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On the set of shadowable measures

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A R T I C L E I N F O

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A B S T R A C T

We first prove that the set of shadowable measures of a homeomorphism of a compact metric space X is an $F_{\sigma\delta}$ set of the space $\mathcal{M}(X)$ of Borel probability measures of X equipped with the weak*-topology. Next that the set of shadowable measures is dense in $\mathcal{M}(X)$ if and only if the set of shadowable points is dense in X. Therefore, if X has no isolated points and every non-atomic Borel probability measure has the shadowing property, then the shadowable points are dense in X (this is false when the space has isolated points). Afterwards, we consider the *almost shadowable measures* (measures for which the shadowable point set has full measure) and prove that all of them are weak* approximated by shadowable ones. In addition the set of almost shadowable measures is a G_δ set of $\mathcal{M}(X)$. Furthermore, the closure of the shadowable points is the union of the supports of the almost shadowable measures. Finally, we prove that every almost shadowable measure can be weak* approximated by ones with support equals to the closure of the shadowable points.

1. Introduction

A well known result in measure theory is that the set of nonatomic Borel probability measures of a compact metric space X is a G_δ in $\mathcal{M}(X)$, the space of all Borel probability measures equipped with the weak* topology. It is known also that the set of ergodic invariant measures of a homeomorphism of X is also a G_δ set of $\mathcal{M}(X)$ ([10]). More recently, it was proved that the set of expansive measures of a homeomorphism of X is a $G_{\delta\sigma}$ set of $\mathcal{M}(X)$ ([7]). In light of these results it is natural to ask whether the same property holds for the set of shadowable measures of a homeomorphism of X (as defined in [6]).

In this paper we will give an answer for this question. Indeed, we shall prove that the set of shadowable measures is an $F_{\sigma\delta}$ set of $\mathcal{M}(X)$. In particular, the set of *invariant* shadowable measures is an $F_{\sigma\delta}$ subset of the space of invariant measures. We will also prove that the set of shadowable measures is dense in $\mathcal{M}(X)$ if and only if the shadowable points set (e.g. [1], [2], [3]) is dense in X. We will consider also a class of measures to be referred to *almost shadowable* and prove that they form together a closed subset of

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$\mathcal{M}(X)$. We also establish relationships between the shadowable points set and the set of almost shadowable measures. Namely we prove that the closure of the former set is the union of the supports of the measures in the latter set. Consequently, every almost shadowable measure can be approximated by ones with support equal to the closure of the shadowable points set. Let us state our results in a precise way.

Hereafter X will denote a compact metric space. The Borel σ-algebra of X is the σ-algebra $\mathcal{B}(X)$ generated by the open subsets of X. A *Borel probability measure* is a σ-additive measure μ defined in $\mathcal{B}(X)$ such that $\mu(X) = 1$. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures of X. This set is a compact metrizable convex space and its topology is the *weak* topology* defined by the convergence $\mu_n \to \mu$ if and only if $\int \phi d\mu_n \to \int \phi d\mu$ for every continuous map $\phi : X \to \mathbb{R}$.

Let $f : X \to X$ denote a homeomorphism of a compact metric space. Given $\delta > 0$ and bi-infinite sequence $(x_n)_{n\in\mathbb{Z}}$ of points of X, say that $(x_n)_{n\in\mathbb{Z}}$ is a *δ-pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for every $n \in \mathbb{Z}$. Also, if there is $y \in X$ such that $d(f^n(y), x_n) < \delta$ for every $n \in \mathbb{Z}$, then we say that $(x_n)_{n\in\mathbb{Z}}$ can be *δ-shadowed*. We say that f has the *shadowing property* [11] if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ϵ-shadowed.

Remark 1.1. Some authors usually use non-strict inequalities in the definitions of pseudo-orbits and shadowing. The definition of the shadowing property with strict inequalities is equivalent to the classical one and allows us to prove our main results.

Next we present one of the main definitions of this work.

Definition 1.2 *([6])*. Let $f : X \to X$ be a homeomorphism of a compact metric space. We say that a Borel probability measure μ of X is *shadowable* with respect to f if for every $\epsilon > 0$ there are $\delta > 0$ and a Borelian $B \subset X$ with $\mu(B) = 1$ such that every δ-pseudo-orbit $(x_n)_{n\in\mathbb{Z}}$ *through* B (i.e., $x_0 \in B$) can be ϵ-shadowed.

