5\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}dx
\end{aligned} \tag{3.9}$$

with $C_5 = C_5(p,\alpha,\beta,\eta,k,\varepsilon_2,\gamma) > 0$, here we use the fact $\beta < 1$. Proceeding to estimate I_3 and obtain

$$\begin{aligned}
I_3 &\leq (p+\alpha+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\alpha+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|dx \\
&\leq \frac{\varepsilon_2}{8}\int_\Omega (u+1)^{p-2}|\nabla u|^2 dx + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx \\
&\quad + C_6\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}dx
\end{aligned} \tag{3.10}$$

with $C_6 = C_6(p,\alpha,\beta,\eta,k,\varepsilon_2,\gamma) > 0$. It is apparent that

$$I_4 \leq \frac{2k+\gamma-2}{\eta^{2k+\gamma-2}}\int_\Omega (u+1)^p dx. \tag{3.11}$$

In view of (3.7)-(3.11), we arrive at

$$\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\alpha+\beta-1}}{v^{2k+\gamma-2}}dx + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\alpha+\beta-1}dx$$
$$\leq \frac{\varepsilon_2}{2}\int_\Omega (u+1)^{p-2}|\nabla u|^2dx + C_7\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}dx + C_7\int_\Omega (u+1)^p dx \tag{3.12}$$

with $C_7 = \max\{C_4 + C_5 + C_6, \frac{2k+\gamma-2}{\eta^{2k+\gamma-2}}\}$. This ends our proof. $\square$

4 Proof of Theorem 1.1

In this concluding section, we will prove our main result in conjunction with the estimates above. Now we give the description that the bound of $u(\cdot,t)$ in $L^p(\Omega)$ with p large enough can be turned into the bound in $L^\infty(\Omega)$, which is exhibited in Lemma 4.1.
Define

$$p_* := \max\Big\{n, (n+2)\beta_+, \lambda_0, 1 - \frac{(1-\alpha)(n+2)}{2}, \frac{n\alpha}{2}\Big\}$$

with $\beta_+ = \max\{\beta, 0\}$ and

$$\lambda_0 := \inf\{\lambda \geq 0 : p^2 + (n - \alpha n - \alpha - (n+2)\beta_+)p + \alpha(n+2)\beta_+ > 0 \text{ for all } p \in (\lambda, \infty)\}.$$

Lemma 4.1 *Under the conditions of Theorem 1.1, if there exists $p > p_*$ such that*

$$\|u(\cdot,t)\|_{L^p} \leq C_p, \quad \textit{for all } t > 0$$

with $C_p > 0$, then we can find $C > 0$ satisfying

$$\|u(\cdot,t)\|_{L^\infty} \leq C, \quad \textit{for all } t > 0.$$

Proof. Since $p > n$, we can invoke the Lemma 2.2 (ii) to $(1.1)_2$ and get

$$\|\nabla v(\cdot,t)\|_{L^\infty} \leq C, \quad \text{for all } t > 0$$

with $C > 0$. Then our statement is supposed to be derived by similar arguments in Lemma 4.1 in [17] and Lemma A.1 in [15]. $\square$

Now we are in a position to prove our theorem.

Proof of Theorem 1.1. For some $p_0 > \max\{p_*, 2-\alpha-\beta\}$, by setting $\varepsilon_1 = \frac{(2k+\gamma_0-1)(2k+\gamma_0-2)}{4}$ in Lemma 3.1 and $\varepsilon_2 = \frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{2}$ in Lemma 3.2, we know that

$$\frac{d}{dt}\Big(\int_\Omega (u+1)^{p_0+\alpha}dx + \int_\Omega \frac{(u+1)^{p_0+\alpha+\beta-1}}{v^{2k+\gamma_0-2}}dx\Big)$$

