J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Normaloid weighted composition operators on H^2

Derek Thompson [1]

Taylor University, Upland, IN, 46989, United States of America

ARTICLE INFO

Article history:
Received 8 December 2017
Available online xxxx
Submitted by R. Curto

Keywords:
Normaloid operator
Convexoid operator
Hyponormal operator
Composition operator
Weighted composition operator
Spectral radius

ABSTRACT

When φ is an analytic self-map of the unit disk with Denjoy–Wolff point $a \in \mathbb{D}$, and $\rho(W_{\psi,\varphi}) = \psi(a)$, we give an exact characterization for when $W_{\psi,\varphi}$ is normaloid. We also determine the spectral radius, essential spectral radius, and essential norm for a class of non-power-compact composition operators whose symbols have Denjoy–Wolff point in $\mathbb{D}$. When the Denjoy–Wolff point is on $\partial\mathbb{D}$, we give sufficient conditions for several new classes of normaloid weighted composition operators.

1. Introduction

In this paper, we are interested in weighted composition operators on the classical *Hardy space* H^2, the Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the open unit disk $\mathbb{D}$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

A *composition operator* C_φ on H^2 is given by $C_\varphi f = f \circ \varphi$. When φ is an analytic self-map of $\mathbb{D}$, the operator C_φ is necessarily bounded. A *Toeplitz operator* T_ψ on H^2 is given by $T_\psi f = P(\psi f)$ where P is the projection back to H^2. When $\psi \in H^\infty$, the space of bounded analytic functions on $\mathbb{D}$, we simply have $T_\psi f = \psi f$, since ψf is guaranteed to be in H^2, and all such Toeplitz operators are bounded. Throughout this paper, we will assume $\psi \in H^\infty$. We write $W_{\psi,\varphi} = T_\psi C_\varphi$ and call such an operator a *weighted composition operator*. We are interested in when such operators are *normaloid*.

E-mail address: theycallmedt@gmail.com.

[1] The author was funded by the Taylor University Distinguished Lecturer award.

https://doi.org/10.1016/j.jmaa.2018.07.061

For a bounded operator T, we have the following definitions:

- $\sigma(T)$ is the spectrum of T.
- $\rho(T)$ is the spectral radius of T.
- $\rho_e(T)$ is the essential spectral radius of T.
- $W(T)$ is the numerical range of T.
- $\|T\|_e$ is the essential norm of T.
- $r(T)$ is the numerical radius of T.

An operator T is:

(1) *Self-adjoint* if $T = T^*$.
(2) *Normal* if $T^*T = TT^*$.
(3) *Hyponormal* if $T^*T \geq TT^*$.
(4) *Cohyponormal* if $T^*T \leq TT^*$.
(5) *Normaloid* if $\|T\| = \rho(T)$.
(6) *Convexoid* if the closure of $W(T)$ is the convex hull of $\sigma(T)$.
(7) *Spectraloid* if $\rho(T) = r(T)$.
(8) *Power-compact* if T^n is compact for some positive integer n.

Here is a list of well-known facts which we will use repeatedly:

(1) We have the following sequences of implications:

$$\text{self-adjoint} \Rightarrow \text{normal} \Rightarrow \text{(co)hyponormal} \Rightarrow \text{normaloid/convexoid} \Rightarrow \text{spectraloid}$$

(2) If $\psi \in H^\infty$, then $\|T_\psi\| = \|\psi\|_\infty$.
(3) For a bounded operator T, we have

$$\|T\|_e = \inf\{\|T - Q\| : Q \text{ is compact}\}$$

$$\rho_e(T) = \lim_{k\to\infty} (\|T^k\|_e)^{1/k}$$

$$\rho(T) = \lim_{k\to\infty} (\|T^k\|)^{1/k}.$$

Throughout this paper, we will focus on weighted composition operators where $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$, where a is the Denjoy–Wolff point of C_φ. This is a large class, including every power-compact weighted composition operator [7, Theorem 4.3], and many weighted composition operators whose compositional symbol converges uniformly to its Denjoy–Wolff point [5, Corollary 10]. Due to the norm inequality $\|W_{\psi,\varphi}\| \leq \|T_\psi\|\|C_\varphi\| = \|\psi\|_\infty\|C_\varphi\|$, we will often assume $|\psi(a)| = \|\psi\|_\infty$. It is unclear whether this is necessary, but we do have $\|\psi\|_2$ as a lower bound for $|\psi(a)|$ when $\rho(C_\varphi) = 1$.