Denote by $\mathcal{M}_{sh}(f)$ the set of all shadowable measures of f. Recall that a subset of a topological space is a G_δ *subset* if it is the intersection of countably many open sets. We say that it is an F_σ *subset* if it is the union of countably many closed sets. Also an $F_{\sigma\delta}$ *subset* is the intersection of countably many F_σ subsets. The $F_{\sigma\delta}$ subsets are precisely the class $\prod_3^0$ in the Borel hierarchy [4].

Now we state our first result.

Theorem 1.3. *The set of shadowable measures of a homeomorphism of a compact metric space X is an $F_{\sigma\delta}$ set of $\mathcal{M}(X)$.*

Recall that a point $x \in X$ is a *shadowable* of $f : X \to X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit through x can be ϵ-shadowed. Denote by $Sh(f)$ the set of all shadowable points of f [1], [2], [3], [8]. The next result establishes a relationship between shadowable points and shadowable measures.

Theorem 1.4. *The shadowable measures of a homeomorphism of a compact metric space X are dense in $\mathcal{M}(X)$ (with respect to the weak* topology) if and only if the shadowable points are dense in X.*

From this theorem we obtain the following corollary. Recall that a metric space *has no isolated points* if no open ball reduces to a singleton. These sets are also referred to as *dense by itself*. A Borel probability measure is *non-atomic* if it has no points of positive mass.

Corollary 1.5. *Let $f : X \to X$ be a homeomorphism of a compact metric space without isolated points. If every non-atomic Borel probability measure has the shadowing property, then the shadowable points are dense in X.*

The next example shows that this corollary is false for general metric spaces.

Example 1.6. There is an uncountable compact metric space X and a homeomorphism $f : X \to X$ for which every non-atomic Borel probability measure is shadowable but $Sh(f)$ is not dense in X.

Proof. Let X be the disjoint of a compact interval $I = [a, b]$ (disjoint from the unit circle S^1) and a bi-infinite sequence $(a_n)_{n\in\mathbb{Z}} \in S^1$ such that $a_n \to a_\infty$ as $n \to \pm\infty$. Put in X the metric induced from the standard metric of $\mathbb{R}^2$ and consider the map $f : X \to X$ so that $f|_I$ is the pole North–South map (i.e. orbits go from a to b), $f(a_\infty) = a_\infty$ and $f(a_n) = a_{n+1}$ for all $n \in \mathbb{Z}$. It follows that f is a homeomorphism and $f|_I$ has the shadowing property since it is Morse–Smale [9]. Since $I \cap S^1 = \emptyset$ and $a_\infty \in S^1$, $a_\infty \notin I$ and so $I \subset Sh(f)$. Since the support of every nonatomic measure is contained in I, we can apply Lemma 2.3 in [8] to conclude that every non-atomic Borel probability measure is shadowable. Since no point a_n is shadowable, $Sh(f)$ is not dense in X. $\square$

Now we will consider the almost shadowable measures.

Definition 1.7. Let $f : X \to X$ be a homeomorphism of a compact metric space. A Borel probability measure μ of X is *almost shadowable* if $\mu(Sh(f)) = 1$.

Kaguaguchi [3] proved that a homeomorphism of a compact metric space has the shadowing property if and only if every *invariant* ergodic measure is almost shadowable [3].

We will see later that the set of shadowable measures is a subset of the set of almost shadowable measures. But how big is the former set inside the latter one? Here we will prove the former set is dense in the latter. And then we will show the set of almost shadowable measures is Baire subspace of $\mathcal{M}(X)$. Denote by $\mathcal{M}_{ash}(f)$ the set of measures having the almost shadowing property of a homeomorphism f. Hereafter we denote by $\overline{B}$ the closure of B.

Theorem 1.8. *If $f : X \to X$ is a homeomorphism of a compact metric space X, then $\mathcal{M}_{ash}(f) \subset \overline{\mathcal{M}_{sh}(f)}$ and $\mathcal{M}_{ash}(f)$ is a G_δ set $\mathcal{M}(X)$.*

We obtain immediately the following corollary of Theorem 1.8.