$$
\begin{aligned}
&+\frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{4}\int_\Omega (u+1)^{p_0-2}|\nabla u|^2dx\\
&\le C_8\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma_0}}dx + C_8\int_\Omega |\nabla v|^2 + C_8\int_\Omega (u+1)^{p_0}dx\\
&\le C_9\int_\Omega |\nabla v|^2 + C_8\int_\Omega (u+1)^{p_0}dx,
\end{aligned}
\tag{4.1}
$$

with $\gamma_0=\frac{2k(1-\alpha-\beta)}{p_0+2\alpha+2\beta-2}>0$, $C_8=C_8(p_0,\alpha,\beta,\eta,k)>0$ and $C_9=C_8+\frac{C_8}{\eta^{2k+\gamma_0}}$. Multiplying the second equation in (1.1) by v, integrating by parts and using Cauchy's inequality result in

$$
\frac{d}{dt}\int_\Omega v^2dx+2\int_\Omega|\nabla v|^2dx=-2\int_\Omega v^2dx+2\int_\Omega uvdx\le-\int_\Omega v^2dx+\int_\Omega u^2dx. \tag{4.2}
$$

By virtue of (4.1) and (4.2),

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega (u+1)^{p_0+\alpha}dx+\int_\Omega \frac{(u+1)^{p_0+\alpha+\beta-1}}{v^{2k+\gamma_0-2}}dx+C_9\int_\Omega v^2dx\Big)\\
&\quad+\frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{4}\int_\Omega (u+1)^{p_0-2}|\nabla u|^2dx+C_9\int_\Omega v^2dx\\
&\le C_8\int_\Omega (u+1)^{p_0}dx+C_9\int_\Omega u^2dx.
\end{aligned}
\tag{4.3}
$$

Through an application of the Gagliardo-Nirenberg inequality,

$$
\begin{aligned}
&\int_\Omega (u+1)^{p_0+\alpha}dx=\|(u+1)^{\frac{p_0}{2}}\|^{\frac{2(p_0+\alpha)}{p_0}}_{L^{\frac{2(p_0+\alpha)}{p_0}}(\Omega)}\\
&\le C_{GN}\Big(\|\nabla(u+1)^{\frac{p_0}{2}}\|^{\theta_1}_{L^2(\Omega)}\|(u+1)^{\frac{p_0}{2}}\|^{1-\theta_1}_{L^{\frac{2}{p_0}}(\Omega)}+\|(u+1)^{\frac{p_0}{2}}\|_{L^{\frac{2}{p_0}}(\Omega)}\Big)^{\frac{2(p_0+\alpha)}{p_0}}
\end{aligned}
$$

with $\theta_1=\frac{p_0(p_0+\alpha)n-p_0n}{(p_0+\alpha)(p_0n+2-n)}\in(0,1)$ due to $p_0>(\frac{n}{2}-1)\alpha$, thus there exist positive constants $C_{10}=C_{10}(p_0,\alpha,\Omega,M)$ and $C_{11}=C_{11}(p_0,\alpha,\Omega,M)$ fulfilling

$$
\begin{aligned}
&C_{10}\big(\int_\Omega (u+1)^{p_0+\alpha}dx\big)^{\frac{p_0n+2-n}{(p_0+\alpha)n-n}}\\
&\le\frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{16}\int_\Omega (u+1)^{p_0-2}|\nabla u|^2dx+C_{11}.
\end{aligned}
\tag{4.4}
$$

Still by utilizing the Gagliardo-Nirenberg inequality, we can find $C_{12}=C_{12}(p_0,\alpha,\beta,\eta,k,\Omega,M)>0$ and $\theta_2=\frac{p_0n-n}{p_0n+2-n}\in(0,1)$ satisfying