Proposition 1.1. *Suppose φ is an analytic self-map of $\mathbb{D}$ with Denjoy–Wolff point a, $\psi \in H^\infty$, $W_{\psi,\varphi}$ is normaloid, and $\rho(W_{\psi,\varphi}) = |\psi(a)|$. Then $\|\psi\|_2 \leq |\psi(a)|$.*

Proof.

$$|\psi(a)| = \rho(W_{\psi,\varphi}) = \|W_{\psi,\varphi}\| \geq \|W_{\psi,\varphi}1\|_2 = \|\psi\|_2. \quad \square$$

Please cite this article in press as: D. Thompson, Normaloid weighted composition operators on H^2, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.07.061

The organization of the rest of the paper is as follows. In Section 2, we consider the case when the Denjoy–Wolff point a of φ belongs to $\mathbb{D}$. In Section 3, we show that if a belongs to $\partial\mathbb{D}$, the set of operators for which $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$ is non-trivial. For such operators, we discover new classes of normaloid weighted composition operators in Section 4. We end with further questions about normaloid weighted composition operators in Section 5.

2. $a \in \mathbb{D}$

When the Denjoy–Wolff point a of φ is in $\mathbb{D}$, C_φ is rarely normaloid, as the next theorem shows.

Theorem 2.1. *If the Denjoy–Wolff point of φ is in $\mathbb{D}$, then C_φ is normaloid if and only if $\varphi(0) = 0$.*

Proof. By [2, Theorem 3.9], the spectral radius of C_φ is 1. By [2, Corollary 3.7], we have

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/2} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}$$

and we have $\|C_\varphi\| = 1$ if and only if $\varphi(0) = 0$. $\square$

Unsurprisingly, then, we show that if $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$, then $W_{\psi,\varphi}$ is normaloid if and only if ψ has a particular form. For the interior fixed point case, since $\rho(C_\varphi) = 1$, that assumption is really $\rho(W_{\psi,\varphi}) = |\psi(a)|$. Since this case also always has $|\psi(a)| \leq \rho(W_{\psi,\varphi}) \leq \|W_{\psi,\varphi}\|$, the next theorem focuses on when $|\psi(a)| = \|W_{\psi,\varphi}\|$.

Theorem 2.2. *Suppose φ is an analytic self-map of the disk with Denjoy–Wolff point $a \in \mathbb{D}$, and $\psi \in H^\infty$. Then $|\psi(a)| = \|W_{\psi,\varphi}\|$ if and only if ψ has the form*

$$\psi = \psi(a)\frac{K_a}{K_a \circ \varphi}.$$

Proof. First, assume the two values are equal. Suppose $\varphi(0) = 0$ so that $\|W_{\psi,\varphi}\| = |\psi(0)|$. However,

$$|\psi(0)| = \|W_{\psi,\varphi}\| \geq \|W_{\psi,\varphi}1\| = \|\psi\| = \sqrt{|\psi(0)|^2 + |\psi'(0)|^2 + \left|\frac{\psi''(0)}{2}\right|^2 + \dots}$$

and we only have equality if every derivative of ψ at 0 is 0, making ψ constant (equal to $\psi(0)$), which trivially fits the required form, since $K_0 = 1$.

Now, suppose $\varphi(a) = a$, for some $a \in \mathbb{D}$ other than 0. The weighted composition operator $W_{\zeta,\tau}$ where $\zeta = \sqrt{1-|a|^2}\frac{1}{1-\bar{a}z}, \tau = \frac{a-z}{1-\bar{a}z}$ is unitary by [1, Theorem 6]. Note that τ switches a and 0 and is an involution and $W_{\zeta,\tau}$ is its own inverse. Therefore $W_{\zeta,\tau}W_{\psi,\varphi}W_{\zeta,\tau}$ is unitarily equivalent to $W_{\psi,\varphi}$, and it is again a weighted composition operator $W_{f,g}$, where $f = (\zeta)(\psi \circ \tau)(\zeta \circ \varphi \circ \tau)$ and $g = \tau \circ \varphi \circ \tau$. Since $g(0) = 0$, by the same logic as above, f is a constant function, and the constant is $f(0) = \psi(a)$. Therefore, we have

$$\psi \circ \tau = \frac{\psi(a)}{(\zeta)(\zeta \circ \varphi \circ \tau)}$$

and now composing both sides with τ, and recalling $\tau \circ \tau = z$, we have

$$\psi = \frac{\psi(a)}{(\zeta \circ \tau)(\zeta \circ \varphi)} \tag{2.1}$$

$$= \psi(a)\left(\frac{1}{1-\overline{a}z}\right)(1-\overline{a}\varphi)$$

$$= \psi(a)\frac{K_a}{K_a \circ \varphi}. \tag{2.2}$$

For the other direction, suppose ψ has the form given in equation (2.2), and we will show that $|\psi(a)| = \|W_{\psi,\varphi}\|$. By the same logic as the other direction, $W_{\psi,\varphi}$ is unitarily equivalent to $W_{\zeta,\tau}W_{\psi,\varphi}W_{\zeta,\tau}$, and again, this is a weighted composition operator of the form $W_{f,g}$, $f = (\zeta)(\psi \circ \tau)(\zeta \circ \varphi \circ \tau), g = \tau \circ \varphi \circ \tau$. Using the form for ψ from Equation (2.1), we see that

$$f = (\zeta)\left(\frac{\psi(a)}{(\zeta \circ \tau \circ \tau)(\zeta \circ \varphi \circ \tau)}\right)(\zeta \circ \varphi \circ \tau) = \psi(a).$$