Corollary 1.9. *If $f : X \to X$ is a homeomorphism of a compact metric space X, then $\mathcal{M}_{ash}(f)$ is a Baire subspace of $\mathcal{M}(X)$.*

Next we analyze the relationship between the set of shadowable points and almost shadowable measures. Namely we prove that the closure of the shadowable points set is closely related with the supports of the almost shadowable measures. Recall that the *support* of $\mu \in \mathcal{M}(X)$ is the set $Supp(\mu)$ of points $x \in X$ such that $\mu(U) > 0$ for any neighborhood U of x. It follows that $Supp(\mu)$ is a nonempty compact subset of X.

Theorem 1.10. *For every homeomorphism of a compact metric space $f : X \to X$ we have $\overline{Sh(f)} = \bigcup\{Supp(\mu) : \mu \in \mathcal{M}_{ash}(f)\}$. Moreover, $\{\mu \in \mathcal{M}_{ash}(f) : Supp(\mu) = \overline{Sh(f)}\}$ is dense in $\mathcal{M}_{ash}(f)$.*

This paper is organized as follows. In Section 2 we prove some preliminary results and Theorem 1.8. In Section 3 we prove the remainder results.

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2. Preliminary results and proof of Theorem 1.8

Let $f : X \to X$ be a homeomorphism of a compact metric space. Given $\epsilon > 0$ and $\delta > 0$, we say a Borelian $B \in \mathcal{B}(X)$ is (ε, δ)-*shadowable* with respect to f if every δ-pseudo-orbit through B can be ε-shadowed. The following lemma deals with the closure of a shadowable set.

Lemma 2.1. *Let $f : X \to X$ be a homeomorphism of a compact metric space. If B is (ϵ, δ)-shadowable, then $\overline{B}$ is $(\epsilon + \gamma, \delta)$-shadowable for each $\gamma > 0$.*

Proof. Fix $\epsilon > 0$, $\delta > 0$, an (ϵ, δ)-shadowable set B and $\gamma > 0$. Let $(x_k)_{k\in\mathbb{Z}}$ be a δ-pseudo-orbit through $\overline{B}$. Since $x_0 \in \overline{B}$, there is $y \in B$ such that

$$d(x_0, y) < \min\{\delta - d\big(f(x_{-1}), x_0\big), \gamma\} \text{ and } d\big(f(x_0), f(y)\big) < \delta - d\big(f(x_0), x_1\big).$$

Define the sequence $(y_k)_{k\in\mathbb{Z}}$ by $y_k = x_k$ for $k \neq 0$ and $y_0 = y$.

We claim that $(y_k)_{k\in\mathbb{Z}}$ is a δ-pseudo-orbit through $y \in B$. From

$$\begin{aligned} d\big(f(y_{-1}), y_0\big) &\le d\big(f(y_{-1}), x_0\big) + d(x_0, y) \\ &< d\big(f(x_{-1}), x_0\big) + \min\big\{\delta - d\big(f(x_{-1}), x_0\big), \gamma\big\} < \delta \end{aligned}$$

and

$$d\big(f(y_0), y_1\big) \le d\big(f(y), f(x_0)\big) + d\big(f(x_0), y_0\big) < \delta$$

we get $(y_k)_{k\in\mathbb{Z}}$ is a δ-pseudo-orbit. $y_0 = y \in B$ implies that such a pseudo-orbit is through B. Since B is (ϵ, δ)-shadowable, $(y_k)_{k\in\mathbb{Z}}$ can be ϵ-shadowed. i.e., there is $z \in X$ such that $d\big(f^i(z), y_i\big) \le \epsilon$ for all $i \in \mathbb{Z}$. This, together with

$$d\big(f^0(z), x_0\big) = d(z, x_0) \le d(z, y) + d(y, x_0) < \epsilon + \delta,$$

implies $d\big(f^i(z), x_i\big) < \epsilon + \delta$ for all $i \in \mathbb{Z}$. □

Next lemma is an useful characterization of the set of shadowable measures.

Given $\epsilon > 0$ and $\delta > 0$, denote

$$C(\epsilon, \delta) = \big\{\mu \in \mathcal{M}(X) : \exists \text{ an } (\epsilon, \delta)\text{-shadowable set } B \text{ with } \mu(B) = 1\big\}.$$

Lemma 2.2. *If $f : X \to X$ is a homeomorphism of a compact metric space, then*

$$\mathcal{M}_{sh}(f) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, m^{-1}\big).$$

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Proof. We claim that

$$\mathcal{M}_{sh}(f) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C\big(n^{-1}, m^{-1}\big).$$

Fix $\mu \in \mathcal{M}_{sh}(f)$ and $n \in \mathbb{N}^+$. There exist $\delta > 0$ and $B \in \beta(X)$ which is (n^{-1}, δ)-shadowable with $\mu(B) = 1$. For $m \in \mathbb{N}^+$ with $m^{-1} < \delta$, it is clear that B is also (n^{-1}, m^{-1})-shadowable. So $\mu \in \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C\big(n^{-1}, m^{-1}\big)$ and the claim follows.