$$
\begin{aligned}
&C_8\int_\Omega (u+1)^{p_0}dx=C_8\|(u+1)^{\frac{p_0}{2}}\|^2_{L^2(\Omega)}\\
&\le C_8C_{GN}\Big(\|\nabla(u+1)^{\frac{p_0}{2}}\|^{\theta_2}_{L^2(\Omega)}\|(u+1)^{\frac{p_0}{2}}\|^{1-\theta_2}_{L^{\frac{2}{p_0}}(\Omega)}+\|(u+1)^{\frac{p_0}{2}}\|_{L^{\frac{2}{p_0}}(\Omega)}\Big)^2\\
&\le C_{12}\big(\int_\Omega (u+1)^{p_0-2}|\nabla u|^2dx\big)^{\theta_2}+C_{12},
\end{aligned}
$$

which along with Young's inequality,

$$C_8 \int_\Omega (u+1)^{p_0} dx \le \frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{16} \int_\Omega (u+1)^{p_0-2}|\nabla u|^2 dx + C_{13} \tag{4.5}$$

with $C_{13} = C_{13}(p_0, \alpha, \beta, \eta, k, \Omega, M) > 0$. Moreover in light of (4.5) and Young's inequality, we have

$$\begin{aligned}\int_\Omega \frac{(u+1)^{p_0+\alpha+\beta-1}}{v^{2k+\gamma_0-2}} dx &\le \frac{1}{\eta^{2k+\gamma_0-2}} \int_\Omega (u+1)^{p_0+\alpha+\beta-1} dx \le C_8 \int_\Omega (u+1)^{p_0} dx + C_{14} \\ &\le \frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{16} \int_\Omega (u+1)^{p_0-2}|\nabla u|^2 dx + C_{13} + C_{14},\end{aligned} \tag{4.6}$$

with $C_{14} = C_{14}(p_0, \alpha, \beta, \eta, k) > 0$. It follows from (4.5) that

$$\begin{aligned}&C_8 \int_\Omega u^2 dx \le C_8 \int_\Omega (u+1)^{p_0} \\ &\le \frac{a_0(p_0+\alpha)(p_0+\alpha-1)}{16} \int_\Omega (u+1)^{p_0-2}|\nabla u|^2 dx + C_{13}\end{aligned} \tag{4.7}$$

due to $p_0 > 2$. As a consequence of (4.3)-(4.7), we see that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega (u+1)^{p_0+\alpha} dx + \int_\Omega \frac{(u+1)^{p_0+\alpha+\beta-1}}{v^{2k+\gamma_0-2}} dx + C_9 \int_\Omega v^2 dx \Big) \\ &\quad + C_{10}\Big(\int_\Omega (u+1)^{p_0+\alpha} dx\Big)^{\frac{pn+2-n}{(p_0+\alpha)n-n}} + \int_\Omega \frac{(u+1)^{p_0+\alpha+\beta-1}}{v^{2k+\gamma_0-2}} dx + C_8 \int_\Omega v^2 dx \\ &\quad \le C_{11} + 3C_{13} + C_{14}.\end{aligned} \tag{4.8}$$

Let

$$y(t) := \int_\Omega (u+1)^{p_0+\alpha}(\cdot,t) dx + \int_\Omega \frac{(u+1)^{p_0+\alpha+\beta-1}(\cdot,t)}{v^{2k+\gamma_0-2}(\cdot,t)} dx + C_9 \int_\Omega v^2(\cdot,t) dx, \quad t > 0.$$

Then (4.8) yields that $y(t)$ satisfies

$$y'(t) + C_{15} y^\kappa(t) \le C_{16}, \qquad \text{for all } t > 0$$

with $\kappa := \min\{\frac{p_0 n+2-n}{(p_0+\alpha)n-n}, 1\}$, $C_{15} = C_{15}(p_0, \alpha, \beta, \eta, k, \Omega, M) > 0$ and $C_{16} = C_{16}(p_0, \alpha, \beta, \eta, k, \Omega, M) > 0$. Therefore by a simple argument of ODI, it can be verified that

$$\sup_{t \in [0, T_{\max})} \int_\Omega (u+1)^{p_0+\alpha}(\cdot,t) dx < \infty,$$

which together with Lemma 4.1 guarantees that

$$\sup_{t \in [0, T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

The proof is complete. □

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