Then $W_{f,g} = \psi(a)C_g$, and since $g(0) = 0$, $\|C_g\| = 1$, so we have $\|W_{\psi,\varphi}\| = \|W_{f,g}\| = \|\psi(a)C_g\| = |\psi(a)|\|C_g\| = |\psi(a)|$. □

From this, we have an immediate corollary about when $W_{\psi,\varphi}$ is normaloid.

Corollary 2.3. *Suppose φ is an analytic self-map of the disk with Denjoy–Wolff point $a \in \mathbb{D}$, $\psi \in H^\infty$, and $\rho(W_{\psi,\varphi}) = |\psi(a)|$. Then $W_{\psi,\varphi}$ is normaloid if and only if ψ has the form*

$$\psi = \psi(a)\frac{K_a}{K_a \circ \varphi}.$$

Proof. Note that we have $|\psi(a)| \le \rho(W_{\psi,\varphi}) \le \|W_{\psi,\varphi}\|$ since $a \in \mathbb{D}$. Then if $W_{\psi,\varphi}$ is normaloid, we have $\|W_{\psi,\varphi}\| = \rho(W_{\psi,\varphi}) = |\psi(a)|$, where the second equality is by hypothesis. If instead we assume ψ has the given form, by Theorem 2.2, we have $|\psi(a)| = \|W_{\psi,\varphi}\|$, therefore $|\psi(a)| = \rho(W_{\psi,\varphi}) = \|W_{\psi,\varphi}\|$, so $W_{\psi,\varphi}$ is normaloid. □

In the previous corollary, we assumed that $\rho(W_{\psi,\varphi}) = |\psi(a)| = |\psi(a)|\rho(C_\varphi)$. The next corollary shows this case includes all power-compact weighted composition operators with $\psi \in H^\infty$, and Section 3 shows it includes several different classes of $W_{\psi,\varphi}$ with Denjoy–Wolff point of φ on $\partial\mathbb{D}$.

Corollary 2.4. *Suppose φ is an analytic self-map of $\mathbb{D}$ with Denjoy–Wolff point $a \in \mathbb{D}$, $\psi \in H^\infty$, and $W_{\psi,\varphi}$ is power-compact. Then $W_{\psi,\varphi}$ is normaloid if and only if ψ has the form*

$$\psi = \psi(a)\frac{K_a}{K_a \circ \varphi}.$$

Proof. By [7, Proposition 4.3], if $W_{\psi,\varphi}$ is compact, then $\rho(W_{\psi,\varphi}) = |\psi(a)|$, so Theorem 2.2 applies. If $W_{\psi,\varphi}$ is power-compact, we still have that the spectral radius is the absolute value of its largest eigenvalue, and $W^*_{\psi,\varphi}(K_a) = \overline{\psi(a)}K_a$, so $\rho(W_{\psi,\varphi}) = |\psi(a)|$. □

This form for ψ is not unexpected, since the same form is required for $W_{\psi,\varphi}$ to be normal when the fixed point of φ belongs to $\mathbb{D}$ [1, Proposition 8], or even for $W_{\psi,\varphi}$ to be cohyponormal (in [4], they are shown to be equivalent when $a \in \mathbb{D}$)). However, while normality requires a much stricter characterization for φ, here we show that this form for ψ is sufficient for $W_{\psi,\varphi}$ to be normaloid, while allowing for many different forms for φ.

At the end of [5], the authors ask how often we have $\sigma(W_{\psi,\varphi}) = \psi(a)\sigma(C_\varphi)$. The work above shows that there are weighted composition operators with φ having Denjoy–Wolff point a where this is false.

Please cite this article in press as: D. Thompson, Normaloid weighted composition operators on H^2, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.07.061

Example 2.5. In [6, Theorem 3.7], examples are given of weights $\psi \in H^\infty$ such that $W_{\psi,\varphi}$ is hyponormal when $\varphi(z) = \frac{sz}{1-(1-s)z}, 0 < s < 1$. Every hyponormal operator is normaloid, but the weights are not as prescribed in Corollary 2.3. Therefore, it must be that $\rho(W_{\psi,\varphi}) > |\psi(0)|$.