Cleary, $C(\epsilon, \delta) \subseteq \bigcap_{l=1}^{\infty} C\big(\epsilon + l^{-1}, \delta\big)$. So the claim implies

$$\begin{aligned} \mathcal{M}_{sh}(f) &\subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C\big(n^{-1}, m^{-1}\big) \\ &\subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, m^{-1}\big). \end{aligned}$$

To verify the reversed inclusion, take

$$\mu \in \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, m^{-1}\big).$$

Then, for every $n \in \mathbb{N}^+$ there is $M \in \mathbb{N}^+$ such that

$$\mu \in \bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, M^{-1}\big).$$

Hence μ has an $\big(n^{-1} + l^{-1}, M^{-1}\big)$-shadowable set B with $\mu(B) = 1$ for every $n, l \in \mathbb{N}^+$. Therefore, for any $\epsilon > 0$, there are $N,\ L \in \mathbb{N}^+$ such that $N^{-1} + L^{-1} < \epsilon$ and $\mu \in C\big(N^{-1} + L^{-1}, M^{-1}\big)$.

Therefore, $\mu \in \mathcal{M}_{sh}(f)$ by taking $\delta = M^{-1}$ for every $\epsilon > 0$ in the definition of shadowing measure. □

Recalling that the set $\mathcal{M}(X)$ is equipped with the weak* topology, we can consider the following lemma.

Lemma 2.3. *If $f : X \to X$ is a homeomorphism of a compact metric space, then $\bigcap_{\gamma > 0} C(\epsilon + \gamma, \delta)$ is closed in $\mathcal{M}(X)$ for every $\epsilon > 0$ and $\delta > 0$.*

Proof. Fix $\epsilon > 0$ and $\delta > 0$. Take $\gamma > 0$ together with $\mu_n \in C(\epsilon + \frac{\gamma}{2}, \delta)$ such that $\mu_n \to \mu$ for some $\mu \in \mathcal{M}(X)$. Choose a sequence $B_n \in \beta(X)$ which is $(\epsilon + \frac{\gamma}{2}, \delta)$-shadowable with $\mu_n(B_n) = 1$. Put $B = \bigcup_{n=1}^{\infty} B_n$. Then $\overline{B}$ is $\big((\epsilon + \frac{\gamma}{2}) + \frac{\gamma}{2}, \delta\big)$-shadowable by Lemma 2.1. As $B \subset \overline{B}$,

$$1 = \mu_n(B_n) \le \mu_n(B) \le \mu_n(\overline{B}).$$

This, together with $\mu_n \to \mu$, implies

$$1 = \limsup_{n \to \infty} \mu_n(\overline{B}) \le \mu(\overline{B}),$$

i.e., $\mu(\overline{B}) = 1$. Therefore, $\mu \in C(\epsilon + \gamma, \delta)$. As γ is arbitrary, the proof follows. □

Lemma 2.4. *If $f : X \to X$ be a homeomorphism of a compact metric space, then $\mathcal{M}_{sh}(f) \subset \mathcal{M}_{ash}(f)$.*

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Proof. For any fixed $\mu \in \mathcal{M}_{sh}(f)$, there exist sequences of numbers δ_n and Borelians B_n with $\mu(X \setminus B_n) = 0$ such that for every $n \in \mathbb{N}^+$ every δ_n-pseudo-orbit through B_n can be $\frac{1}{n}$-shadowed. Put $B = \bigcap_{n\in\mathbb{N}^+} B_n$. Since

$$\begin{aligned}\mu(X \setminus B) &= \mu\big(X \setminus \bigcap_{n\in\mathbb{N}^+} B_n\big)\\ &= \mu\big(\bigcup_{n\in\mathbb{N}^+} (X \setminus B_n)\big)\\ &\le \sum_{n\in\mathbb{N}^+} \mu(X \setminus B_n) = 0,\end{aligned}$$