When φ has Denjoy–Wolff point $a \in \mathbb{D}$, the primary differentiation of spectrum comes from whether or not φ has a fixed point or a periodic point on $\partial\mathbb{D}$. Here, we obtain a partial result, that gives the spectral radius in Example 2.5. The proof of the following theorem is heavily borrowed from theorems about C_φ in [2].

Remark 2.6. Let φ_n denote the nth iterate of φ, i.e. $\varphi_n = \varphi \circ \varphi \cdots \circ \varphi$, n times. By the discussion ahead of [2, Theorem 7.36], when φ has Denjoy–Wolff point in $\mathbb{D}$, is analytic in a neighborhood of the closed disk, and is not an inner function, there is an integer n so that the set $S_n = \{w : |w| = 1 \text{ and } |\varphi_n(w)| = 1\}$ is either empty or consists only of the finitely many fixed points of φ_n on the circle. The essential spectral radius of C_φ is

$$\rho_e(C_\varphi) = \max\{\varphi_n'(w)^{-1/2n} : w \in S_n\}.$$

If S_n is empty, then C_φ is power-compact, which we have covered, so we will assume S_n is nonempty. We will say that the chosen element b of S_n *establishes* $\rho_e(C_\varphi)$.

Theorem 2.7. *Suppose φ, not an inner function, is an analytic self-map of $\mathbb{D}$ which is univalent on $\mathbb{D}$ and analytic in a neighborhood of $\overline{\mathbb{D}}$, with Denjoy–Wolff point $a \in \mathbb{D}$. Suppose $b \in \partial\mathbb{D}$ establishes $\rho_e(C_\varphi)$. Let $\psi \in H^\infty$ be continuous at b and let $|\psi(b)| = \|\psi\|_\infty$. Then*

(1) $\|W_{\psi,\varphi}\|_e = |\psi(b)|\|C_\varphi\|_e$,
(2) $\rho_e(W_{\psi,\varphi}) = |\psi(b)|\rho_e(C_\varphi)$, *and*
(3) $\rho(W_{\psi,\varphi}) = \max\{|\psi(a)|, |\psi(b)|\rho_e(C_\varphi)\}$.

Proof. By [2, Theorem 7.31], we have that

$$\rho_e(C_\varphi) = \lim_{k\to\infty}\left(\limsup_{|w|\to 1} \frac{\|K_{\varphi_k(w)}\|}{\|K_w\|}\right)$$

and this happens in particular as w approaches the element b of $\partial\mathbb{D}$ that gives the maximum value in the definition in Remark 2.6.

Now, we adapt the proof from [2, Proposition 3.13.] to show that

$$\|W_{\psi,\varphi}\|_e \geq |\psi(b)|\|C_\varphi\|_e.$$

Let w_j be a sequence in $\mathbb{D}$ tending to $\partial\mathbb{D}$. Then the normalized weight sequence $k_j = \frac{K_{w_j}}{\|K_{w_j}\|}$ tends to 0 weakly as j approaches infinity. If Q is an arbitrary compact operator on H^2, then $Q^*(k_j) \to 0$.

Now, $\|W_{\psi,\varphi}\|_e = \inf\{\|W_{\psi,\varphi} - Q\| : Q \text{ is compact}\}$, and for Q compact,

$$\begin{aligned}\|W_{\psi,\varphi} - Q\| &\geq \limsup_{j\to\infty} \|(W_{\psi,\varphi} - Q)^* k_j\| \\ &= \limsup_{j\to\infty} \|W_{\psi,\varphi}^* k_j\| \\ &= \limsup_{j\to\infty} |\psi(w_j)|\|C_\varphi^* k_j\|.\end{aligned}$$

Since $\|C_\varphi\|_e = \limsup_{j\to\infty} \|C_\varphi^* k_j\|$ is achieved by taking w_j tending towards b, and likewise $\limsup_{j\to\infty} |\psi(w_j)| = |\psi(b)| = \|\psi\|_\infty$ is achieved by taking w_j towards b, we have $\limsup_{j\to\infty} |\psi(w_j)|\|C_\varphi^* k_j\| = |\psi(b)|\|C_\varphi\|_e$ and $\|W_{\psi,\varphi}\|_e \geq |\psi(b)|\|C_\varphi\|_e$.

For the other direction, note that since the compact operators are an ideal, $T_\psi Q$ is compact for any compact operator Q, and if $B \subseteq A$, then $\inf A \leq \inf B$. Then

$$\begin{aligned}
\|W_{\psi,\varphi}\|_e &= \inf\{\|W_{\psi,\varphi} - Q\| : Q \text{ is compact}\} \\
&\leq \inf\{\|W_{\psi,\varphi} - T_\psi Q\| : Q \text{ is compact}\} \\
&= \inf\{\|T_\psi(C_\varphi - Q)\| : Q \text{ is compact}\} \\
&\leq \inf\{\|T_\psi\|\|C_\varphi - Q\| : Q \text{ is compact}\} \\
&= \inf\{\|\psi\|_\infty\|C_\varphi - Q\| : Q \text{ is compact}\} \\
&= \inf\{|\psi(b)|\|C_\varphi - Q\| : Q \text{ is compact}\} \\
&= |\psi(b)| \inf\{\|C_\varphi - Q\| : Q \text{ is compact}\} \\
&= |\psi(b)|\|C_\varphi\|_e.
\end{aligned}$$

Therefore we have $\|W_{\psi,\varphi}\|_e = |\psi(b)|\|C_\varphi\|_e$.

Suppose momentarily that b is a fixed point of φ. Since φ is analytic in a neighborhood of $\overline{\mathbb{D}}$ (i.e. continuous at b), the above gives the same result if ψ is replaced by $\psi \circ \varphi_k$ for any k. Then,

$$\begin{aligned}
\rho_e(W_{\psi,\varphi}) &= \lim_{k\to\infty} \left(\|{W_{\psi,\varphi}}^k\|_e\right)^{1/k} \\
&= \lim_{k\to\infty} \left(\|T_{(\psi)(\psi\circ\varphi)(\psi\circ\varphi_2)\cdots(\psi\circ\varphi_{k-1})} C_{\varphi_k}\|_e\right)^{1/k} \\
&= \lim_{k\to\infty} \left(|\psi(b)|^k \|C_{\varphi_k}\|_e\right)^{1/k} \\
&= |\psi(b)| \lim_{k\to\infty} \left(\|C_{\varphi_k}\|_e\right)^{1/k} \\
&= |\psi(b)|\rho_e(C_\varphi).
\end{aligned}$$

Now, while b may not be a fixed point of φ, we know it is the fixed point of some nth iterate φ_n. Furthermore, we know that $\rho_e(W_{\psi,\varphi}) = \lim_{k\to\infty} \left(\|{W_{\psi,\varphi}}^k\|_e\right)^{1/k}$ is a convergent sequence. Therefore, every subsequence converges to $\rho_e(W_{\psi,\varphi})$. By taking the subsequence with indices nk as $k \to \infty$, we see that $\rho_e(W_{\psi,\varphi}) = |\psi(b)|\rho_e(C_\varphi)$.

The complement in $\sigma(W_{\psi,\varphi})$ of the essential spectrum consists of eigenvalues of finite multiplicity. By [7, Proposition 4.3], any eigenvalue of $W_{\psi,\varphi}$ must be of the form

$$\{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \ldots\}$$

and the largest of those values in magnitude is $\psi(a)$. This value is necessarily in $\sigma(W_{\psi,\varphi})$, since $W_{\psi,\varphi}^*(K_a) = \overline{\psi(a)}K_a$.

Therefore, $\rho(W_{\psi,\varphi}) = \max\{|\psi(a)|, |\psi(b)|\rho_e(C_\varphi)\}$. $\square$

Corollary 2.8. *Suppose φ, not an inner function, is an analytic self-map of $\mathbb{D}$ which is univalent on $\mathbb{D}$ and analytic in a neighborhood of $\overline{\mathbb{D}}$, with Denjoy–Wolff point $a \in D$. Suppose $b \in \partial\mathbb{D}$ establishes $\rho_e(C_\varphi)$. Let $\psi \in H^\infty$ be continuous at b and let $|\psi(b)| = \|\psi\|_\infty$.*

Suppose further that $W_{\psi,\varphi}$ is normaloid. Then, either $\|W_{\psi,\varphi}\| = \rho(W_{\psi,\varphi}) = |\psi(a)|$ and $\psi = \psi(a)\frac{K_a}{K_a\circ\varphi}$, or $\|W_{\psi,\varphi}\| = \rho(W_{\psi,\varphi}) = \rho_e(W_{\psi,\varphi}) = |\psi(b)|\rho_e(C_\varphi)$.

Proof. By Theorem 2.7, $\rho(W_{\psi,\varphi}) = \max\{|\psi(a)|, |\psi(b)|\rho_e(C_\varphi)\}$. If $\rho(W_{\psi,\varphi}) = |\psi(a)|$, by Theorem 2.2, ψ has the form $\psi = \psi(a)\frac{K_a}{K_a\circ\varphi}$. Otherwise, $\rho(W_{\psi,\varphi}) = |\psi(b)|\rho_e(C_\varphi)$. □

Example 2.9. Let $W_{\psi,\varphi}$ be the hyponormal operator given by $\psi(z) = \frac{2e^z}{(2-z)}, \varphi(z) = \frac{z}{2-z}$ [6, Example 3.8]. Then $\rho(W_{\psi,\varphi}) = |\psi(1)|\varphi'(1)^{-1/2} = \sqrt{2}e$.