$\mu(B) = 1$. Now take $x \in B$ and $\epsilon > 0$. Fix $N \in \mathbb{N}^+$ with $\frac{1}{N} < \epsilon$. Define $\delta = \delta_N$ and take a δ pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ through x. As $x \in B = \bigcap_{n\in\mathbb{N}^+} B_n$, $x \in B_N$ and so $(x_k)_{k\in\mathbb{Z}}$ intersects B_N. Then, $(x_k)_{k\in\mathbb{Z}}$ can be ϵ-shadowed proving $x \in Sh(f)$. Therefore $B \subset Sh(f)$ and so $\mu(Sh(f)) \ge \mu(B) = 1$. It follows that $\mu \in \mathcal{M}_{ash}(f)$. $\square$

We say that a Borel probability measure μ is *supported on* B if $Supp(\mu) \subset B$.

Lemma 2.5. *Let $f : X \to X$ be a homeomorphism of a compact metric space. Then, every Borel probability measure supported on $Sh(f)$ has the shadowing property.*

Proof. Fix $\epsilon > 0$ and a Borel probability measure μ supported on $Sh(f)$. By Lemma 2.2 in [8] for every $x \in Supp(\mu)$ there is $\delta_x > 0$ such that every δ_x-pseudo-orbit through the closed ball $B[x, \delta_x]$ can be ϵ-shadowed. The collection of open balls $\{B(x, \delta_x) : x \in Supp(\mu)\}$ is an open covering of $Supp(\mu)$ which is compact. Then, there are finitely many points $x_1, \cdots, x_l \in Supp(\mu)$ such that $Supp(\mu) \subset \bigcup_{i=1}^{l} B[x_i, \delta_{x_i}]$. Put

$$B = \bigcup_{i=1}^{l} B[x_i, \delta_{x_i}].$$

$Supp(\mu) \subset B$ implies $\mu(B) = 1$, i.e., $\mu(X \setminus B) = 0$. Define $\delta = \min(\delta_{x_1}, \cdots, \delta_{x_l})$. Then, every δ-pseudo-orbit through B is a δ_{x_i}-pseudo-orbit through $B[x_i, \delta_{x_i}]$ for some $1 \le i \le l$ and, then, it can be ϵ-shadowed. $\square$

The next lemma is a characterization of the shadowable points in terms of Dirac measures. Denote by m_x the *Dirac measure* supported on $x \in X$, namely, $m_x(A) = 0$ or 1 depending on whether $x \notin A$ or $x \in A$.

Lemma 2.6. *If $f : X \to X$ is a homeomorphism of a compact metric space, then $Sh(f) = \{x \in X : m_x \in \mathcal{M}_{sh}(f)\}$.*

Proof. If $x \in Sh(f)$ then $Supp(m_x) = \{x\} \subseteq Sh(f)$. Lemma 2.5 implies $m_x \in \mathcal{M}_{sh}(f)$. Now suppose $m_x \in \mathcal{M}_{sh}(f)$. Fix $\epsilon > 0$. Take $\delta > 0$ and Borelian B with $m_x(B) = 1$ given by the shadowableness of m_x. Take a δ-pseudo-orbit through x. Since $m_x(B) = 1$, $x \in B$. So the δ-pseudo-orbit is through B and then it can be ϵ shadowed. $\square$

The following lemma seems to be well known and will be proved for the sake of completeness.

Lemma 2.7. *Let X a compact metric space. If $E \subset X$, every measure with finite support and supported on E is a finite convex combination of Dirac measures supported on points of E.*

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Proof. Let μ be a Borel probability measure with $Supp(\mu) = \{p_1, p_2, \cdots, p_n\} \subseteq E$. Define $\hat{\mu} = \mu(\{p_1\}) \cdot m_{p_1} + \cdots + \mu(\{p_n\}) \cdot m_{p_n}$. Then $\hat{\mu}$ is a finite convex combination of Dirac measures over points in E. Notice $\mu(\{p_i\}) > 0$ for any $i \in \{1, 2, \cdots, n\}$. For every Borelian B, we have

$$\begin{aligned}\mu(B) &= \mu\big(Supp(\mu) \cap B\big) \\ &= \sum_{i=1}^{n} \mu\big(\{p_i\} \cap B\big) \\ &= \sum_{i=1}^{n} \mu\big(\{p_i\}\big) \cdot m_{p_i}(B) \\ &= \hat{\mu}(B).\end{aligned}$$