While neither Theorem 2.7 nor Corollary 2.8 gives sufficient conditions for $W_{\psi,\varphi}$ to be normaloid, Theorem 2.7 does accomplish something else. An operator T is said to be *essentially normaloid* if $\rho_e(T) = \|T\|_e$.

Corollary 2.10. *Suppose φ, not an inner function, is an analytic self-map of $\mathbb{D}$ which is univalent on $\mathbb{D}$ and analytic in a neighborhood of $\overline{\mathbb{D}}$, with Denjoy–Wolff point $a \in D$. Let b be a fixed point of φ on ∂D such b that establishes $\rho_e(C_\varphi)$. Let $\psi \in H^\infty$ be continuous at b and let $|\psi(b)| = \|\psi\|_\infty$. Then $W_{\psi,\varphi}$ is essentially normaloid if and only if C_φ is essentially normaloid.*

Proof. This is an immediate consequence of (1) and (2) in Theorem 2.7. □

Example 2.11. Let $\varphi(z) = \frac{z}{2-z}, \psi = e^{-z}$. Then $|\psi(0)| = 1$ and $|\psi(1)|\varphi'(1)^{-1/2} = \frac{\sqrt{2}}{2e} < 1$. Since ψ is not of the form $|\psi(0)|\frac{K_0}{K_0\circ\varphi} \equiv 1$, $W_{\psi,\varphi}$ is not normaloid. However, since C_φ is essentially normaloid [2, Theorem 7.31, 7.36], therefore so is $W_{\psi,\varphi}$.

3. Uniformly convergent iteration (UCI)

We now turn our attention to when φ has Denjoy–Wolff point $a \in \partial\mathbb{D}$. We wish to continue to assume that $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$. Our goal in this section, before determining when such operators are normaloid, is to show that this class is non-trivial. To do so, we will put restrictions on how the iterates of φ converge to the Denjoy–Wolff point. This definition is from [5], where this hypothesis is used to determine the spectrum of weighted composition operators in this setting.

The Denjoy–Wolff Theorem [2, Theorem 2.51] states that all analytic self-maps of $\mathbb{D}$ other than elliptic automorphisms have a point in $\overline{\mathbb{D}}$ that they converge to under iteration on compact subsets of $\mathbb{D}$. Here, we ask the convergence to be stronger.

Definition 3.1 *(Uniformly convergent iteration).* We say φ is UCI if φ is an analytic self-map of $\mathbb{D}$ and the iterates of φ converge uniformly to the Denjoy–Wolff point uniformly on *all* of $\mathbb{D}$, rather than compact subsets of $\mathbb{D}$.

If φ is UCI and the Denjoy–Wolff point a of φ belongs to $\mathbb{D}$, then $W_{\psi,\varphi}$ is power-compact [5, Corollary 2], so we have already covered that scenario in Section 2 without requiring this additional hypothesis.

Analytic self-maps of $\mathbb{D}$ that exhibit UCI while having Denjoy–Wolff point a on $\partial\mathbb{D}$ are a non-trivial set, and include maps whose derivative at the Denjoy–Wolff point are both less than 1 (e.g. $\varphi(z) = (z+1)/2$) and equal to 1 (e.g. $\varphi(z) = 1/(2-z)$) [5, Example 5]. A simple sufficient condition for UCI when $\varphi'(a) < 1$ is that $\varphi_N(\overline{\mathbb{D}}) \subseteq \mathbb{D} \cup \{a\}$ for some N [5, Theorem 4]. This includes, then, any linear-fractional map with Denjoy–Wolff point on the boundary and $\varphi'(a) < 1$.

The main reason to now introduce this definition is the following theorem, proved in [5].

Theorem 3.2. *Suppose φ is UCI with Denjoy–Wolff point $a \in \partial\mathbb{D}$, $\psi \in H^\infty$ is continuous at a, and $\psi(a) \neq 0$. Then:*

(1) $\overline{\sigma_p(\psi(a)C_\varphi)} \subseteq \overline{\sigma_{ap}(T_\psi C_\varphi)} \subseteq \sigma(T_\psi C_\varphi) \subseteq \sigma(\psi(a)C_\varphi)$,
(2) *if* $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$, *then* $\sigma(T_\psi C_\varphi) = \sigma(\psi(a)C_\varphi)$,
(3) *if* $\varphi'(a) < 1$, *then* $\sigma(T_\psi C_\varphi) = \sigma(\psi(a)C_\varphi)$ **and** $\sigma_p(W_{\psi,\varphi}) = \sigma_p(\psi(a)C_\varphi)$.