Therefore $\mu = \hat{\mu}$. □

Lemma 2.8. *Let X be a compact metric space and $K \subseteq X$. If $\mu \in \mathcal{M}(X)$ is supported on $\overline{K}$, then there is a sequence $\nu_n \in \mathcal{M}(X)$ supported on K and converging to μ with respect to the weak* topology.*

Proof. $Supp(\mu) \subseteq \overline{K}$ allows to define $\hat{\mu} \in \mathcal{M}(\overline{K})$ as $\hat{\mu}(B) = \mu(B)$ for every Borelian B of $\overline{K}$. Since X is compact, $\overline{K}$ also is. So $\overline{K}$ is a separable metric space and K is dense in K. Then, the set of all measures whose supports are (finite) subsets of K is dense $\mathcal{M}(\overline{K})$ by Theorem 6.3 p. 44 in [10]. That is, there is a sequence (ν_n) of Borel probability measure in $\mathcal{M}(\overline{K})$ such that $Supp(\hat{\nu}_n) \subseteq K$ and $\hat{\nu}_n \to \hat{\nu} \in \mathcal{M}(\overline{K})$ with respect to the weak* topology on $\overline{K}$. $Supp(\mu) \subseteq \overline{K}$ implies $\nu_n \to \mu$ where $\nu_n(B) = \hat{\nu}_n(\overline{K} \cap B)$ for every Borelian B of X. Since $Supp(\hat{\nu}_n) \subseteq K$, $Supp(\nu_n) \subseteq K$ too and the result follows. □

Proof of Theorem 1.8. Take $\mu \in \mathcal{M}_{ash}(f)$ then $\mu(Sh(f)) = 1$ and $Supp(\mu) \subseteq \overline{Sh(f)}$. By Lemma 2.8, there is a sequence $\nu_n \in \mathcal{M}(X)$ supported on $Sh(f)$ such that $\nu_n \to \mu$. Since ν_n is supported on $Sh(f)$, $\nu_n \in \mathcal{M}_{sh}(f)$ by Lemma 2.5. Then, $\mu \in \overline{\mathcal{M}_{sh}(f)}$ proving $\mathcal{M}_{ash}(f) \subset \overline{\mathcal{M}_{sh}(f)}$.

On the other hand, by Corollary 2.10 in [1], $Sh(f)$ is a G_δ set. Then $Sh(f) = \bigcap_{k\in\mathbb{N}} G_k$ where each G_k is open set such that $G_1 \supseteq G_2 \supseteq \cdots$. We are going to prove

$$\mathcal{M}_{ash}(f) = \bigcap_{k=1}^{\infty} \bigcap_{r=1}^{\infty} \Big\{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) < \frac{1}{r}\Big\}. \tag{1}$$

Indeed,

$$\begin{aligned}\mu \in \mathcal{M}_{ash}(f) &\Leftrightarrow \mu(Sh(f)) = 1 \\ &\Leftrightarrow \mu\Big(\bigcap_{k=1}^{\infty} G_k\Big) = 1 \\ &\Leftrightarrow \inf_{k\in\mathbb{N}^+} \mu(G_k) = 1 \\ &\Leftrightarrow \mu(G_k) = 1, \qquad \forall k \\ &\Leftrightarrow \mu(X \setminus G_k) = 0, \qquad \forall k.\end{aligned}$$

$$\text{i.e., } \mathcal{M}_{ash}(f) = \bigcap_{k=1}^{\infty} \Big\{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) = 0\Big\}.$$

Besides,

$$\{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) = 0\} = \bigcap_{r=1}^{\infty} \{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) < \frac{1}{r}\}$$

then (1) follows.

Now we prove $\{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) \geq \frac{1}{r}\}$ is closed in $\mathcal{M}(X)$. If $\mu_n \to \mu$ and $\mu_n(X \setminus G_k) \geq \frac{1}{r}$ then

$$\frac{1}{r} \leq \limsup_{n\to\infty} \mu_n(X \setminus G_k) \leq \mu(X \setminus G_k).$$

So, $\mu(X \setminus G_k) \geq \frac{1}{r}$ proving that $\{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) \geq \frac{1}{r}\}$ is closed. Hence, $\{\mu \in \mathcal{M}(X) : \mu(X \setminus G_k) < \frac{1}{r}\}$ is open in $\mathcal{M}(X)$. Therefore, $\mathcal{M}_{ash}(f)$ is G_δ in $\mathcal{M}(X)$ by (1). □

Lemma 2.9. *Let $f : X \to X$ be a homeomorphism on a compact metric space. Then, the set of almost shadowable measures μ satisfying*

$$Supp(\mu) = \bigcup \{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\}$$

is dense in $\mathcal{M}_{ash}(f)$.

Proof. Let $K(X)$ be the set of compact subsets of X endowed with the Hausdorff metric

$$d_H(A, C) = \inf\{\epsilon > 0 : A \subset B(C, \epsilon) \text{ and } C \subset B(A, \epsilon)\},$$

where $B(H, \delta) = \bigcup_{x \in H} B(x, \delta)$ for every $\delta > 0$ and $H \subset X$. Since X is compact, $K(X)$ also is. We say that a map $\phi : \mathcal{M}_{ash}(f) \to K(X)$ is:

- *lower-semicontinuous at* $\mu \in \mathcal{M}_{ash}(f)$ if for any open $V \subset X$ with $V \cap \phi(\mu) \neq \emptyset$ there is a neighborhood $\mathcal{U}$ of μ in $\mathcal{M}_{ash}(f)$ such that $V \cap \phi(\nu) \neq \emptyset$ for every $\nu \in \mathcal{U}$;
- *upper-semicontinuous at* $\mu \in \mathcal{M}_{ash}(f)$ if for any open $V \subset X$ containing $\phi(\mu)$, there is a neighborhood $\mathcal{U}$ of μ in $\mathcal{M}_{ash}(f)$ such that V contains $\phi(\nu)$ for every $\nu \in \mathcal{U}$;
- *lower-semicontinuous* (*upper-semicontinuous*) if it is lower-semicontinuous (respectively upper-semicontinuous) at every $\mu \in \mathcal{M}_{ash}(f)$.

It is easy to check that the map $\phi : \mathcal{M}_{ash}(f) \to K(X)$ defined by $\phi(\mu) = Supp(\mu)$ is lower-semicontinuous. By Theorem 1.8, $\mathcal{M}_{ash}(f)$ is G_δ in $\mathcal{M}(X)$. Since $\mathcal{M}(X)$ is compact, $\mathcal{M}_{ash}(f)$ is Baire space. From this and well-known results [5] we obtain that the set $\mathcal{R}$ of measures $\mu \in \mathcal{M}_{ash}(f)$ where ϕ is lower-semicontinuous is both dense and G_δ subset of $\mathcal{M}_{ash}(f)$. Let us prove

$$Supp(\mu) = \bigcup \{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\}, \qquad \forall \mu \in \mathcal{R}.$$

Take $\mu \in \mathcal{R}$ and $\nu \in \mathcal{M}_{ash}(f)$. Define $\mu_t = (1-t)\mu + t\nu$ for $0 \leq t \leq 1$. Clearly $\mu_t \to \mu$ as $t \to 0$ and then $\phi(\mu_t) \to \phi(\mu)$ with respect to the Hausdorff metric as $t \to 0$. But $\phi(\mu_t) = Supp(\mu) \cup Supp(\nu)$ implying $Supp(\nu) \subset Supp(\mu)$. Thus

$$Supp(\mu) \supset \bigcup \{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\}.$$

The reversed inclusion is obvious since $\mu \in \mathcal{R} \subset \mathcal{M}_{ash}(f)$. □

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3. Proof of remainder theorems

Proof of Theorem 1.3. Let $f : X \to X$ be a homeomorphism of a compact metric space X. By Lemma 2.2,

$$\mathcal{M}_{sh}(f) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, m^{-1}\big). \tag{2}$$

Lemma 2.3 implies that

$$\bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, m^{-1}\big)$$

is closed in $\mathcal{M}(X)$. Therefore

$$\bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} C\big(n^{-1} + l^{-1}, m^{-1}\big)$$

is a F_σ subset of $\mathcal{M}(X)$ for every $n \in \mathbb{N}^+$. So $\mathcal{M}_{sh}(f)$ is a $F_{\sigma\delta}$ subset of $\mathcal{M}(X)$ by (1). $\square$

Proof of Theorem 1.4. First we prove that a finite convex combination of shadowable measures is shadowable.