We have an immediate corollary regarding the spectral radius.

Corollary 3.3. *Suppose φ is UCI with Denjoy–Wolff point $a \in \partial\mathbb{D}$, $\psi \in H^\infty$ is continuous at a, and $\psi(a) \neq 0$. Then $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$.*

Proof. If $\varphi'(a) < 1$, then (3) of the Theorem 3.2 makes this clear. If $\varphi'(a) = 1$, note that $1 \in \sigma_p(C_\varphi)$, so $\psi(a) \in \sigma_p(\psi(a)C_\varphi)$, and therefore by (1) of Theorem 3.2, $\rho(W_{\psi,\varphi}) \geq |\psi(a)|$. Again by (1), we also have $\sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(a)C_\varphi)$, so $\rho(W_{\psi,\varphi}) \leq \rho(\psi(a)C_\varphi) = |\psi(a)|\rho(C_\varphi) = |\psi(a)|$, since $\rho(C_\varphi) = 1$ [2, Theorem 3.9]. Therefore $\rho(W_{\psi,\varphi}) = |\psi(a)| = |\psi(a)|\rho(C_\varphi)$. □

4. $a \in \partial\mathbb{D}$

Here we follow the same path as Section 2. We will continue to assume that we have $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$, and will seek to determine conditions for which $\|W_{\psi,\varphi}\|$ is the same. We will also obtain a few corollaries for when φ is explicitly UCI.

Theorem 4.1. *Suppose φ is an analytic self-map of $\mathbb{D}$ with Denjoy–Wolff point $a \in \partial\mathbb{D}$, $\psi \in H^\infty$, and ψ is continuous at the Denjoy–Wolff point a of φ, with $\|\psi\|_\infty = |\psi(a)|$. Furthermore, assume $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$. If C_φ is normaloid, then $W_{\psi,\varphi}$ is normaloid.*

Proof. Note that

$$\begin{aligned} \|W_{\psi,\varphi}\| &\leq \|T_\psi\|\|C_\varphi\| \\ &= \|\psi\|_\infty\|C_\varphi\| \\ &= |\psi(a)|\|C_\varphi\| \\ &= |\psi(a)|\rho(C_\varphi) \\ &= \rho(W_{\psi,\varphi}) \leq \|W_{\psi,\varphi}\|. \quad \square \end{aligned}$$

Example 4.2. If $\psi = e^z$ and $\varphi = (z+1)/2$, then $\|\psi\|_\infty = e = \psi(1)$. Since C_φ is cohyponormal and therefore normaloid [2, Theorem 8.7], $W_{\psi,\varphi}$ is normaloid and also convexoid.

While examples generated by Theorem 4.1 are reasonable to come by when $\varphi'(a) < 1$, they are actually impossible to come by when $\varphi'(a) = 1$.

Theorem 4.3. *Suppose φ is an analytic self-map of $\mathbb{D}$ with Denjoy–Wolff point $a \in \partial\mathbb{D}$ and $\varphi'(a) = 1$. Then C_φ is not normaloid.*

Proof. The proof is analogous to Theorem 2.1. The spectral radius for C_φ is $\varphi'(a)^{-1/2}$ when $a \in \partial\mathbb{D}$ [2, Theorem 3.9], so here we have $\rho(C_\varphi) = 1$. Since $\varphi(0) \neq 0$, we know $\|C_\varphi\| > 1$, therefore C_φ is not normaloid. □

However, there *are* known weighted composition operators where $\varphi'(a) = 1$ and $W_{\psi,\varphi}$ is normaloid – even self-adjoint [3]. We make a minor adjustment to Theorem 4.1 to generate new examples of normaloid weighted composition operators in this setting.

Corollary 4.4. *Suppose φ is an analytic self-map of $\mathbb{D}$ with Denjoy–Wolff point a, $\psi \in H^\infty$ is continuous at a, and $f \in H^\infty$ is also f is continuous at a, with $\|f\|_\infty = |f(a)|$. If $W_{\psi,\varphi}$ is normaloid and $\rho(W_{f\psi,\varphi}) = |f(a)|\rho(W_{\psi,\varphi})$, then $W_{f\psi,\varphi}$ is normaloid.*