Take $\mu_i, \cdots, \mu_k \in \mathcal{M}_{sh}(f)$ and $t_1, \cdots, t_n \in (0, 1]$ such that $\sum_{i=1}^{k} t_i = 1$. We are going to prove $\mu = \sum_{i=1}^{k} t_i \mu_i \in \mathcal{M}_{sh}(f)$. Take $\epsilon > 0$, $\delta_1, \cdots, \delta_k > 0$ and Borelian $B_1, \cdots, B_k$ corresponding $\mu_i, \cdots, \mu_k \in \mathcal{M}_{sh}(f)$. Put $\delta = \min\{\delta_1, \cdots, \delta_k\}$ and $B = \cup_{i=1}^{k} B_i$. From

$$\begin{aligned} \mu(B) &= \sum_{i=1}^{k} t_i \mu_i\Big(\bigcup_{j=1}^{k} B_j\Big) \\ &\geq \sum_{i=1}^{k} t_i \mu_i(B_i) \\ &= \sum_{i=1}^{k} t_i = 1, \end{aligned}$$

we get $\mu(B) = 1$. Take δ-pseudo-orbit ξ through B. It follows that ξ is through B_i for some $1 \leq i \leq k$. Since $\delta \leq \delta_i$, ξ is also a δ_i-pseudo-orbit. That is, ξ can be ϵ-shadowed by the choice of δ_i and B_i.

To complete the proof of the theorem, we first assume that $Sh(f)$ is dense in X. By Lemma 2.7, the set of finite convex combinations of Dirac measures supported on points of $Sh(f)$ is dense in $\mathcal{M}(X)$. Lemma 2.6 and the above claim imply that the latter linear combinations are in $\mathcal{M}_{sh}(f)$. So we conclude that $\mathcal{M}_{sh}(f)$ is dense in $\mathcal{M}(X)$.

Conversely, assume that $\mathcal{M}_{sh}(f)$ is dense in $\mathcal{M}(X)$ and by contradiction that $Sh(f)$ is not dense in X. Then, there exists $x \notin \overline{Sh(f)}$. Pick an open neighborhood O of x with $O \cap \overline{Sh(f)} = \emptyset$. Since $\mathcal{M}_{sh}(f)$ is dense in $\mathcal{M}(X)$, there is a sequence $\mu_n \in \mathcal{M}_{sh}(f)$ converging to δ_x. By Lemma 2.4 we have $Supp(\mu_n) \subseteq \overline{Sh(f)}$ and so $\mu_n(O) = 0$ for every n. Then,

$$0 = \liminf_{n \to \infty} \mu_n(O) \geq \delta_x(O) = 1$$

which is absurd, and $Sh(f)$ is dense in X. $\square$

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Proof of Theorem 1.10. By Lemma 2.9 we have $\bigcup\{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\}$ is closed. From Lemma 2.4, observe that $\mathcal{M}_{sh}(f) \subseteq \mathcal{M}_{ash}(f)$. Since $x \in Sh(f)$ if and only if $m_x \in M_{sh}(f)$ by Lemma 2.6, we get $m_x \in \mathcal{M}_{ash}(f)$ for every $x \in Sh(f)$. We conclude that

$$Sh(f) = \bigcup_{x \in Sh(f)} Supp(m_x) \subseteq \bigcup_{\nu \in \mathcal{M}_{ash}(f)} Supp(\nu).$$

On the other hand, $\nu(Sh(f)) = 1$ for every $\nu \in \mathcal{M}_{ash}(f)$. So $Supp(\nu) \subset \overline{Sh(f)}$ for all such measures. Therefore,

$$Sh(f) \subseteq \bigcup\{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\} \subseteq \overline{Sh(f)}.$$

Since $\bigcup\{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\}$ is closed,

$$\bigcup\{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\} = \overline{Sh(f)}.$$

This proves the first part of the theorem.

By Lemma 2.9, the set of measures $\mu \in \mathcal{M}_{ash}(f)$ such that

$$Supp(\mu) = \bigcup\{Supp(\nu) : \nu \in \mathcal{M}_{ash}(f)\}$$

is dense in $\mathcal{M}_{ash}(f)$. Then, the set of measures $\mu \in \mathcal{M}_{ash}(f)$ such that $Supp(\mu) = \overline{Sh(f)}$ is also dense in $\mathcal{M}_{ash}(f)$. This completes the proof of the second part of the theorem. □

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