Proof. The proof is identical to Theorem 4.1, with a mere adjustment of symbols:

$$\begin{aligned}\|W_{f\psi,\varphi}\| &\le \|T_f\|\|W_{\psi,\varphi}\| \\ &= \|f\|_\infty \|W_{\psi,\varphi}\| \\ &= |f(a)|\|W_{\psi,\varphi}\| \\ &= |f(a)|\rho(W_{\psi,\varphi}) \\ &= \rho(W_{f\psi,\varphi}) \\ &\le \|W_{f\psi,\varphi}\|. \quad \square\end{aligned}$$

Example 4.5. Suppose $\psi(z) = \frac{1}{2-z}$, $f(z) = e^z$. Then $W_{\psi,\psi}$ is self-adjoint and therefore normaloid by [3, Theorem 6]. Since ψ is UCI [5, Example 5], we have $\rho(W_{f\psi,\psi}) = |f(1)||\psi(1)| = |f(1)|\rho(W_{\psi,\psi})$. Since $|f(1)| = e = \|f\|_\infty$, we have that $W_{f\psi,\psi}$ is normaloid.

We end this section with a few extra facts for when φ is UCI and $\varphi'(a) < 1$.

Theorem 4.6. *Suppose φ is UCI, the Denjoy–Wolff point a of φ is on $\partial\mathbb{D}$, and $\varphi'(a) < 1$. Then $W_{\psi,\varphi}$ is convexoid if and only if $W_{\psi,\varphi}$ is spectraloid.*

Proof. Every convexoid operator is spectraloid. In the other direction, assume $W_{\psi,\varphi}$ is spectraloid, so that $\rho(W_{\psi,\varphi}) = r(W_{\psi,\varphi})$. Note that by (3) of Theorem 3.2, the spectrum of $W_{\psi,\varphi}$ is a closed disk centered at the origin, completely filling in the set $\{\lambda \in \mathbb{C} : |\lambda| \le \rho(W_{\psi,\varphi})\}$. Since $\rho(W_{\psi,\varphi}) = r(W_{\psi,\varphi})$, this set is necessarily also the closure of the numerical range. Therefore, $W_{\psi,\varphi}$ is convexoid. $\square$

Corollary 4.7. *Suppose φ is UCI, the Denjoy–Wolff point a of φ is on $\partial\mathbb{D}$, and $\varphi'(a) < 1$. If $W_{\psi,\varphi}$ is normaloid, then $W_{\psi,\varphi}$ is convexoid.*

Proof. Every normaloid operator is spectraloid, so $W_{\psi,\varphi}$ is spectraloid. By Theorem 4.6, if $W_{\psi,\varphi}$ is spectraloid, it is also convexoid. $\square$

5. Further questions

Here we summarize the questions raised by the work of this paper.

(1) If $W_{\psi,\varphi}$ is normaloid and $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$, is it necessary that $|\psi(a)| = \|\psi\|_\infty$?
(2) Can the many hypotheses of Theorem 2.7 be weakened, to identify ρ_e in the general setting when φ has interior Denjoy–Wolff point and C_φ is not power-compact?
(3) Can we then characterize all normaloid weighted composition operators where φ has Denjoy–Wolff point in $\mathbb{D}$?
(4) What are the necessary conditions for $W_{\psi,\varphi}$ to be normaloid when the Denjoy–Wolff point of φ is on $\partial\mathbb{D}$?
(5) Ultimately, can we get an exact characterization of when $W_{\psi,\varphi}$ is normaloid?

Please cite this article in press as: D. Thompson, Normaloid weighted composition operators on H^2, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.07.061

Acknowledgments

The author would like to thank the Taylor University Distinguished Lecturer program for funding this research, and the reviewer for his extremely helpful comments.

References

[1] P.S. Bourdon, S.K. Narayan, Normal weighted composition operators on the Hardy space $H^2(\mathbb{D})$, J. Math. Anal. Appl. 367 (2010) 278–286.

[2] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995, MR 97i:47056.

[3] C.C. Cowen, G. Gunatillake, E. Ko, Hermitian weighted composition operators and Bergman extremal functions, Complex Anal. Oper. Theory 7 (1) (2013) 69–99.

[4] C.C. Cowen, S. Jung, E. Ko, Normal and cohyponormal weighted composition operators on H^2, Oper. Theory Adv. Appl. 240 (2014) 69–85.

[5] C.C. Cowen, E. Ko, D. Thompson, F. Tian, Spectra of some weighted composition operators on H^2, Acta Sci. Math. (Szeged) 82 (2016) 221–234.

[6] M. Fatehi, M. Haji Shaabani, D. Thompson, Quasinormal and hyponormal weighted composition operators on H^2 and A^2_α with linear fractional compositional symbol, Complex Anal. Oper. Theory (2017), https://doi.org/10.1007/s11785-017-0683-3.

[7] C. Hammond, On the Norm of a Composition Operator, Thesis, University of Virginia, 2003.